Chapter 2

Example: Handling Inconsistency

In this chapter a first adaptive logic will be presented. Or rather, starting from the question how inconsistencies should be handled, I shall describe the heuristic process that led to the first adaptive logics.
After arguing that inconsistent theories require being approached in terms of logics, I shall present a simple and basic paraconsistent logic. This will provide the basis for devising inconsistency-adaptive logics, viz. logics that 'interpret' inconsistent theories as consistently as possible. We shall see that there are choices to be made, and that some of the choices are equally sensible from a logical point of view.
This and the next chapter are meant to introduce adaptive logics in an intuitive way. I want to show how one arrives at such logics and also that one needs such logics. A decent formal treatment will be presented in Chapters 4–6. The theoretical underpinning of dynamic proofs is presented in Chapter 4.
While the idea behind adaptive logics was arrived at by considering proofs, we shall see (in the present chapter) that these logics have an elegant semantics.

2.1 A Paradigmatic Problem

Why should one handle inconsistency? Most people think that 'the world is consistent'—let us call them consistentists. For them, so it seems, inconsistent theories, inconsistent beliefs, etc., are bound to be false and hence need not be handled. Other people claim that the world is inconsistent and call themselves dialetheists—Graham Priest and the late Richard Sylvan, né Routley, are leading figures of this group. For them, so it seems, inconsistencies need not to be handled, but need to be lived with.

The semblances described in the previous paragraph are misleading. By endorsing consistentism, one cannot eliminate inconsistencies from one's knowledge, not even from one's best knowledge. Inconsistencies occurred in the history of the sciences. Examples from mathematics are well-known: Cantor's set theory, Frege's set theory, Newton's infinitesimal calculus, ...; for some examples from the empirical sciences see [Bro90, Meh93, Meh99a, Meh02a, Ner02, Nor87, Nor93, Smi88]. Moreover, it appears from the history of the sciences that scientists do not simply give up inconsistent theories in order to start from scratch. Even if a theory T is inconsistent, and so according to consistentists cannot be a correct description of the structure of the world, it may still correctly describe a great deal of the structure of the world. So T may be taken as a valuable starting point in the search of a consistent theory—it often is the only available starting point. Moreover, T may have an important function with respect to other theories or with respect to our knowledge or our action in some domain. So merely giving up T has disastrous consequences. All this shows that the historically documented behaviour of scientists is sensible: they *reason from* inconsistent theories in order to locate the inconsistencies and to find consistent replacements for the theory.

The situation for dialetheists is pretty much the same. They claim that some inconsistencies are true, but consider most inconsistencies as false (and repeatedly stressed this). So when a dialetheist comes across an inconsistency and there is no serious justification for holding it true, the dialetheist will try to eliminate it, locally restoring consistency. The same holds obviously for those who are neither dialetheists nor consistentists.

My personal view on the matter is not very relevant here, but there is no reason why I should hide it. First, consistency does not concern the world, but rather the world as described in terms of a given conceptual system—see for example [Bat80]. Next, there is no guarantee that humans are able to handle a conceptual system in which the world can be described consistently. For example, the structure of the world might be so complex that it cannot be consistently described by a denumerable language. Nevertheless, consistency may be taken as a methodological maxim: one should try to eliminate inconsistencies because, if one succeeds, one obtains in general a theory that is not only consistent but that is also richer than the original paraconsistent theory. However, one should also consider that there may be more urgent tasks at a given moment. I defended this view in [Bat02a], but shall not expand on it here. The point I wanted to make is that, irrespective of one's view on the relation between consistency and truth, one sometimes has to reason from inconsistent theories.

Let us turn to the following paradigm case. Consider a theory $T = \langle \Gamma, \mathbf{CL} \rangle$, i.e. a couple that has a set of non-logical axioms, Γ, as its first element and the logic **CL** as its second element. The theorems[1] of T are the sentences derivable from Γ by **CL**, i.e. the members of $Cn_{\mathbf{CL}}(\Gamma)$. From the fact that the second element of T is **CL** we know that the theory is or was meant and believed to be consistent. Suppose, however, that we are able to derive an inconsistency from Γ, and hence find out that every sentence of the language is a theorem of T as it stands. The trouble is what we should do next. A first alternative is that we simply reject T because it is false and trivial, i.e. contains all sentences as theorems. We have seen that this is not a viable choice. Another alternative is that we replace **CL**, the second element of the theory, by a paraconsistent logic. This means that we move to a theory which has the same non-logical axioms as T, but has a considerably weaker logical basis. The resulting theory, however will in general be awfully weak with respect to T—I shall show this at the beginning of Section 2.2. I do not only mean that the theory is not trivial,

[1]In general, where $T = \langle \Gamma, \mathbf{L} \rangle$, A is a theorem of T iff $A \in Cn_{\mathbf{CL}}(\Gamma)$.

but, more importantly, that it is much weaker than 'what T was intended to be', much weaker than 'T except for the pernicious consequences of its inconsistency'. I realize that such expressions may be intuitively appealing but are nevertheless vague. However, I shall show in Section 2.3 that we can make them fully precise.

Given the failures of the two aforementioned alternatives, let us look for something better. In doing so, we should keep in mind that we have to perform two steps. The first is that, for the time being, i.e. as long as we have no decent alternative for T, we should find a way to 'interpret' T in a way that is as close as possible to the original intention: as consistently as possible. In other words, we need a theory that is closed under **CL** wherever possible without being trivial. The second step consists in looking for a theory T' that can replace T. T itself is heuristically important in this connection: we want a large number of theorems of T, viz. all 'good' ones, to be theorems of T'. In order to make sense of this requirement we need to know which sentences are theorems of T. Alas, if we stick to **CL**, then all sentences of the language are theorems of T, and if we replace **CL** by some paraconsistent logic, we end up with a set of theorems which is too weak. Summarizing the situation: with respect to both problems we need 'T, in its full richness, except for the pernicious consequences of its inconsistency'—let us call this T^*. Neither **CL** nor any paraconsistent logic that has static proofs is able to provide us with T^*. It is this problem that triggered the search for logics that were later called inconsistency-adaptive.

There are indeed several reasons to *locate* the inconsistencies in the inconsistent theory T. A first reason is that we want to eliminate the inconsistencies. However, there is more. The theory we are after, T^*, should be as close as possible to T, except that it should be non-trivial. In other words, we do not want T^* to comprise the **CL**-consequences of the inconsistencies that occur in T, but we want it to comprise all other **CL**-consequences of T. In order to even make sense of this intuitive idea, it is essential that one locates the inconsistencies in T. Adaptive logics are capable of doing so.

Before moving on, let me make explicit a convention that was used implicitly until now. A set of formulas Γ will be said to be inconsistent with respect to a certain logic **L** and with respect to a certain negation ξ iff there is a closed or open formula A such that the existential closure of the conjunction of A and ξA is **L**-derivable from Γ. This sounds complicated but is simple whenever one applies it to a specific logic and language. If, for example, the logic is **CL**, Γ is inconsistent iff there is an $A \in \mathcal{F}_s$ such that $\Gamma \vdash_{\mathbf{CL}} \exists(A \wedge \neg A)$, in which $\exists$ abbreviates an existential quantifier over every variable free in $A \wedge \neg A$. The unqualified words "consistent", "inconsistency", etc., will always refer to the logic and negation under consideration—where confusion may arise, I shall always qualify. In the sequel of this chapter, the paraconsistent logic **CLuN** will be introduced and will play a central role. Let me reassure the reader: it is a trivial consequence of Lemma 2.2.1, which is proved below, that $\Gamma \subseteq \mathcal{W}_s$ is consistent with respect to **CLuN** just in case it is consistent with respect to **CL**. So if you have any intuitions about the latter, just stick to them.

2.2 A Regular Paraconsistent Logic

In order to devise inconsistency-adaptive logics, we need paraconsistent logics of the usual sort, logics that have static proofs. As stated in Section 1.1, a logic

L is *paraconsistent* iff there are inconsistent premise sets to which it does not assign the trivial consequence set. So we can safely define **L** to be paraconsistent iff $A, \neg A \nvdash_{\mathbf{L}} B$.[2]

I shall consider a paraconsistent logic that is very close to **CL**. The problem considered here concerns inconsistency and I shall change **CL** just as much to handle this problem. Put differently, I shall suppose that all presuppositions of **CL** are correct, except that inconsistencies need not lead to triviality and that, in semantic terms, some inconsistent theories have models. Let us call a logic **L** *regular* iff it differs only from **CL** with respect to negation and is moreover a (proper or improper) fragment of **CL**, which means that $Cn_{\mathbf{L}}(\Gamma) \subseteq Cn_{\mathbf{CL}}(\Gamma)$ for all Γ. In this section, I describe the basic regular paraconsistent logic. Its propositional fragment was presented in [Bat80] and was there called **PI**. In [Bat89], the corresponding predicate logic was introduced and named **CLuN**.[3] In that paper, the language schema of **CLuN** is $\mathcal{L}_s$ and so I shall introduce it here. I shall retain the name of the logic when it pertains to $\mathcal{L}_S$. Other regular paraconsistent logics will be considered in Section 7.2.

Consistentists will tend to complain about models verifying both A and $\neg A$ (for some A) because such models do not correspond, on their views, to possible states of the world. This complaint is unfair. If a theory turns out to be inconsistent, we often have to reason from it in order to find a consistent replacement. In order to reason from a theory, we have to consider it as possible; in technical terms: it needs to have models. If we want to reason about our knowledge, we should not only introduce models for possible states of the world, but also for the states corresponding to our theories.

Consider the clause for negation, C¬, in the **CL**-semantics from Section 1.7. This clause may be split into two parts, the consistency requirement

$$\text{If } v_M(A) = 1, \text{ then } v_M(\neg A) = 0. \tag{2.1}$$

and the (negation-)completeness requirement

$$\text{If } v_M(A) = 0, \text{ then } v_M(\neg A) = 1. \tag{2.2}$$

A semantics for **CLuN** is obtained by replacing, in the **CL**-semantics, C¬ by (2.2), which means that the consistency requirement is dropped.

This semantics is indeterministic. A semantics is *deterministic* iff the value that the valuation function assigns to a formula is a function of the model. Unlike the **CL**-semantics, the **CLuN**-semantics defined before is indeterministic. Indeed, if, for some A, $v_M(A) = 1$, the semantics does not specify the value of $v_M(\neg A)$. Of course $v_M(\neg A)$ has a value in every model, because $v_M \colon \mathcal{W}_{\mathcal{O}} \to \{0, 1\}$. It follows that one cannot say, for this semantics, that v_M is *determined* by a model $M = \langle D, v\rangle$. Indeed, $v_M(A) = 1$ is compatible with both $v_M(\neg A) = 0$ and $v_M(\neg A) = 1$. Nor can one say that a *model* verifies or falsifies a formula, because the model (in the strict sense of $M = \langle D, v\rangle$) does not by itself determine the truth value of every formula.

There is nothing wrong with an indeterministic semantics, provided one adjusts a few definitions. For example, a valuation v_M should be said to be *associated* with (rather than determined by) a model $M = \langle D, v\rangle$. Several valuations

[2] Note that $A, \neg A \nvdash_{\mathbf{L}} B$ does not exclude that $A, \neg A \vdash_{\mathbf{L}} B$ holds for some A and B. Thus some logics **L** are paraconsistent because $p, \neg p \nvdash_{\mathbf{L}} q$, even if $(p \wedge r), \neg(p \wedge r) \vdash_{\mathbf{L}} q$—see [Bat80].

[3] The logic **CLuN** is like **CL**, except that it allows for gl*u*ts with respect to *N*egation.

may be associated with the same model if the semantics is indeterministic. Also a valuation (rather than a model) should be said to verify or falsify a formula, and the semantic consequence relation should be defined in terms of valuations.[4] I refer to [Avr05, ABNK07, AK05] for some interesting technical studies of indeterministic semantics. In this book, indeterministic semantic systems will reappear in Chapter 8.

CLuN also has a deterministic semantics. What I mean by this is that the logic **CLuN** as defined by the above indeterministic semantics is also defined by a deterministic semantics. The latter strikingly resembles the **CL**-semantics. There is one difference, which I shall clarify after presenting the semantics. As for **CL**, this semantics is formulated in the language $\mathcal{L}_\mathcal{O}$, which again is $\mathcal{L}_s$ extended with the set of pseudo-constants $\mathcal{O}$. A **CLuN**-model $M = \langle D, v\rangle$, in which D is a set and v an assignment function, is an interpretation of $\mathcal{W}_\mathcal{O}$ and hence of $\mathcal{W}_s$. The assignment function v is defined by:

C1 $v\colon \mathcal{W}_\mathcal{O} \to \{0,1\}$
C2 $v\colon \mathcal{C} \cup \mathcal{O} \to D$ (where $D = \{v(\alpha) \mid \alpha \in \mathcal{C} \cup \mathcal{O}\}$)
C3 $v\colon \mathcal{P}^r \to \wp(D^r)$

The valuation function $v_M\colon \mathcal{W}_\mathcal{O} \to \{0,1\}$ determined by M is defined as follows:

C$\mathcal{S}$ where $A \in \mathcal{S}$, $v_M(A) = 1$ iff $v(A) = 1$
C$\mathcal{P}^r$ $v_M(\pi^r\alpha_1 \ldots \alpha_r) = 1$ iff $\langle v(\alpha_1), \ldots, v(\alpha_r)\rangle \in v(\pi^r)$
C= $v_M(\alpha = \beta) = 1$ iff $v(\alpha) = v(\beta)$
C¬ $v_M(\neg A) = 1$ iff $v_M(A) = 0$ or $v(\neg A) = 1$
C⊃ $v_M(A \supset B) = 1$ iff $v_M(A) = 0$ or $v_M(B) = 1$
C∧ $v_M(A \wedge B) = 1$ iff $v_M(A) = 1$ and $v_M(B) = 1$
C∨ $v_M(A \vee B) = 1$ iff $v_M(A) = 1$ or $v_M(B) = 1$
C≡ $v_M(A \equiv B) = 1$ iff $v_M(A) = v_M(B)$
C∀ $v_M(\forall\alpha A(\alpha)) = 1$ iff $\{v_M(A(\beta)) \mid \beta \in \mathcal{C} \cup \mathcal{O}\} = \{1\}$
C∃ $v_M(\exists\alpha A(\alpha)) = 1$ iff $1 \in \{v_M(A(\beta)) \mid \beta \in \mathcal{C} \cup \mathcal{O}\}$

$M \Vdash A$ iff $v_M(A) = 1$. In view of Section 1.6, this semantics defines $\Gamma \vDash_{\mathbf{CLuN}} A$ and $\vDash_{\mathbf{CLuN}} A$ for all $\Gamma \subseteq \mathcal{W}_s$ and $A \in \mathcal{W}_s$.

Note that one cannot tell whether a *model* $M = \langle D, v\rangle$ belongs to the **CL**-semantics or to the **CLuN**-semantics. However, depending on the logic, a different valuation function is determined by the model. This is sensible. The model concerns the possible states of 'the world' whereas the valuation function concerns the behaviour of the logical symbols. In the **CL**-semantics the value of $v_M(\neg A)$ is a function of the value of $v_M(A)$; in the **CLuN**-semantics the value of $v_M(\neg A)$ is a function of the value of $v_M(A)$ *and* of the value of $v(\neg A)$.[5] This modification is related to the specificity of **CLuN**, viz. to the fact that it is

[4] An alternative is to rename a model M to, say, a structure S, to let v_S (rather than v_M) be a valuation determined by a structure, and to define a model as $M = \langle S, v_S\rangle$. In this case the definition of the semantic consequence relation can be retained, but my definition of an indeterministic semantics (see the first sentence of the previous paragraph in the text) has to be adjusted.

[5] In the **CL**-semantics, the values $v(A)$ have only an effect on the valuation function iff $A \in \mathcal{S}$; in the **CLuN**-semantics, the values $v(A)$ have only an effect on the valuation function iff $A \in \mathcal{S} \cup \{\neg A \mid A \in \mathcal{W}_\mathcal{O}\}$. In the semantics of other logics, $v(A)$ may have an effect on the valuation function for different sets of formulas—see Chapter 8.

a paraconsistent logic and hence should allow for the possibility that a model verifies A as well as $\neg A$.

As usual, two semantic systems will be called *equivalent* iff to every model M of one system corresponds a model M' of the other system such that M and M' verify exactly the same formulas of the considered language schema. If one of the semantic systems is indeterministic, the formulation has to be adjusted as it is done in the following theorem.

Theorem 2.2.1 *For every model M of the deterministic* **CLuN**-*semantics, there is, in the indeterministic* **CLuN**-*semantics, a model M' and a valuation function $v^i_{M'}$ associated with M' such that, for all $A \in \mathcal{W}_{\mathcal{O}}$, $v_M(A) = v^i_{M'}(A)$, and* vice versa.

Proof outline. Starting from $M = \langle D, v\rangle$: define $M' = \langle D', v'\rangle$ with $D' = D$ and $v' = v$; define, for all $A \in \mathcal{W}_{\mathcal{O}}$, $v^i_{M'}(\neg A) = v_M(\neg A)$; show by an induction on the complexity of A that, for all $A \in \mathcal{W}_{\mathcal{O}}$, $v^i_{M'}(A) = v_M(A)$.

Starting from $M' = \langle D', v'\rangle$ and $v^i_{M'}$: define $M = \langle D, v\rangle$ in such a way that $D = D'$ and v is exactly like v' except that, for all $A \in \mathcal{W}_{\mathcal{O}}$, $v(\neg A) = 1$ iff $v^i_{M'}(\neg A) = v^i_{M'}(A) = 1$;[6] show by an induction on the complexity of A that, for all $A \in \mathcal{W}_{\mathcal{O}}$, $v_M(A) = v^i_{M'}(A)$. ∎

An adequate axiomatic system for **CLuN** is obtained by dropping A¬2 from the axiomatic system for **CL** presented in Section 1.7 and by attaching to A=2 the restriction "provided $A(\alpha) \in \mathcal{W}^p_s$", where $\mathcal{W}^p_s$ is the set of *primitive* closed formulas (those not containing any logical symbols other than identity).[7] Adding the same restriction to the **CL**-axiom schema A=2 would result in an axiom system that is equivalent to that for **CL** because all so lost axioms would still be theorems. Dropping the restriction from the axiom system for **CLuN** results precisely in what one expects: **CLuN** extended with (universal) Replacement of Identicals. The **CLuN**-semantics is a natural weakening of the **CL**-semantics and it requires that the restriction is attached to A=2 in order for the axiomatization to be adequate. Of course, there is a version of **CLuN** that validates A=2 without restriction—the way in which the semantics is adjusted to this effect is the same as for the logic **CLuNs** from Section 7.2.

CLuN is well axiomatized in the sense of [AB75, p. 374] by the preceding axiomatic system ("in interesting cases, all theorems in a set of [logical symbols] can be derived using *only* axioms and rules which contain those [symbols]"). The same obviously holds for the axiom system for **CL** that appears in Section 1.7.

A nice result on alternative axiomatizations of **CLuN** was presented by Sergei Odintsov in [Odi06]—see also [Odi03] for the context and motivation. Remove negation from the primitive symbols of the language and add the unary operator $\circledcirc$ to it. Remove the axiom on negation, viz. excluded middle, from the previous axiom system. Define negation explicitly by $\neg A =_{df} A \supset \circledcirc A$. Alternatively, remove negation and add the unary operator $\circledast$; add the axiom $\circledast A \supset A$ and define $\neg A =_{df} A \supset \circledast A$. As $A \vee (A \supset B)$ is a theorem of the positive fragment of **CLuN**, both definitions give us $A \vee \neg A$. Moreover, in the original axiomatization, $\circledcirc A$ can be defined by $\neg A$ and $\circledast A$ by $A \wedge \neg A$.

[6] If $A \notin \mathcal{S}$ and A does not have the form $\neg B$, then the value of $v(A)$ has no effect on the valuation function of the deterministic **CLuN**-semantics.

[7] Primitive formulas are sometimes called atomic formulas, but I shall need the word "atom" in a different sense.

This approach has a direct connection to the deterministic semantics. The expression $\circledcirc A$ corresponds to $v(A)$: $v_M(\neg A) = 1$ iff $v_M(A \supset \circledcirc A) = 1$, and the latter reduces to $v_M(A) = 0$ or $v(\neg A) = 1$. The expression $\circledast A$ corresponds to "$v_M(A) = 1$ and $v(\neg A) = 1$". This gives us: $v_M(\neg A) = 1$ iff $v_M(A) = 0$ or $(v_M(A) = 1$ and $v(A) = 1)$.

Each of the following theorems is provable. The proofs are easily adjusted from the corresponding proofs for **CL** in Section 1.7 (sometimes by first proving the corresponding lemmas).

Theorem 2.2.2 **CLuN** *has static proofs.*

Corollary 2.2.1 **CLuN** *is reflexive, transitive, monotonic, compact, and there is a positive test for it.*[8]

Theorem 2.2.3 *If* $\Gamma \vdash_{\mathbf{CLuN}} B$ *and* $A \in \Gamma$, *then* $\Gamma - \{A\} \vdash_{\mathbf{CLuN}} A \supset B$. *(Generalized Deduction Theorem for* **CLuN***)*

Theorem 2.2.4 *If* $\Gamma \vdash_{\mathbf{CLuN}} A$, *then* $\Gamma \vDash_{\mathbf{CLuN}} A$. *(Soundness of* **CLuN** *with respect to its semantics)*

Theorem 2.2.5 *If* $\Gamma \vDash_{\mathbf{CLuN}} A$, *then* $\Gamma \vdash_{\mathbf{CLuN}} A$. *(Completeness of* **CLuN** *with respect to its semantics)*

A few comments on **CLuN** will be useful. **CLuN** is just like **CL** except that the properties of negation are weakened to $A \vee \neg A$. So **CLuN** contains positive **CL**, which is obtained from **CL** by removing all properties of negation.[9] Actually it is identical to positive **CL** to which a negation is added with $A \vee \neg A$ as its only property—the weakness of this negation will be highlighted when we shall consider extensions of **CLuN**. Note that $\vdash_{\mathbf{CLuN}} p \equiv (p \vee p)$ whereas $\nvdash_{\mathbf{CLuN}} \neg p \equiv \neg(p \vee p)$. So Replacement of Equivalents does not hold in **CLuN**—RoE: if $\vdash A \equiv B$, then $\vdash (\ldots A \cdots) \equiv (\ldots B \cdots)$. Also $a = b \nvdash_{\mathbf{CLuN}} \neg Pa \equiv \neg Pb$, which illustrates that Replacement of Identicals does not hold either—RoI: $\alpha = \beta, A(\alpha) \vdash A(\beta)$.[10] Incidentally, Replacement of Identicals can easily be restored, as we shall see when the logic **CLuNs** is introduced in Section 7.2.

None of the usual reduction theorems for negation ($\neg\neg A \equiv A$, $\neg(A \vee B) \equiv (\neg A \wedge \neg B)$, etc.) holds in **CLuN**. There are richer paraconsistent logics in which those theorems hold (although not all of them together). Those logics play a role in other chapters of this book, but in the present chapter I shall only consider the weakest and most basic paraconsistent logic, which is **CLuN**.

Material implication, viz. $\supset$, is detachable in **CLuN**, which means that Modus Ponens holds for it. Disjunctive Syllogism does not hold in **CLuN**: $A, \neg A \vee B \nvdash_{\mathbf{CLuN}} B$.

My final comment is not technical but rather conceptual and philosophical. Many people have been baffled by the claim that Disjunctive Syllogism is incorrect according to some logics; some were even riled by the fact that such logics

[8] The propositional fragment of **CLuN** is as decidable as that of **CL**.

[9] In the literature, "positive logic" is sometimes used for positive intuitionistic logic. This is obviously weaker than positive **CL**.

[10] RoE is valid provided A does not occur within the scope of a negation. RoI is valid provided α does not occur within the scope of a negation in A. The scope of a negation is the formula that immediately follows the negation; thus the scope of the negation in $p \wedge (\neg(q \vee r) \supset s)$ is $(q \vee r)$. Similarly for the scope of a quantifier. The scope of a binary connective comprises the formula immediately preceding it as well as the formula immediately following it.

were presented. And yet the matter is very simple and no big deal at all. We all agree that $A \vee B$ and $\neg A$, the two (local) premises of Disjunctive Syllogism, are jointly equivalent to $(A \wedge \neg A) \vee (B \wedge \neg A)$. If the underlying logic is **CL**, then $A \wedge \neg A$ is bound to be false. So if $(A \wedge \neg A) \vee (B \wedge \neg A)$ is true, then so is B. However, we have seen in Section 2.1 that one sometimes needs to reason from inconsistent premises and that this requires that one considers some inconsistencies as true. This is why we need paraconsistent logics. Thus, according to **CLuN**, $A \wedge \neg A$ *may* be true. But if $A \wedge \neg A$ is true, then $(A \wedge \neg A) \vee (B \wedge \neg A)$ is true, even if B is false. So, if the logic is paraconsistent, the joint truth of $A \vee B$ and $\neg A$ does not warrant the truth of B. So, if the logic is paraconsistent, Disjunctive Syllogism is *not* truth preserving.

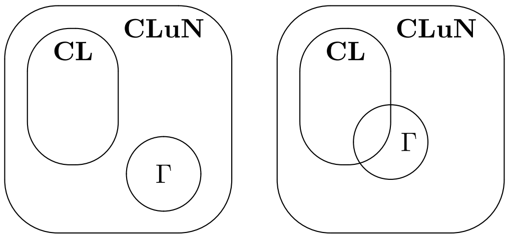
It is instructive to realize that the situation may be described as follows. According to both **CL** and **CLuN**, $A \vee B, \neg A \vDash B \vee (A \wedge \neg A)$. So all **CLuN**-models of $\{A \vee B, \neg A\}$ that falsify $A \wedge \neg A$ verify B. This result will be generalized to the Theorem 2.2.6 below. This theorem is extremely important because it
motor? will function as the motor of the adaptive logics devised in Section 2.3.

Even if two models, M and M', belong to different semantic systems, they will be called *equivalent* iff they verify the same formulas, i.e. $\{A \in \mathcal{W}_{\mathcal{O}} \mid M \Vdash A\} = \{A \in \mathcal{W}_{\mathcal{O}} \mid M' \Vdash A\}$. This presupposes obviously that both semantic systems are defined with respect to the same language. If they are not, one might still call both models $\mathcal{L}$-equivalent iff they verify the same members of $\mathcal{W}$. A model M is called *consistent* iff there is no $B \in \mathcal{W}_{\mathcal{O}}$ for which $B, \neg B \in \{A \in \mathcal{W}_{\mathcal{O}} \mid M \Vdash A\}$.[11] The set of consistent **CLuN**-models form a semantics for **CL** in view of the following lemma.

Lemma 2.2.1 *Every consistent* **CLuN**-*model is equivalent to a* **CL**-*model and, for every* **CL**-*model* M*, there is an equivalent* **CLuN**-*model.*

Proof outline. Given a consistent **CLuN**-model $M = \langle D, v\rangle$, consider the **CL**-model $M' = \langle D', v'\rangle$ with $D' = D$ and $v' = v$ and verify that M and M' are equivalent. The only crucial case concerns the clause for negation, C¬. If there were a $A \in \mathcal{L}_{\mathcal{O}}$ for which $v(\neg A) = v_M(A) = 1$, then M would be inconsistent.

Given the **CL**-model $M = \langle D, v\rangle$, an equivalent **CLuN**-model $M' = \langle D', v'\rangle$ is obtained by setting $D' = D$ and choosing a v' that is exactly like v except that $v(\neg A) = 0$ for all $A \in \mathcal{W}_{\mathcal{O}}$. ■



Abnormal Γ Normal Γ

In view of the lemma, the **CL**-models may be identified with the consistent **CLuN**-models, as the drawing illustrates. Abnormal Γ, which here means inconsistent Γ, have inconsistent **CLuN**-models only. Normal Γ have **CL**-models,

[11]Given the nature of models, there is no need to refer to the logical closure of the set of verified formulas. Note that, if M is inconsistent, there need not be a $B \in \mathcal{W}_s$ for which $B, \neg B \in \{A \in \mathcal{W}_{\mathcal{O}} \mid M \Vdash A\}$. However, there will be a $B \in \mathcal{F}_s$ such that the existential closure of $B \wedge \neg B$, which is a member of $\mathcal{W}_s$ is verified by M.

but they also have *inconsistent* **CLuN**-models. Consider for example, the set $\{p\}$. Its **CLuN**-models are those that verify p. So all **CLuN**-models that verify $p \wedge \neg p$ are models of $\{p\}$.

Let sub(A), a set of $\mathcal{L}_\mathcal{O}$-subformulas of A, be defined as the smallest Σ such that (i) $A \in \Sigma$, (ii) if $\xi B \in \Sigma$, where ξ is a unary connective of $\mathcal{L}_\mathcal{O}$, then $B \in \Sigma$, (iii) if $(B\xi C) \in \Sigma$, where ξ is a binary connective of $\mathcal{L}_\mathcal{O}$, then $B, C \in \Sigma$, and (iv) if $\xi\alpha B(\alpha) \in \Sigma$, where ξ is a quantifier of $\mathcal{L}_\mathcal{O}$, then $B(\alpha) \in \Sigma$. Define $\mathrm{sub}(\Gamma) =_{df} \{\mathrm{sub}(A) \mid A \in \Gamma\}$. Let $\exists A$ be the existential closure of A, which is A preceded by an existential quantifier over every variable that is free in A (possibly in some preferred order). Let $\bigvee(\Gamma)$ be the disjunction of the members of Γ, provided Γ is a finite set. As sub(A) is a finite set, $A \vee \bigvee\{\exists(B \wedge \neg B) \mid B \in \mathrm{sub}(A)\}$ is a closed formula.

Let $I = \{f\colon \mathcal{C} \cup \mathcal{V} \rightarrow \mathcal{C} \cup \mathcal{O} \mid \text{if } \alpha \in \mathcal{C}, \text{ then } f(\alpha) = \alpha\}$,[12] and let, for all $f \in I$ and $A \in \mathcal{W}_\mathcal{O}$, $f(A)$ be the result of replacing in A every $\alpha \in \mathcal{C} \cup \mathcal{V}$ by $f(\alpha)$.

As both **CL** and **CLuN** are sound and complete with respect to their semantics (Theorems 1.7.4, 1.7.5, 2.2.4, and 2.2.5), I shall switch without warning between the derivability relation and the semantic consequence relation of both logics (in the proof of the following theorem as well as in other proofs).

Theorem 2.2.6 $\Gamma \vdash_{\mathbf{CL}} A$ *iff there are* $C_1, \ldots, C_n$ *such that* $\Gamma \vdash_{\mathbf{CLuN}} A \vee (\exists(C_1 \wedge \neg C_1) \vee \ldots \vee \exists(C_n \wedge \neg C_n))$. *(Derivability Adjustment Theorem)*[13]

Proof outline. $\Rightarrow$ Suppose that $\Gamma \vdash_{\mathbf{CL}} A$. By the compactness of **CL** (see Corollary 1.7.1), there are $B_1, \ldots, B_m \in \Gamma$ for which $B_1, \ldots, B_m \vdash_{\mathbf{CL}} A$, whence $\vdash_{\mathbf{CL}} B_1 \supset (\ldots \supset (B_m \supset A)\ldots)$ by the Deduction Theorem (Theorem 1.7.2). Let X abbreviate $(B_1 \supset (\ldots \supset (B_m \supset A)\ldots))$. Suppose that a **CLuN**-model $M = \langle D, v\rangle$ falsifies

$$X \vee \bigvee\{\exists(C \wedge \neg C) \mid C \in \mathrm{sub}(X)\}$$

and hence falsifies all members of $\{\exists(C \wedge \neg C) \mid C \in \mathrm{sub}(X)\}$. Let $M' = \langle D, v\rangle$ be a **CL**-model, with D and v as for M. I show, by an induction on the complexity of formulas, that M and M' verify the same closed subformulas of X and the same instances of open subformulas of X. In other words, I shall show that

$$\text{for all } f \in I,\ v_M(f(C)) = v_{M'}(f(C))\,. \tag{2.3}$$

holds whenever $C \in \mathrm{sub}(X)$. As for the basis, (2.3) obviously holds if the complexity of C is 0. As for the induction step, all cases are obvious except for the following.

Case $\neg$. Let C be $\neg D$. As $D \in \mathrm{sub}(X)$, $\exists(D \wedge \neg D) \in \{\exists(C \wedge \neg C) \mid C \in \mathrm{sub}(X)\}$. For all $f \in I$, if $v_M(f(D)) = v_{M'}(f(D)) = 0$, then $v_M(f(\neg D)) = v_{M'}(f(\neg D)) = 1$, and if $v_M(f(D)) = v_{M'}(f(D)) = 1$, then, as $v_{M'}(\exists(D \wedge \neg D)) = 0$, $v_M(f(\neg D)) = 0 = v_{M'}(f(\neg D))$.

Cases $\forall$ *and* $\exists$. Let C be $\forall\alpha D$, respectively $\exists\alpha D$. For all $f \in I$, if $v_M(f(D)) = v_{M'}(f(D))$. So $v_M(f(\forall\alpha D)) = v_{M'}(f(\forall\alpha D))$ and $v_M(f(\exists\alpha D)) = v_{M'}(f(\exists\alpha D))$.

[12] The f differ from each other in $f(\alpha)$ for $\alpha \in \mathcal{V}$. The set I is obviously non-denumerable.

[13] The corresponding theorem is proved for all adaptive logics in standard format as Theorem 4.6.2 in Section 4.6. As we shall see in that Section, the contextual meaning of the theorem is different in both cases, and so is the proof.

So $v_M(X) = v_{M'}(X)$. But $v_{M'}(X) = 1$ because X is a **CL**-theorem. This contradicts the supposition.

$\Leftarrow$ Suppose that $\Gamma \vdash_{\mathbf{CLuN}} A \vee ((C_1 \wedge \neg C_1) \vee \ldots \vee (C_n \wedge \neg C_n))$. It follows that, if Γ has **CLuN**-models, then they all verify either A or $(C_1 \wedge \neg C_1) \vee \ldots \vee (C_n \wedge \neg C_n)$. So, if Γ has consistent **CLuN**-models, they all verify A. So, if Γ has **CL**-models, they all verify A in view of Lemma 2.2.1. ■

2.3 Enter Dynamics

To keep the exposition as clear as possible, I shall first deal with the dynamics at the propositional level, even if I will refer to the predicative logic **CLuN**. Complications arising at the predicative level will be introduced in Section 2.3.4.

In Section 2.1, I promised to return to the following claim. If a theory T was intended (and believed) to be consistent and hence was given **CL** as its underlying logic, but turns out inconsistent, then replacing **CL** by monotonic paraconsistent logics offers a theory that is much weaker than 'what T was intended to be'. Consider the premise set $\{\neg p \vee q, p, r \vee s, \neg r, \neg p\}$. Obviously, this is a toy example. My aim is to illustrate the point, not to offer a historical case study. As the premise set is inconsistent, it requires a formula to behave inconsistently, viz. p. So Disjunctive Syllogism can obviously not be applied in general. In this sense, **CLuN** reacts sensibly. The premise $\neg p \vee q$ is itself a consequence of the premise $\neg p$, but so are $\neg p \vee \neg q$, $\neg p \vee t$, and so on. If Disjunctive Syllogism were added to **CLuN**, then not only q but every formula would be derivable. So the consequence set would be trivial. But clearly, **CLuN** does not require r to behave inconsistently. So T, as originally intended, should have s as a theorem. In this sense, **CLuN** and other paraconsistent logics are too weak. If one takes a more sophisticated premise set, the situation becomes only worse.

So the reasoning from T should proceed in such a way that one obtains an 'interpretation' of it that is as consistent as possible. Precisely for this reason, the reasoning cannot proceed in terms of a monotonic paraconsistent logic. Indeed, every such logic invalidates certain **CL**-rules, for example Disjunctive Syllogism.[14] However, as we saw from the previous example, the requested reasoning should not invalidate certain rules of inference of **CL**, but only certain *applications* of these rules. Let me express this more precisely. For certain rules, an application should be valid if specific involved formulas behave consistently on the theory, and invalid otherwise. Precisely this proviso causes the reasoning to be internally dynamic: there is no positive test for the consistent behaviour of some formula on a set of premises.[15]

Here is the idea to approach this kind of reasoning. Let us proceed in terms of the basic paraconsistent logic **CLuN**, leaving variants for later. On the one hand, **CLuN** determines a set of unconditional rules of inference, viz. the usual ones: if $A_1, \ldots, A_n \vdash_{\mathbf{CLuN}} B$, then one may infer B from $A_1, \ldots, A_n$. On the other hand, **CLuN** determines a set of conditional rules of inference in view of Theorem 2.2.6: if $A_1, \ldots, A_n \vdash_{\mathbf{CLuN}} B \vee ((C_1 \wedge \neg C_1) \vee \ldots \vee (C_n \wedge \neg C_n))$, one may

[14] Some paraconsistent logics validate Disjunctive Syllogism but avoid Ex Falso Quodlibet by invalidating some other **CL**-rule. I shall spell out such a logic in Section 7.2.

[15] Explicating this kind of reasoning was at the origin of the adaptive logic programme—see [Bat89], [Bat99b], and many other papers.

infer B from $A_1, \ldots, A_n$ on the condition that $C_1, \ldots C_n$ behave consistently. This seems a viable idea, provided we are able to articulate the meaning of "behaves consistently". This phrase should obviously refer to the premise set. So the following convention seems a good starting point:

(*) A sentence behaves consistently according to a premise set Γ iff either the sentence itself or its negation is not derivable from Γ.

In view of this, the conditional rule might be phrased as:

(†) Where $A_1, \ldots, A_n \vdash_{\mathbf{CLuN}} B \vee ((C_1 \wedge \neg C_1) \vee \ldots \vee (C_n \wedge \neg C_n))$ and, for each C_i, either C_i or $\neg C_i$ is not derivable from the premise set under consideration, to infer B from $A_1, \ldots, A_n$.

Although (†) is intuitively appealing, it does not tell us how to define an inconsistency-adaptive logic from **CLuN**. It is actually a circular statement. It states that a sentence E is derivable if another sentence, D, is not derivable. But this comes to: D is derivable if E is not. In order to define an inconsistency-adaptive logic we need a way to circumvent this circularity.

A well-tried means to avoid circularity is to come down from the heaven of systematic and abstract definition to the earthly level of concrete actions. As far as logic is concerned this means that we should concentrate on actual proofs instead of derivability or semantic consequence. The construction of proofs is determined by instructions, viz. commands and permissions; if you have written down this and that, then (you may) do so and so. It is obvious, however, that (*) is altogether inappropriate to formulate instructions for proof construction, if these instructions themselves are to be based upon concrete matters such as the lines of which the proof consists at some time. Looking for an appropriate substitute for (*) we gain some hints again by keeping in mind the object we pursue: T, 'in its full richness', 'except for the pernicious consequences of its inconsistency'. We want all rules of inference that are validated by **CL** to apply, except in those cases where they lead to triviality. In other words, we suppose that all statements of the theory (or all consequences of the premise set) are consistent, with the exception of those that the theory requires to be inconsistent. This suggests that we replace (*) by:

(**) A sentence behaves consistently unless and until proven otherwise.

As a consequence we obtain the following corresponding conditional rule of inference:

(††) If $A_1, \ldots, A_n \vdash_{\mathbf{CLuN}} B \vee ((C_1 \wedge \neg C_1) \vee \ldots \vee (C_n \wedge \neg C_n))$ then, unless both some C_i and its negation have actually been derived, to infer B from $A_1, \ldots, A_n$.

This move from the syntactic and semantic level to the pragmatic level eliminates the circularity involved in (†).

The reference to time in (††) is essential. Suppose p has not been proved at some time (stage in the proof) and hence behaves consistently at that time; suppose moreover that, given the consistent behaviour of p, we derive q from $p \vee q$ and $\neg p$. It is possible that we later derive p, for example from r and $r \supset p$. From this time on, q is no longer derivable from $p \vee q$ and $\neg p$. A very simple example is displayed in the following proof:

1	$p \vee q$	premise
2	$\neg p$	premise
3	$r \supset p$	premise
4	r	premise
5	q	from 1 and 2 and the consistent behaviour of p
6	p	from 3 and 4

At stage 4, viz. after line 4 has been written, and also at stage 5, q is derivable from the premises; at stage 6 q is no longer derivable, because p has turned out to behave inconsistently. Obviously, it cannot merely depend on the accidental way in which we construct a proof, whether or not some sentence belongs to 'T except for the pernicious consequences of its inconsistency'. Hence, from stage 6 on, we should consider q as not derivable, and we should keep in mind not to use it for further inferential steps. To assist our memory, we might delete line 5 after line 6 was written, or even add "deleted at stage 6". When I first discovered these logics, I claimed to delete lines and actually put them in square brackets for readability. Later it turned out, however, that deleting lines leads to useless trouble for some adaptive logics. This is why I shall *mark* lines when the formula derived in them is no longer derivable. The advantage of marking is that lines may not only be unmarked at a stage and marked at a later one, but may also be unmarked at a still later stage.

Some readers probably still wonder whether something sensible is going to come out. I beg their patience. I shall prove some nice properties of the logics I am at the point of articulating, but first I have two further clarifications. The first concerns speech. The 'stages' I need are merely members of an ordered series of intervals, each of which occurs 'at the moment' that some line in the proof is written down. They are named after the line number of this line. The proof at that stage is a sequence of lines, starting with the first and ending with the line after which the stage is named. In the above proof, line 5 should be marked at stage 6. The second clarification concerns notation. The lines of a standard (explicit) proof consist of three elements:

(i) a line number,
(ii) a formula derived,
(iii) a justification: the line numbers of the sentences from which (ii) is derived, and the rule of inference that justifies the derivation.

In inconsistency-adaptive logic proofs, it is preferable to add a fourth element to each line:

(iv) the set of formulas that have to behave consistently in order for (ii) to be derivable by the rule mentioned in (iii) and from the lines enumerated in (iii).

By adding (iv) it will be easy to detect at any time which lines have to be marked. Moreover, we do not have to mention the awkward "and the consistent behaviour of" as in line 5 of the preceding proof. So much for preparation. Let's move ahead to the first attempt to formulate the inconsistency-adaptive logic.

2.3.1 A Failing Strategy

Let us first look at the rules for adding lines to a proof.

Prem At any stage, one may write down a line consisting of (i) an appropriate line number, (ii) a premise, (iii) "Prem", and (iv) "$\emptyset$".

RU If $A_1, \ldots, A_n \vdash_{\mathbf{CLuN}} B$ and each A_i occurs as the second element of a line that has, say, Δ_i as its fourth element, then one may add a line consisting of (i) an appropriate line number, (ii) B, (iii) the numbers of the lines of which the A_i are the second element, followed by "RU", (vi) $\Delta_1 \cup \ldots \cup \Delta_n$.

RC If $A_1, \ldots, A_n \vdash_{\mathbf{CLuN}} B \vee ((C_1 \wedge \neg C_1) \vee \ldots \vee (C_m \wedge \neg C_m))$ and each A_i occurs as the second element of a line that has, say, Δ_i as its fourth element, then one may add a line consisting of (i) an appropriate line number, (ii) B, (iii) the numbers of the lines of which the A_i are the second element, followed by "RC", (vi) $\{C_1 \wedge \neg C_1, \ldots, C_m \wedge \neg C_m\} \cup \Delta_1 \cup \ldots \cup \Delta_n$.

RU is short for Unconditional Rule and RC for Conditional Rule. Only the second introduces new elements to the condition. I decided to phrase object-level proofs in terms of these two 'generic rules' in order to avoid dealing with a plethora of specific (primitive and derived) rules. In all examples I shall give, I shall take care that the applied specific rule is straightforward.

The three rules are fully spelled out. They actually become more transparent when written in schematic form. To simplify the notation even more, I shall uniformly use Γ to refer to the premise set and I shall use $Dab(\Delta)$ to abbreviate the disjunction of a *finite* set of contradictions—so in a *Dab*-formula, such as $Dab(\Delta)$, Δ is (in the present context) always a finite set of contradictions[16] and $Dab(\Delta)$ is the disjunction of the members of Δ.

Prem If $A \in \Gamma$:
$$\begin{array}{ll} \ldots & \ldots \\ \hline A & \emptyset \end{array}$$

RU If $A_1, \ldots, A_n \vdash_{\mathbf{CLuN}} B$:
$$\begin{array}{ll} A_1 & \Delta_1 \\ \ldots & \ldots \\ A_n & \Delta_n \\ \hline B & \Delta_1 \cup \ldots \cup \Delta_n \end{array}$$

RC If $A_1, \ldots, A_n \vdash_{\mathbf{CLuN}} B \vee Dab(\Theta)$:
$$\begin{array}{ll} A_1 & \Delta_1 \\ \ldots & \ldots \\ A_n & \Delta_n \\ \hline B & \Delta_1 \cup \ldots \cup \Delta_n \cup \Theta \end{array}$$

I still need to specify marking. Note that this cannot be governed by a rule. A rule is an instruction that comes with a universal permission and marking is not a matter to be decided by the person who constructs the proof. So it should be governed by a definition. I shall, however, change the convention underlying (*) and implicitly also underlying (**) in the previous section. Contradictions will be required to explicitly occur in the proof before they cause lines to be marked. In a sense, the author of the proof should *see* that there is a contradiction before the marks are affected—see Section 4.10.

Definition 2.3.1 *A line l that has Δ as its fourth element is marked iff a member of Δ has been derived.*

[16]Later, this expression will obtain a more general meaning. Its name abbreviates "disjunction of abnormalities". At the present point, abnormalities are contradictions. We shall see later that the notion of abnormality has a much more general meaning.

Let us consider a simple proof that illustrates the matter. I shall write the marks, if any, at the right of the lines as a check followed by the number of the stage at which the line is marked.

1	$\neg p \wedge r$	Prem	$\emptyset$	
2	$q \supset p$	Prem	$\emptyset$	
3	$q \vee \neg r$	Prem	$\emptyset$	
4	$r \supset p$	Prem	$\emptyset$	
5	$\neg p$	1; RU	$\emptyset$	
6	r	1; RU	$\emptyset$	
7	$\neg q$	2, 5; RC	$\{p \wedge \neg p\}$	
8	$\neg r$	3, 7; RC	$\{p \wedge \neg p, q \wedge \neg q\}$	
9	q	3, 6; RC	$\{r \wedge \neg r\}$	$\checkmark^{10}$
10	$r \wedge \neg r$	6, 8; RU	$\{p \wedge \neg p, q \wedge \neg q\}$	

Both 5 and 6 are derivable unconditionally from 1, whence the fourth element of these lines is empty. 7 is derivable from 2 and 5 because p behaves consistently at stage 6 of the proof. 8 is derivable from 3 and 7 because q behaves consistently at stage 7. However, as $\neg q$ was only derivable because p behaves consistently—see line 7—we should add $p \wedge \neg p$ in the fourth element of line 8; and indeed RC forces us to do exactly so. If p had not behaved consistently, then we would not have been able to derive $\neg r$ in the way we did. Finally, q is derivable from 3 and 6 because r behaves consistently at stage 8. At stage 10 r does not behave consistently any more, whence line 9 is marked at stage 10.

The proof can be continued, however. I repeat the lines from 7 on.

7	$\neg q$	2, 5; RC	$\{p \wedge \neg p\}$	$\checkmark^{11}$
8	$\neg r$	3, 7; RC	$\{p \wedge \neg p, q \wedge \neg q\}$	$\checkmark^{11}$
9	q	3, 6; RC	$\{r \wedge \neg r\}$	
10	$r \wedge \neg r$	6, 8; RC	$\{p \wedge \neg p, q \wedge \neg q\}$	$\checkmark^{11}$
11	$p \wedge \neg p$	4, 5, 6; RU	$\emptyset$	

At stage 11 p does not behave consistently any more. So lines 7, 8, and 10 are marked at stage 11. But then r behaves consistently at stage 11. So line 9 is unmarked at that stage.

This proof illustrates nicely the dynamic character of inconsistency-adaptive logics. From stage 2 to 10 $\neg q$ is derivable, and at stages 7–10 it is derived. From stage 11 on it is not derived any more, and not derivable either. On the other hand, q is not derivable from 3 and 6 at stage 10, because r behaves inconsistently at that stage. But at stage 11 r behaves consistently again, and consequently q is derivable. Hence, the dynamics with respect to the sentences that are derived at some stage and with respect to the sentences that are derivable at some stage actually occurs.

The dynamics with respect to the rules of inference may be illustrated as follows: up to stage 11 the rule of modus tollens may be applied to $q \supset p$ and $\neg p$ in order to infer $\neg q$. From stage 11 on, however, the rule can no longer be applied to those formulas. In general, the dynamics with respect to the rules concerns their range of application. This range may change several times as the proof proceeds, and depends on the inconsistencies that are (still) derived at some stage. It is obvious that this logic is an inconsistency-adaptive logic in the sense explained in Section 2.1. It is easily demonstrated that the set of consequences of some set Γ is generally richer than the **CLuN**-consequence set of Γ. For example, q is not a **CLuN**-consequence of the premises 1-4. If Γ is consistent, all **CL**-consequences of Γ are consequences of Γ and *vice versa*. If Γ is inconsistent, its set of consequences does not contain all sentences and hence is weaker than its set of **CL**-consequences.

The reader may wonder what would happen if we were to continue the proof after line 11. The answer is that nothing worth mentioning will happen. Indeed, no line that is not deleted in the present proof will be deleted in any of its extensions, and no further atoms (primitive formulas and their negations) will be derived. The only moves that are still possible are applications of such rules as Adjunction ($A, B/A \wedge B$), Addition, and similar uninteresting stuff.

I did not properly define the logic that was at work in the previous proof. The reason for this is that this logic has a serious drawback, which I shall spell out from the next paragraph on. The drawback is so serious that the logic cannot be considered as viable. Nevertheless, the logic at work in the above proof or a logic close to it has an extremely interesting property, viz. its dynamic character. No logic having static proofs leads to the same set of consequences from the premises 1-4. Indeed, in the preceding proof q is derived from $q \vee \neg r$ and r, and will not be deleted in any extension of this proof, whereas s is not derivable from the premises, notwithstanding the fact that $s \vee \neg p$ and p are both derivable.

As I announced, the logic has a serious drawback. Consider the premise set $\Gamma_1 = \{\neg p, \neg q, p \vee r, q \vee s, p \vee q\}$ and the following proof:

1	$\neg p$	Prem	$\emptyset$	
2	$\neg q$	Prem	$\emptyset$	
3	$p \vee r$	Prem	$\emptyset$	
4	$q \vee s$	Prem	$\emptyset$	
5	$p \vee q$	Prem	$\emptyset$	
6	r	1, 3; RC	$\{p \wedge \neg p\}$	$\checkmark^8$
7	s	2, 4; RC	$\{q \wedge \neg q\}$	
8	$p \wedge \neg p$	1, 2, 5; RC	$\{q \wedge \neg q\}$	

As p behaves inconsistently at stage 8, it is obvious that it will not be possible to derive $q \wedge \neg q$ in any extension of this proof. The only possible way to derive $q \wedge \neg q$ is by adding the following line, but the line is marked as soon as it is added.

9	$q \wedge \neg q$	1, 2, 5; RC	$\{p \wedge \neg p\}$	$\checkmark^8$

So the proof is final, $p \wedge \neg p$ and s are 'finally derivable'—see below—and nothing interesting is going to happen after this.

However, as everyone will have seen, there is something very wrong here. Indeed, suppose that we *exchange* lines 8 and 9. The proof—I rewrite it from line 6 on—would look as follows.

6	r	1, 3; RC	$\{p \wedge \neg p\}$	
7	s	2, 4; RC	$\{q \wedge \neg q\}$	$\checkmark^8$
8	$q \wedge \neg q$	1, 2, 5; RC	$\{p \wedge \neg p\}$	
9	$p \wedge \neg p$	1, 2, 5; RC	$\{q \wedge \neg q\}$	$\checkmark^8$

As q behaves inconsistently at stage 8, it is obvious that it will not be possible to derive $p \wedge \neg p$ in any extension of this proof. So now *this* proof is final, $q \wedge \neg q$ and r are finally derivable, and nothing interesting is going to happen after this.

By comparing the two proofs, we find out that it depends on the accidental way in which we start off the proof whether p and s is derivable whereas q and r are not, or the other way around. So the logic is not proof invariant: once a certain consequence is derived in a proof, it is impossible to extend the proof in such a way that a certain other consequence of the premises is derived in it. While a logic as defined in Section 1.1 does not require proof invariance, proof invariance is sufficiently important a property to be retained whenever possible. And we shall see that it is possible to retain it.

Some people who consider this kind of logics interesting might argue—some did orally to me—that this indeterminacy displays the creative aspects of human reasoning. It seems to me, however, that sheer accident should not be confused with creativity. No doubt, if you have a good reason to prefer the derivation of p to the derivation of q, and if the logic tells you whenever you have the choice, then it is quite all right to derive p. Alas, none of these conditions applies. The 'logic' at work here (as it stands) does not tell you when you have the choice. Moreover, the reasons you might have to prefer the derivation of p (and hence of its inconsistency) over that of q, are by no means logical reasons. As far as the 'logic' is concerned the choice for either of the two proofs is equally unjustifiable. Consequently, the argument from creativity does not hold water. The situation is even worse from the more general point of view, which is that the logic we are looking for should provide us, for any $T = \langle \Gamma, \mathbf{CL} \rangle$, with '$T$ except for the pernicious consequences of its inconsistency'. From the premises 1-5 (of the last proof) one may arrive at two different results; from other premise sets one may arrive at two hundred different results. Indeterminacy might be palatable if we were guaranteed at least an overview of the alternatives; but the present 'logic' does not provide us with such an overview. We should look for something better.

2.3.2 The Reliability Strategy

Actually, the indeterminacy we met in the previous subsection is very simple in nature. If a line has a non-empty condition, this condition can only spring from one or more applications of RC. What this means is this. Whenever A occurs in a proof on a condition Δ—note that Δ is always a finite set of contradictions—then $A \vee Dab(\Delta)$ can be derived on the condition $\emptyset$ and *vice versa*. I shall later formally prove this (and the proof is unsophisticated) but a little reflection will convince you right here that the claim holds true.

The claim holds for every A, also when A is a contradiction. So let A be $B \wedge \neg B$ and let Δ be a finite set of contradictions. Whenever a proof from Γ contains the line at which $B \wedge \neg B$ is derived on the condition Δ,[17] one can also derive $(B \wedge \neg B) \vee Dab(\Delta)$ on the empty condition in the same proof. Moreover, in the same proof one can derive each member of Δ on the condition that comprises $B \wedge \neg B$ as well as the *other* members of Δ. This will later be proved as Lemma 4.4.1.

[17] That is: of which $B \wedge \neg B$ is the second element and Δ the fourth.

What this comes to is that deriving a contradiction on a non-empty condition is not very sensible. Whenever one can do so, the situation is as follows. In view of the premises certain formulas are connected with respect to their consistency. The premises do not provide sufficient information to decide which of the sentences behaves inconsistently and which consistently. Reconsider the premise set from the last example of the previous subsection: $\{\neg p, \neg q, p \vee r, q \vee s, p \vee q\}$. As either p or q is true, one of them behaves inconsistently, but the premises do not provide enough information to decide which of the two behaves inconsistently. Nor does it provide sufficient information to decide that both behave inconsistently. So, a first remedy seems straightforward: prevent the derivation of $p \wedge \neg p$ as well as the derivation of $q \wedge \neg q$. Alas, this is not the end of our worries. As a consequence of this move, both p and q behave consistently, and this is not quite all right either. Indeed, $r \vee (p \vee q)$ is (unconditionally) derivable from $p \vee q$, and in view of the consistent behaviour of p and of q, one is justified in deriving r from $r \vee (p \vee q)$, $\neg p$ and $\neg q$. But then everything is derivable from the premise set. So this leads us back to Ex Falso Quodlibet. As the premises state that either p or q behaves inconsistently, to take both as behaving consistently is mistaken and abortive. The problem we are facing is the following: if some formulas are connected with respect to their consistency, one should be prevented from deriving the inconsistency of one of them, but at the same time one should be prevented from relying on the consistent behaviour of one of them in deriving other formulas.

This seems a good idea, provided we are able to spell out precisely what is meant by connected inconsistencies. At first sight, that (2.4) is derivable might be taken as the expression of the fact that $A_1, \ldots, A_n$ are connected with respect to their consistency.

$$(A_1 \wedge \neg A_1) \vee \ldots \vee (A_n \wedge \neg A_n) \tag{2.4}$$

However, (2.5) is derivable whenever (2.4) is, even if the consistency of B is not in any sense related to the inconsistency of $A_1, \ldots, A_n$.

$$(A_1 \wedge \neg A_1) \vee \ldots \vee (A_n \wedge \neg A_n) \vee (B \wedge \neg B) \tag{2.5}$$

On the other hand, whenever (2.5) is derivable by relying on the consistent behaviour of certain sentences, then (2.4) is derivable by relying on the consistent behaviour of B and of those other sentences.

This suggests that we take (2.5) to be the expression of the fact that A_1, $\ldots$, A_n, and B are connected with respect to their consistency, unless (2.4) is derivable *without* relying on the consistent behaviour of B. Although this criterion contains the solution, the latter may be phrased much more transparently. If (2.4) is derivable on some non-empty condition, then, as we have seen, it is possible to derive *on the empty condition* the disjunction of (2.4) and of the members of its condition. So let us rephrase the matter in terms of the disjunctions of contradictions that are derivable on the empty condition.

Let Δ be a finite set of contradictions. That the disjunction of the members of Δ is derived on the empty condition reveals that the premises require one of these members to be true. It is still possible, however, that the disjunction of the members of a proper subset of Δ is also derivable. Suppose for example that both (2.4) and (2.5) are derivable on the empty condition. Then we cannot conclude from (2.5) that the consistent behaviour of B is *not* connected to the consistent behaviour of $A_1, \ldots, A_n$. If (2.5) is derivable on the empty condition, and the result of dropping one or more disjuncts from (2.5) is not derivable on the empty condition, then we are justified in concluding that the consistent behaviour of B is connected to the consistent behaviour of $A_1, \ldots, A_n$.

Let me phrase this in general terms. If Δ is a finite set of contradictions, $Dab(\Delta)$ is derivable on the empty condition, and there is no $\Delta' \subset \Delta$ such that $Dab(\Delta')$ is derivable on the empty condition, then the members of Δ are connected with respect to their consistent behaviour.

We have a decent criterion now, but again at the abstract level of derivability. How can we formulate this with respect to concrete proofs? All that is given in proofs is which disjunctions of contradictions *have been derived* on the empty condition. These are known to us, not those that are derivable from the premises. So we have to consider them as an estimate, by present insights, of the disjunctions of contradictions that are derivable from the premises. From the previous paragraph we know that we should take into account only the *minimal* disjunctions of contradictions. $Dab(\Delta)$ is a minimal Dab-formula at a stage of a proof iff $Dab(\Delta)$ is derived on the empty condition and there is no $\Delta' \subset \Delta$ such that $Dab(\Delta')$ is also derived on the empty condition.

What we basically need is an improved replacement for Definition 2.3.1. Consider a proof from a premise set Γ and let $Dab(\Delta_1), \ldots, Dab(\Delta_n)$ be the minimal Dab-formulas that are derived at stage[18] s of the proof. Define $U_s(\Gamma) =_{df} \Delta_1 \cup \ldots \cup \Delta_n$. In view of the information available at stage s, the members of $U_s(\Gamma)$ are the contradictions that should be deemed *unreliable*: we cannot rely on their consistent behaviour at stage s of the proof. So, if one of these abnormalities occurs in the condition of a line, the line is marked.

Definition 2.3.2 *Line i is marked at stage s iff, where Δ is its condition, $\Delta \cap U_s(\Gamma) \neq \emptyset$. (Marking for Reliability.)*

So we retain the rules from the previous subsection and combine it with the present marking definition. Note that the rules and the definition are independent of each other. For example, nothing prevents one to derive a formula that refers to marked lines. In view of the rules and present definition, the added line will also be marked as soon as it is added—but this will not even be the case for other strategies as we shall soon see. Let us now look at some examples of proofs in order to see the dynamics at work. The examples are as simple as possible in view of what I want to illustrate.

1	p	Prem	$\emptyset$	
2	q	Prem	$\emptyset$	
3	$\neg p \vee r$	Prem	$\emptyset$	
4	$\neg q \vee s$	Prem	$\emptyset$	
5	$\neg q$	Prem	$\emptyset$	
6	r	1, 3; RC	$\{p \wedge \neg p\}$	
7	s	2, 4; RC	$\{q \wedge \neg q\}$	$\checkmark^{8}$
8	$q \wedge \neg q$	2, 5; RU	$\emptyset$	

Nothing unexpected happens in this proof. Line 7 is unmarked at stage 7 of the proof and is marked at stage 8 on. If the proof is continued, nothing interesting

[18]The letter s is used both as a sentential letter and as a variable for stages. The context always disambiguates.

happens. More importantly, line 7 will remain marked and lines 1–6 and 8 will remain unmarked.

The following proof from $\{(p \wedge q) \wedge t, \neg p \vee r, \neg q \vee s, \neg p \vee \neg q, t \supset \neg p\}$ illustrates that a line may be marked at a stage and unmarked at subsequent stages. Let us first consider stage 8 of the proof, at which lines 6 and 7 are marked in view of the minimal *Dab*-formula derived at line 8.

1	$(p \wedge q) \wedge t$	Prem	$\emptyset$	
2	$\neg p \vee r$	Prem	$\emptyset$	
3	$\neg q \vee s$	Prem	$\emptyset$	
4	$\neg p \vee \neg q$	Prem	$\emptyset$	
5	$t \supset \neg p$	Prem	$\emptyset$	
6	r	1, 2; RC	$\{p \wedge \neg p\}$	$\checkmark^8$
7	s	1, 3; RC	$\{q \wedge \neg q\}$	$\checkmark^8$
8	$(p \wedge \neg p) \vee (q \wedge \neg q)$	1, 4; RU	$\emptyset$	

Consider, however, the following continuation of the proof—I repeat the lines from 6 on.

6	r	1, 2; RC	$\{p \wedge \neg p\}$	$\checkmark^8$
7	s	1, 3; RC	$\{q \wedge \neg q\}$	
8	$(p \wedge \neg p) \vee (q \wedge \neg q)$	1, 4; RU	$\emptyset$	
9	$p \wedge \neg p$	1, 5; RU	$\emptyset$	

Now the formula of line 8 is not a minimal *Dab*-formula any more in view of line 9, whence line 7 is unmarked. Again, nothing interesting happens in a continuation of the proof.

Let me summarize the insights gained. Where s and s' are two subsequent stages of a proof from Γ, $U_{s'}(\Gamma) - U_s(\Gamma)$ may be non-empty because some new minimal *Dab*-formula has been derived at stage s'. This may cause a line that was unmarked at stage s to be marked at stage s'. The set $U_s(\Gamma) - U_{s'}(\Gamma)$ may also be non-empty because a *Dab*-formula that was minimal at stage s need not be minimal at stage s'—this is illustrated by line 8 at stage 9 of the example proof. This may cause a line that was marked at stage s to be unmarked at stage s'.

In order to show that a viable logic is at work here, we need quite a few metatheoretic proofs. These are postponed to later chapters. For now, let me just mention that the logic is viable and that it does exactly what we expected it to do, viz. interpret an inconsistent premise set as consistently as possible.

The distinctive feature of the logic at work here is the marking definition, Definition 2.3.2. As we shall see later, the definition is typical for what is called the *Reliability* strategy. There are indeed other ways to handle connected inconsistencies and one of them is actually doing a little bit better, as we shall see in the next subsection.

2.3.3 The Minimal Abnormality Strategy

Let us return to the premise set $\{\neg p, \neg q, p \vee r, q \vee s, p \vee q\}$, which also appeared in Subsection 2.3.1. Here is a proof the lines of which are marked according to the Reliability strategy.

1	$\neg p$	Prem	$\emptyset$	
2	$\neg q$	Prem	$\emptyset$	
3	$p \vee r$	Prem	$\emptyset$	
4	$q \vee s$	Prem	$\emptyset$	
5	$p \vee q$	Prem	$\emptyset$	
6	r	1, 3; RC	$\{p \wedge \neg p\}$	$\checkmark^8$
7	s	2, 4; RC	$\{q \wedge \neg q\}$	$\checkmark^8$
8	$(p \wedge \neg p) \vee (q \wedge \neg q)$	1, 2, 5; RU	$\emptyset$	

The markings are quite all right. We cannot rely on the consistent behaviour of either p or q, whence we cannot derive either r or s. Consider the following continuation of the proof.

9	$r \vee s$	6; RU	$\{p \wedge \neg p\}$	$\checkmark^8$
10	$r \vee s$	7; RU	$\{q \wedge \neg q\}$	$\checkmark^8$

As promised, nothing unexpected happens here. Line 9 is obtained by Addition from a marked line, whence it has the same condition as that line and hence is also marked. Similarly for line 10.

And still, one might be discontent with the result. As 8 is a minimal *Dab*-formula from stage 8 on, we should not rely on the consistency of either p or q. Quite all right. Recall the reason to do so: the premise set informs us that either p or q behaves inconsistently but does not provide enough information to decide which of the two does. In line with the Reliability strategy, we consider both p and q as unreliable and we cannot derive formulas by relying on the consistent behaviour of unreliable contradictions.

We might, however, also reason as follows. The premise set informs us that either p or q behaves inconsistently. If we really interpret the premises as consistently as possible, we might take this to mean that p or q behaves inconsistently and that *at most* one of them does. This changes the picture. Indeed, if p behaves inconsistently, then q does not and hence $r \vee s$ follows from the premises in view of line 10; if q behaves inconsistently, then p does not and hence $r \vee s$ follows from the premises in view of line 9. So either way it follows from the premises. This too leads to a viable strategy, only a different one from Reliability. The new strategy will be called *Minimal Abnormality.* If we apply it to the previous proof, lines 9 and 10 are both unmarked at stage 10.

While it seems simple enough to apply the Minimal Abnormality strategy in the presence of a single minimal *Dab*-formula, the general formulation of the marking definition requires a lot of attention.

Consider first stage 9 of the previous proof. Note that line 9 is marked at this stage. Indeed, 8 informs us that either p or q behaves inconsistently. So the information provided by line 9 alone is insufficient to consider $r \vee s$ as derivable from the premises. At stage 10 of the proof, however, line 9 is unmarked in view of the information provided by line 10. So, in the case of a single minimal *Dab*-formula, marking seems to be governed by the following idea: A is derivable if, for each disjunct of the minimal *Dab*-formula, A is derivable on a condition in which this disjunct does not occur. *If* this holds for all disjuncts, then the lines at which A is derived on a condition that does not contain one of the disjuncts are unmarked.

Next consider a case in which there are two minimal *Dab*-formulas. Let us take $(p \wedge \neg p) \vee (q \wedge \neg q)$ and $(r \wedge \neg r) \vee (s \wedge \neg s) \vee (t \wedge \neg t)$ as an example. The information provided by such a proof is this: the premises require that one out of six combinations of abnormalities holds: $\{p \wedge \neg p, r \wedge \neg r\}$, $\{p \wedge \neg p, s \wedge \neg s\}$, $\{p \wedge \neg p, t \wedge \neg t\}$, $\{q \wedge \neg q, r \wedge \neg r\}$, $\{q \wedge \neg q, s \wedge \neg s\}$, and $\{q \wedge \neg q, t \wedge \neg t\}$. So, in order for a formula A to be derivable at this stage of the proof, we need, for each of the six combinations, that A is the second element of a line of which the condition does not overlap with the combination. Of course, one line might be sufficient:

i	A	$\ldots$	Δ

in which Δ is $\emptyset$ or another set that does not overlap with any of the six combinations. Also sufficient would be the occurrence of the following two lines.

i	A	$\ldots$	$\{r \wedge \neg r\}$
j	A	$\ldots$	$\{s \wedge \neg s\}$

In all such cases, these lines would be unmarked. If only line j occurs in the proof, then line A is marked as explained before. If A is also the second element of a line that has condition $\{p \wedge \neg p, q \wedge \neg q\}$, then that line is marked because it overlaps with each of the six combinations.

One possible complication has still to be considered. Suppose again that two minimal *Dab*-formulas occur, but let them be $(p \wedge \neg p) \vee (q \wedge \neg q)$ and $(r \wedge \neg r) \vee (s \wedge \neg s) \vee (q \wedge \neg q)$. The number of combinations now reduces to three: $\{p \wedge \neg p, r \wedge \neg r\}$, $\{p \wedge \neg p, s \wedge \neg s\}$, and $\{q \wedge \neg q\}$. Indeed, if we interpret the premises as normally as possible, then other combinations are not minimally abnormal. If $q \wedge \neg q$ holds true, both minimal *Dab*-formulas hold true and no other contradiction is required to hold true. The criterion for marking lines is as before, except that only these three combinations have to be taken into account.

Let us move to the general case. Suppose that the following minimal *Dab*-formulas occur in a proof from Γ: $Dab(\Delta_1)$, $\ldots$, $Dab(\Delta_n)$ at stage s. Recall that the Δ_i are sets of contradictions. Consider the set of all choice sets of $\{\Delta_1, \ldots, \Delta_n\}$, viz. all sets that comprise a member of every Δ_i. The minimal choice sets are those that are not proper subsets of another choice set. Let $\Phi_s(\Gamma)$ be the set of minimal choice sets of $\{\Delta_1, \ldots, \Delta_n\}$.

Definition 2.3.3 *Line i is marked at stage s iff, where A is derived on the condition Δ at line i, (i) there is no $\varphi \in \Phi_s(\Gamma)$ such that $\varphi \cap \Delta = \emptyset$, or (ii) for some $\varphi \in \Phi_s(\Gamma)$, there is no line at which A is derived on a condition Θ for which $\varphi \cap \Theta = \emptyset$. (Marking for Minimal Abnormality.)*

The definition sounds somewhat complicated, but the following reads more easily: where A is derived on the condition Δ on line l, line l is *unmarked* at stage s iff (i) there is a $\varphi \in \Phi_s(\Gamma)$ for which $\varphi \cap \Delta = \emptyset$ and (ii) for every $\varphi \in \Phi_s(\Gamma)$, there is a line at which A is derived on a condition Θ for which $\varphi \cap \Theta = \emptyset$.

The reader may check that, on this definition, lines 9 and 10 of the proof at the beginning of this subsection are unmarked at stage 10 of the proof. There is no need for further examples as that proof illustrates all that there is to illustrate. The logic defined by the generic rules from the previous subsection and by the marking definition is a viable logic. I shall show this in Chapters 4 and 5.

The Minimal Abnormality strategy assigns (slightly) stronger consequence sets to most premise sets. At the same time, it is not only more complicated, but also computationally more complex as we shall see later.

2.3.4 The Predicative Cases

Two decent adaptive logics were intuitively described in the previous pages. The predicative versions of these logics will be called $\mathbf{CLuN}^r$ and $\mathbf{CLuN}^m$ respectively, in which the superscripts refer to the strategy. It is now time to have a look at these predicative versions. How to do so, we have seen already in Section 2.2, viz. in Theorem 2.2.6. The only change required with respect to the propositional case is that not only plain contradictions but also *existentially closed* contradictions have to be considered as abnormalities. Note for example that $\forall x \neg Qx, \exists x(Px \vee Qx) \vdash_{\mathbf{CL}} \exists x Px$ whereas $\forall x \neg Qx, \exists x(Px \vee Qx) \vdash_{\mathbf{CLuN}} \exists x Px \vee \exists x(Qx \wedge \neg Qx)$.

Let us start with an straightforward example.

1	$\forall x \neg Qx$	Prem	$\emptyset$	
2	$\exists x(Px \vee Qx)$	Prem	$\emptyset$	
3	Qa	Prem	$\emptyset$	
4	$\exists x Px$	1, 2; RC	$\{\exists x(Qx \wedge \neg Qx)\}$	$\checkmark^6$
5	$Qa \wedge \neg Qa$	1, 3; RU	$\emptyset$	
6	$\exists x(Qx \wedge \neg Qx)$	5; RU	$\emptyset$	

Line 4 is unmarked at stages 4 and 5 of the proof, and marked at stage 6.

In order to clearly illustrate the interplay between quantified and non-quantified (closed) formulas, I add a further example.

1	$\forall x(Px \supset Qx)$	Prem	$\emptyset$
2	$\exists x(Px \wedge Rx)$	Prem	$\emptyset$
3	$\neg Qa$	Prem	$\emptyset$
4	$\forall x(Rx \supset \neg Qx)$	Prem	$\emptyset$
5	$Pa \supset Qa$	1; RU	$\emptyset$
6	$\neg Pa$	3, 5; RC	$\{Qa \wedge \neg Qa\}$
7	$\exists x \neg Px$	6; RU	$\{Qa \wedge \neg Qa\}$
8	$\exists x(Qx \wedge \neg Qx)$	1, 2, 4; RU	$\emptyset$

If the premise $\neg Qa$ is removed, we can still derive $\exists x \neg Qx$, viz. from 2 and 4. As a result, $\exists x \neg Px$ can only be derived on the condition $\{\exists x(Qx \wedge \neg Qx)\}$. So the resulting line will be marked at stage 8. And justly so: the very object which, in that case, is known to be Q is also known to be inconsistent with respect to Q-hood. In the displayed proof, line 7 is unmarked. So this crucially depends on the presence of the premise $\neg Qa$. Although we know from 8 that some objects are inconsistent with respect to Q-hood, there is no reason to consider a as one of them. So $\neg Pa$ is derivable and hence also $\exists x \neg Px$.

Both examples illustrate $\mathbf{CLuN}^r$ as well as $\mathbf{CLuN}^m$ because the respective marking definitions lead to the same marks in both examples—there are no connected abnormalities.

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2.3.5 Final Derivability

An aspect that was left implicit until now concerns the notion of derivability involved in adaptive proofs. Actually, it is wiser to distinguish between two notions of derivability. In the previous subsections, I called the formula (second element) of a line derived or not derived according as the line is unmarked or marked. The notion involved here is *derivability at a stage*. A formula may be derived at one stage and not derived at the next, or *vice versa*.

Clearly, we also want to have a stable notion of derivability. I shall call it final derivability. One way to describe it is by saying that A is finally derived iff it is derived on a line that is unmarked and will remain unmarked in every extension of the proof. A different but equivalent description reads that A is derived on an unmarked line l and that, whenever l is marked in an extension of the proof, there is a further extension in which l is again unmarked. All this will be described in a rigourous way in Section 4.4.

2.4 Semantics

We obviously are interested in a semantics of which the semantic consequence relation corresponds to final derivability. Pinning down this semantics was not an easy matter, but the semantics itself is astonishingly simple.

Let us first consider Reliability. For any Γ there is a set of *Dab*-formulas that are **CLuN**-derivable from Γ and hence are derivable on the empty condition in a $\mathbf{CLuN}^r$-proof from Γ. Some of these *Dab*-formulas are not minimal. Let the minimal ones be called *minimal Dab-consequences of* Γ. Their set is recursive in the propositional case and semi-recursive in the predicative case. Let $U(\Gamma)$ be the set of abnormalities that comprises every disjunct of every minimal *Dab*-consequences of Γ. Note that $U(\Gamma)$ is defined in terms of the minimal *Dab*-consequences of Γ in the same way as $U_s(\Gamma)$ was defined before in terms of the minimal *Dab*-formulas that occur at stage s in the considered proof from Γ.

It is rather obvious that A is finally $\mathbf{CLuN}^r$-derivable from Γ iff there is a finite set of abnormalities, say Δ, such that $\Gamma \vdash_{\mathbf{CLuN}} A \vee Dab(\Delta)$ and no member of Δ is a member of $U(\Gamma)$. Indeed, if this is the case, it is possible, first, to derive A on the condition Δ in a $\mathbf{CLuN}^r$-proof from Γ. Let A be so derived on line l. Next, whenever some *Dab*-formula $Dab(\Theta)$ is derived that causes l to be marked—this means that $\Delta \cap \Theta \neq \emptyset$—$Dab(\Theta)$ is not a minimal *Dab*-consequence of Γ. So the proof can be extended by deriving a minimal *Dab*-consequence of Γ, $Dab(\Theta')$ for which $\Theta' \subset \Theta$ and $\Delta \cap \Theta' = \emptyset$. In the so-extended proof, line l is unmarked.

To obtain the semantics for $\mathbf{CLuN}^r$, we simply map this idea in two steps. As a first step we define the minimal *Dab*-consequences of Γ semantically. They are the minimal *Dab*-formulas that hold true in all **CLuN**-models of Γ. From these we define $U(\Gamma)$ as before. Note that some **CLuN**-models of Γ verify no other abnormalities than members of $U(\Gamma)$.[19] Formally: where $Ab(M)$ is the set of abnormalities verified by M, there are models M of Γ for which $Ab(M) \subseteq U(\Gamma)$.

For the second step, recall the finite set of abnormalities, say Δ, such that $\Gamma \vdash_{\mathbf{CLuN}} A \vee Dab(\Delta)$ and no member of Δ is a member of $U(\Gamma)$. Obviously, every **CLuN**-model of Γ verifies $A \vee Dab(\Delta)$. So the **CLuN**-models of Γ that verify no member of Δ verify A.

So defining the $\mathbf{CLuN}^r$-semantics is a simple matter. Call a **CLuN**-model M of Γ a *reliable* model of Γ iff $Ab(M) \subseteq U(\Gamma)$ and define $\Gamma \vDash_{\mathbf{CLuN}^r} A$ iff every

[19]Don't worry if you don't see this at once. It is proved as Corollary 5.2.2.

reliable model of Γ verifies A.

The semantics of $\mathbf{CLuN}^m$ is even simpler. Call a **CLuN**-model M of Γ a *minimal abnormal model* of Γ iff there is no **CLuN**-model M' such that $Ab(M') \subset Ab(M)$ and define $\Gamma \vDash_{\mathbf{CLuN}^m} A$ iff every minimal abnormal model of Γ verifies A. Proofs that the logics are sound and complete with respect to their semantics follow in Chapter 5.

2.5 The Classical Symbols

What would happen if we upgrade the language from $\mathcal{L}_s$ to $\mathcal{L}_S$, which contains all classical logical symbols? Actually not much. I shall show that there are a few advantages and, in the present context, no disadvantages. Of course, classical negation is the only really new symbol after this upgrade. All other logical symbols have the same meaning in **CLuN** and **CL**.

Before discussing the matter from a technical point of view, let me stress that there is no need to introduce the classical symbols. This is important for dialetheists, who consider classical negation a nonsensical operator. Nevertheless, all classical symbols, including classical negation, play an important role in this book in view of the advantages I announced. Some of the advantages are illustrated below. Others will become clear later. The classical logical symbols will enable us to phrase the standard format of adaptive logics, in Chapter 4, and to offer simple metatheoretic proofs for their properties in Chapters 4 and 5. Most of these proofs may be rephrased without using classical negation, but only at the expense of sometimes serious complications. I shall present the simpler and more transparent proofs. Other advantages not argued for in this section concern the use of classical negation for characterizing certain existing consequence relations in terms of adaptive logics. Incidentally, in proceeding thus, I am not unfaithful to my philosophical views. I do not believe in the existence of a true logic. I am a pluralist, who holds that logics may be used in contexts in which it is justified to use them, even if it is unsuitable for other contexts. For me logics are instruments and it is not because a violin is no good to hammer nails in the wall, that one should ban it from the orchestra.

So what becomes of **CLuN** if we upgrade the language to $\mathcal{L}_S$ and give all classical symbols their classical meaning? The only gain in expressive power comes from adding the classical $\check{\neg}$. To the axiom system one adds

A$\check{\neg}$1 $(A \supset \check{\neg}A) \supset \neg A$
A$\check{\neg}$2 $A \supset (\check{\neg}A \supset B)$

and to the (deterministic as well as the indeterministic) semantics one adds the clause

C$\check{\neg}$ $v_M(\check{\neg}A) = 1$ iff $v_M(A) = 0$

stipulating that the pseudo-language $\mathcal{L}_{\mathcal{O}}$ extends $\mathcal{L}_S$. For many purposes it is sufficient that $\mathcal{L}_{\mathcal{O}}$ extends $\mathcal{L}_s$ and that $\check{\neg}$ is added to this—we shall come to that in Section 4.3. In the present context, however, the excess in expressive power is not a hindrance and involves no specific complications. The other classical symbols may be introduced accordingly in the semantics and axiom system; one simply duplicates the relevant axiom schemata and clauses. The upgrade is standard.

An alternative way of axiomatizing this version of **CLuN** reveals an interesting property. Start from the **CL**-axiom system from Section 1.7 but replace every standard logical symbol by the classical one (also in the names of the axioms). To this add the following axioms: for negation $\check{\neg}A \check{\supset} \neg A$; for each of the other logical symbols a suitable equivalence: $(A \supset B) \check{\equiv} (A \check{\supset} B)$, ..., $\exists\alpha A \check{\equiv} \check{\exists}\alpha A$, and $\alpha = \beta \check{\equiv} \alpha \check{=} \beta$. Note that the negation axiom is equivalent to $A \check{\vee} \neg A$ and that $(A \check{\vee} \neg A) \check{\equiv} (A \vee \neg A)$ is a theorem of this version of **CLuN**.

To see that this is a sound and complete axiomatization of **CLuN**, let us move to the semantics. Take the **CL**-semantics from Section 1.7 but replace every standard logical symbol by the classical one (also in the names of the clauses). To this add the following clauses: for negation "if $v_M(\check{\neg}A) = 1$ then $v_M(\neg A) = 1$", and for the other logical symbols a suitable identity, for example $v_M(A \supset B) = v_M(A \check{\supset} B)$ for the implication. Note that, in this semantics $v_M(\check{\neg}A) = 1$ holds true just in case $v_M(A) = 0$ holds true. Replacing the former by the latter in the clause for negation gives us literally the indeterministic **CLuN**-semantics.

I leave it to the reader to check that Theorems 2.2.1–2.2.5 and Corollary 2.2.1 are provable for the upgraded systems. So I shall simply refer to those theorems when I need the results about the upgraded systems.

I Now come to a point where the use of $\mathcal{L}_S$ makes a real difference. I have stated in Section 2.2 that consistent Γ have **CL**-models, which are consistent **CLuN**-models, but they also have *inconsistent* **CLuN**-models. There are no exceptions to this statement in $\mathcal{L}_s$, but that there are if one moves to $\mathcal{L}_S$. Indeed, if M is a **CL**-model, then $\Gamma = \{A \in \mathcal{W}_s \mid M \Vdash A\}$ has **CLuN**-models that verify $p \wedge \neg p$, but $\Gamma = \{A \in \mathcal{W}_S \mid M \Vdash A\}$ has only consistent models. Consider a **CL**-model M of Γ that verifies p and hence falsifies $\neg p$. However, that M falsifies p cannot be expressed within $\mathcal{L}_s$. Obviously, one can write $M \nVdash \neg p$ in the semantic metalanguage, but no member of $\{A \in \mathcal{W}_s \mid M \Vdash A\}$ witnesses this fact. The matter is different in $\mathcal{L}_S$ because, if this is our language, $M \nVdash \neg p$ iff $M \Vdash \check{\neg}\neg p$. So $\check{\neg}\neg p \in \{A \in \mathcal{W}_S \mid M \Vdash A\}$. In general, if $M \nVdash A$ iff $M \Vdash \check{\neg}A$. So $\{A \in \mathcal{W}_s \mid M \Vdash A\}$ has no inconsistent **CLuN**-models. I obviously mean that the set has no $\neg$-inconsistent **CLuN**-models, because there *are* no $\check{\neg}$-inconsistent **CLuN**-models.

At this point, a warning has to be issued. Some readers will realize that all logical symbols of **CL** may be defined by means of, for example, the symbols in the set $\{\check{\neg}, \check{\vee}, \check{\forall}\}$ and may think this gives one a way to present **CLuN** in a most economical way—skip to the next paragraph if you are not one of those readers. This is all right, provided the symbols of the standard language $\mathcal{L}_s$ are not themselves defined in terms of $\{\check{\neg}, \check{\vee}, \check{\forall}\}$. If, for example, one *defines* $A \supset B$ as $\check{\neg}A \check{\vee} B$, then one cannot avoid $\neg(A \supset B) \check{\equiv} \neg(\check{\neg}A \check{\vee} B)$ and this is not a theorem of **CLuN** upgraded to $\mathcal{L}_S$. Even $\neg(A \supset B) \equiv \neg(\check{\neg}A \vee B)$ is not a theorem. Introducing the implication of the standard language by $(A \supset B) \check{\equiv} (A \check{\supset} B)$ is all right, however, because Replacement of $\check{\equiv}$-Equivalences is invalid within the scope of a negation of the standard language. Indeed, the standard negation is characterized axiomatically by $\check{\neg}A \supset \neg A$. So even if $A \check{\equiv} B$ is a theorem of the upgraded **CLuN**, one cannot derive $\neg A \check{\equiv} \neg B$ from it. Semantically: if A and B have the same truth-value, but this truth-value happens to be true, then it is very well possible that $\neg A$ is true whereas $\neg B$ is false.

The idea to characterize a paraconsistent negation by $\check{\neg}A \supset \neg A$ is natural and attractive. One may read $\check{\neg}A$ as A is *false* (on a classical understanding) and $\neg A$ as not-A, the negation of A. If A is *false* $\neg A$ is true, but A and $\neg A$ may also be true together. This is the minimal meaning of negation needed to reason from inconsistencies and precisely this is the meaning of the standard negation in **CLuN**. So in that sense too, **CLuN** is the basic paraconsistent logic if one starts from **CL**.

The classical symbols also throw light on the functioning of $\mathbf{CLuN}^r$ and $\mathbf{CLuN}^m$. Obviously the conditional rules enable one to make the following move.

i	$\neg A$	...	Δ
j	$\check{\neg} A$	i; RC	$\{A \wedge \neg A\} \cup \Delta$

More remarkable is that it can be proved that the generic rule RC is contextually equivalent with the following "Basic Schema" rule—don't make too much of the name, which is largely an accident.

$$\text{BS} \quad \frac{\neg A \qquad \Delta}{\check{\neg} A \qquad \{A \wedge \neg A\} \cup \Delta}$$

To see the impact of BS, note that, for example, $A \vee B, \check{\neg} A \vdash_{\mathbf{CLuN}} B$. So the idea is that BS is used to replace negations by classical negations, meanwhile introducing new conditions, and next to apply only RU, which actually comes to applying **CL**-rules.

This reveals an idea behind the functioning of inconsistency-adaptive logics: negations are interpreted as classical whenever possible, and as paraconsistent where the premises require so. Of course, the "whenever possible" and the "the premises require so" are ambiguous and are only disambiguated by choosing a strategy. More important, however, is that, once a strategy is chosen, we are dealing with logics in which a logical symbol, in this case negation, has a variable meaning. This is rather unheard of, but it is a natural consequence of interpreting a premise set as consistently as possible. It illustrates the innovative character of adaptive logics. Although these logics are formulated for formal languages and are described by strict metatheoretic means, they enable one to move in a significant way into the direction of the dynamics of reasoning that is typical for natural languages. In this sense, they bring us closer to what is usually called "argumentation", a discipline which proceeds in terms of natural language and lacks metatheory altogether.

2.6 Concluding Comments

The main result of this chapter is that we have two logics that offer a maximal consistent interpretation of a premise set. These logics solve the problem described in Section 2.1.

In the previous chapter I used the phrases "lower limit logic" and "upper limit logic". For both $\mathbf{CLuN}^r$ and $\mathbf{CLuN}^m$ the lower limit logic is **CLuN**. Whatever is **CLuN**-derivable is finally derivable by the adaptive logics and is derivable on the empty condition in their proofs. The upper limit logic is **CL**. This is witnessed by the fact that all **CL**-consequences of a consistent premise set Γ are $\mathbf{CLuN}^r$-derivable as well as $\mathbf{CLuN}^m$-derivable from Γ. The proof of this is obvious in view of Theorem 2.2.6 and the fact that no *Dab*-formula is **CLuN**-derivable from a consistent premise set.

Allow me to end this chapter with a philosophical comment. Adaptive logics are tools. They may be used by everyone, independent of his or her philosophical convictions. If you believe that the world is consistent, you still need an inconsistency-adaptive logic to solve the problem sketched in Section 2.1. As I explained there, even dialetheists consider most inconsistencies as false and consider an inconsistency only then as true when there is a good reason to do so. Dialetheists have, however, another reason to like inconsistency-adaptive logics. Most dialetheists believe that there is a true logic and that it is paraconsistent. But if this is so, how can one understand the vast mass of classical reasoning, in mathematics and elsewhere, and the long tradition, ascribed to Aristotle, of abhorrence for inconsistency. Inconsistency-adaptive logics provide an explanation which is acceptable for dialetheists. Although a dialetheist will not consider, for example, Disjunctive Syllogism as a logically valid schema, she will admit that B follows from $A \vee B$ and $\neg A$ if one has the supplementary information that A behaves consistently. So, if inconsistencies are indeed only exceptionally true—Graham Priest uses the phrase "the improbability of inconsistency"—it is sensible to consider a statement as consistent, unless and until proven otherwise. This is what Priest called "the classical recapture". verborgen
The enthusiasm of the leading dialetheist Priest is witnessed, for example, by introductory chapters of [PRN89] and by [Pri91, Pri06].

For consistentists, inconsistency-adaptive logics are *corrective*. The standard of deduction is **CL** (or another logic validating Ex Falso Quodlibet) and inconsistency-adaptive logics are a tool to handle premise sets that inadvertently turn out to conflict with the standard of deduction. So for them the inconsistency-adaptive logic has to restrict applications of some correct rules of inference. For dialetheists, the situation is exactly the reverse. Their standard of induction—for Priest this is **LP**, see Chapter 7—functions as the lower limit logic and the inconsistency-adaptive logic is *ampliative*. It validates certain inferences that are not correct for logical reasons but are quasi-valid in that their justification requires logical considerations together with the information that a certain statement behaves consistently.

2.7 Appendix: Proof of Theorems 1.5.9 and 1.5.10

Let us start with Theorem 1.5.10. I shall introduce a logic $\mathbf{CLuN}^+$ that is Reflexive, Transitive, Monotonic, Uniform and Compact, and for which there is a positive test. $\mathbf{CLuN}^+$ is defined over the standard propositional language and actually is an extension of the propositional fragment of **CLuN**, but does not have static proofs in the standard propositional language. In this section $\mathcal{L}_p$ will denote the standard *propositional* language and $\mathcal{W}_p$ its set of formulas.

Define $(\vee_0 A) =_{df} A$ and $(\vee_{n+1} A) =_{df} ((\vee_n A) \vee A)$—for example $(\vee_2 A) =_{df} ((A \vee A) \vee A)$. Note that, if $i \neq j$, then $\neg(\vee_i A) \nvdash_{\mathbf{CLuN}} \neg(\vee_j A)$. Let Σ be the semi-recursive set comprising the Gödel numbers of the **CL**-theorems (given an encoding)—see for example [BBJ02] on Gödel numbers. Define the logic $\mathbf{CLuN}^+$ as the result of extending the propositional fragment of **CLuN** with an infinite set of axioms, viz. all formulas of the form $\neg(A \wedge \neg(\vee_i A)) \supset ((A \wedge \neg A) \supset B)$ for which $i \in \Sigma$. All these axioms are **CLuN**-independent of each other; this means that no such axiom is **CLuN**-derivable from any set of such axioms that does not comprise it. Moreover, there is a positive test for "$\neg(A \wedge \neg(\vee_i A)) \supset ((A \wedge \neg A) \supset B)$ is a **CLuN**$^+$-axiom" because there is a positive test for "i is the Gödel number of a **CL**-theorem".

Given that all the new axioms (and all formulas of the same form) are **CLuN**$^+$-independent, there is no recursive subset of them from which the others may be derived. So **CLuN**$^+$ has no static proofs in $\mathcal{L}_p$: no recursive set of S-rules characterizes **CLuN**$^+$-proofs, if the formulas in these proofs have to be members of $\mathcal{W}_p$. Of course, one might introduce the new axioms by a rule of the form "from $\emptyset$ to derive $\neg(A \wedge \neg(\vee_i A)) \supset ((A \wedge \neg A) \supset B)$ provided i is the Gödel number of a formula C for which $\vdash_{\mathbf{CL}} C$". This, however, is not a S-rule because it cannot be decided whether the restriction is fulfilled by inspecting the list of lines to which the application of the rule belongs (viz. the **CLuN**$^+$-proof). Indeed, it cannot even be decided by any means whether the condition is fulfilled because the set of **CL**-theorems is only semi-recursive. Note that in the rule, which is not a S-rule anyway, the C is not a metavariable for members of $\mathcal{W}_p$, but for (possibly predicative) members of $\mathcal{W}_s$.

Obviously **CLuN**$^+$ is Reflexive, Transitive, Monotonic, Uniform and Compact. I quickly show that there is a positive test for it.

Consider a finite $\Gamma \subset \mathcal{W}_p$ and an $A \in \mathcal{W}_p$. If $\Gamma \vdash_{\mathbf{CLuN}^+} A$, there is a **CLuN**$^+$-proof of A from Γ and this proof has a Gödel number. So we consider one by one every natural number n and check whether, for a given encoding, it is the Gödel number of a list of formulas[20] (members of $\mathcal{W}_p$) in which A occurs. There are four cases.

Case 1: n is not the Gödel number of a list of formulas in which A occurs. So n is not the Gödel number of a **CLuN**$^+$-proof of A from Γ.

Case 2: n is the Gödel number of a **CLuN**-proof of A from Γ. As $Cn_{\mathbf{CLuN}}(\Gamma) \subset Cn_{\mathbf{CLuN}^+}(\Gamma)$, it follows that $\Gamma \vdash_{\mathbf{CLuN}^+} A$. So we are done.

Case 3: n is the Gödel number of a list of formulas that is not a **CLuN**-proof of A from Γ and that would not be a proof of A from Γ even if the formulas of the form $\neg(A \wedge \neg(\vee_i A)) \supset ((A \wedge \neg A) \supset B)$, which occur in the list, would be **CLuN**$^+$-axioms.[21] So n is not the Gödel number of a **CLuN**$^+$-proof of A from Γ.

Case 4: n is the Gödel number of a list of formulas that is not a **CLuN**-proof of A from Γ, the list contains formulas of the form $\neg(A \wedge \neg(\vee_i A)) \supset ((A \wedge \neg A) \supset B)$, and the list is a proof of A from Γ iff these formulas are **CLuN**$^+$-axioms. So, for each such i, we start the positive test for "i is the Gödel number of a **CL**-theorem". If the test succeeds we are done.

The procedure forms indeed a positive test for $\Gamma \vdash_{\mathbf{CLuN}^+} A$. Supposing that we search for the proof by means of Turing machines, a finite number of Turing machines running together is sufficient to find a **CLuN**$^+$-proof of A from Γ if there is one. Theorem 1.5.10 is hereby proven.

Theorem 1.5.9 can use some preparation and **CLuN**$^+$ is helpful in that respect too. **CLuN**$^+$ has static proofs if the language is extended. Let $\mathcal{W}_{\check{s}}$ be obtained from $\mathcal{W}_s$ by replacing all standard logical symbols by their classical

[20] One could consider a list of annotated lines, as in Lemma 1.5.2, but a non-annotated proof will do just as well.

[21] Whether the list would be such a proof is decidable. It comes to deciding whether that list is a proof of A from $\Gamma \cup \Gamma'$, in which Γ' is the set of formulas that have the form $\neg(A \wedge \neg(\vee_i A)) \supset ((A \wedge \neg A) \supset B)$, occur in the proof, and are not members of Γ.

counterparts and consider the language that has $\mathcal{W}_p \cup \mathcal{W}_{\check{s}}$ as its set of closed formulas. We may now phrase the following rule "From $\vdash C$ to derive $\vdash \neg(A \wedge \neg(\vee_i A)) \supset ((A \wedge \neg A) \supset B)$, provided $C \in \mathcal{W}_{\check{s}}$, $\neg(A \wedge \neg(\vee_i A)) \supset ((A \wedge \neg A) \supset B) \in \mathcal{W}_p$, and i is the Gödel number of C". The condition of this rule is decidable. The price paid for devising static proofs for $\mathbf{CLuN}^+$ is high, however. We are interested in a logic $\mathbf{CLuN}^+$ defined on $\mathcal{W}_p$. In order to devise static proofs for $\mathbf{CLuN}^+$, we have to allow that these proofs contain formulas of a different language, viz. members of $\mathcal{W}_{\check{s}}$. Moreover, in order to apply the rule, one needs to prove that C is a **CL**-theorem. So the recursive set of S-rules for $\mathbf{CLuN}^+$ has to comprise a subset that forms a set of S-rules for **CL**. The rules of the latter set are restricted to formulas from $\mathcal{W}_{\check{s}}$, except that the Prem applied to both sorts of formulas.

over?

What is described in the last sentence is *not* similar to the fact that, by moving **CLuN** from $\mathcal{L}_s$ to $\mathcal{L}_S$, **CL** is included in **CLuN**—see Section 2.2. The logic **CLuN** is defined over $\mathcal{W}_s$, has static proofs in $\mathcal{L}_s$, and is extended to $\mathcal{W}_S$ for technical reasons. The logic $\mathbf{CLuN}^+$ is defined over $\mathcal{W}_p$ but has only static proofs in a different language. That is odd, but precisely this oddity enables one to prove Theorem 1.5.9.

Let us now turn to the proof of Theorem 1.5.9. Consider a logic **L** that is defined over a (further unspecified) language $\mathcal{L}$. We shall also need a version of **CL** that is defined over a possibly completely different language. Let this language be $\mathcal{L}_s$ and keep in mind, while reading the subsequent paragraphs, that $\mathcal{L}$ and $\mathcal{L}_s$ need not have any formula in common. To avoid unnecessary complications, let us suppose that $\mathcal{W} \cap \mathcal{W}_s = \emptyset$.

Suppose that **L** is Reflexive, Transitive, Monotonic, and Compact, and that there is a positive test for it. It is sufficient that there is a positive test for $B_1, \ldots, B_n \vdash_{\mathbf{L}} A$ because **L** is compact. In view of what is often called Turing's Thesis, there is a Turing Machine T that halts with the answer YES (represented, for example, by a single stroke on an otherwise blank tape) iff indeed $B_1, \ldots, B_n \vdash_{\mathbf{L}} A$. There is a formula of the form $M \supset S \in \mathcal{W}_s$, for which the following is provable: $\vdash_{\mathbf{CL}} M \supset S$ iff T, started in standard position on a tape containing the Gödel numbers of $A, B_1, \ldots, B_n$ (in that order), halts with the answer YES after finitely many steps. The way in which $M \supset S$ is constructed is described, for example, in [BBJ02] and the statement is there proved.[22] The M and S in $M \supset S$ depend on the specific premises $B_1, \ldots, B_n$ and conclusion A. Let $M \supset S = f(A, B_1, \ldots, B_n)$ denote that $M \supset S$ is the suitable formula for these premises and conclusion. The function f is effectively computable. Note that f is actually composed from an infinity of functions, say f_0 which has one argument ($n = 0$), f_1 which has two arguments ($n = 1$), and so on. I now summarize:

(†) Where $M \supset S = f(A, B_1, \ldots, B_n)$, $B_1, \ldots, B_n \vdash_{\mathbf{L}} A$ iff $\vdash_{\mathbf{CL}} M \supset S$.

Consider the set $\mathcal{R}$ of rules comprising (i) a sufficient number of S-rules, restricted to members of $\mathcal{W}_s$ and not containing Prem, to prove all **CL**-theorems

[22] The statement is proved in the chapter on the undecidability of **CL** (chapter 11 of the cited edition). The reader might have the impression that [BBJ02] use the language of arithmetic, but there is no need to do so. The formula $M \supset S$ may easily be constructed within the predicative language *schema* $\mathcal{L}_s$—the schema contains more than enough symbols of the required kinds. When consulting [BBJ02], note that there no other logic than **CL** is ever considered.

in $\mathcal{W}_s$, (ii) the Prem rule, restricted to members of $\mathcal{W}$, and (iii) the recursive set of rules formed by all rules of the form

R_i From $M \supset S$, B_1, ..., and B_i to derive A, provided $M \supset S \in \mathcal{W}_s$, $A, B_1, \ldots, B_i \in \mathcal{W}$, and $M \supset S = f_i(A, \ldots, B_i)$.

in other words, the rules:

R_0 From $M \supset S$ to derive A, provided $M \supset S \in \mathcal{W}_s$, $A \in \mathcal{W}$, and $M \supset S = f_0(A)$.
R_1 From $M \supset S$ and B_1 to derive A, provided $M \supset S \in \mathcal{W}_s$, $A, B_1 \in \mathcal{W}$, and $M \supset S = f_1(A, B_1)$.
R_2 From $M \supset S$, B_1, and B_2 to derive A, provided $M \supset S \in \mathcal{W}_s$, $A, B_1, B_2 \in \mathcal{W}$, and $M \supset S = f_2(A, B_1)$.
⋮

In view of $\mathcal{R}$, any $B \in \mathcal{W}_s$ which occurs in a $\mathcal{R}$-proof from any $\Gamma \subseteq \mathcal{W}$ is a **CL**-theorem. The rules R_0, R_1, ... are S-rules—I mean the rules (metalinguistic expressions) as I list them above, not their instances.

Now we are ready to show that **L** has static proofs, viz. that $\Gamma \vdash_{\mathbf{L}} A$ iff $\Gamma \vdash_{\mathcal{R}} A$.

Part 1: Completeness: If $A, B_1, \ldots, B_n \in \mathcal{W}$ and $B_1, \ldots, B_n \vdash_{\mathbf{L}} A$, then $B_1, \ldots, B_n \vdash_{\mathcal{R}} A$. Suppose that the antecedent is true. So $M \supset S = f(A, B_1, \ldots, B_n)$ is a **CL**-theorem by (†), whence there is a $\mathcal{R}$-proof of it. Extend this proof by introducing $B_1, \ldots, B_n$ by application of Prem and next apply R_n to obtain A.

Part 2: Soundness: If $A, B_1, \ldots, B_n \in \mathcal{W}$ and $B_1, \ldots, B_n \vdash_{\mathcal{R}} A$, then $B_1, \ldots, B_n \vdash_{\mathbf{L}} A$. Suppose that the antecedent is true. So there is a $\mathcal{R}$-proof of A from $B_1, \ldots, B_n$. We proceed by an induction on the length of this proof. Three cases have to be considered; (i) The line's formula is $C \in \mathcal{W}_s$. So C is a **CL**-theorem (as was noted before). (ii) The line's formula $C \in \{B_1, \ldots, B_n\}$ is introduced by application of Prem. So $B_1, \ldots, B_n \vdash_{\mathbf{L}} C$ because **L** is reflexive. Moreover, as **L** is monotonic, all members of $\mathcal{W}$ that are the formula of a previous line remain derivable from the introduced premises (and obviously all members of $\mathcal{W}_s$ that are the formula of a previous line remain **CL**-theorems). (iii) The line's formula $C \in \mathcal{W}$ is introduced by applying of the rule R_m to $D_1, \ldots, D_m \in \mathcal{W}$ that are the formulas of previous lines. So $D_1, \ldots, D_m \vdash_{\mathbf{L}} C$ by (†). By the induction hypothesis, $B_1, \ldots, B_n \vdash_{\mathbf{L}} D_i$ for every D_i $(1 \leq i \leq m)$ because $B_1, \ldots, B_n$ are the formulas of previous lines. So $B_1, \ldots, B_n \vdash_{\mathbf{L}} C$ follows by the transitivity of **L**.

Theorem 1.5.9 is established: **L** has static proofs. The price to pay is even worse than in the case of $\mathbf{CLuN}^+$. The **L**-proof of A from $B_1, \ldots, B_n$ proceeds mainly in terms of the S-rules of **CL**, applied to formulas that have no symbol in common with $\mathcal{L}$, and the only S-rules that comprise formulas from $\mathcal{L}$ are Prem and the R_i. Every logic **L** that is Reflexive, Transitive, Monotonic, and Compact, and for which there is a positive test may be handled in this way. But clearly doing so hardly reveals anything about the logic **L**.

over?

What is to be concluded from all this? If a logic is defined over a certain language, loosening the definition of static proofs by allowing for semi-recursive sets of S-rules seems preferable over extending the language as done in the previous paragraphs. I shall not do so in this book because I do not know

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interesting logics that require this. The only aim of the present section was to prove the theorems, and to justify that I shall in the future prefer to say that a logic has static proofs with respect to a given language, rather than to describe it in terms of Tarski-like properties.