Chapter 4

The Standard Format

In this chapter, the standard format is introduced in a precise way and illustrated. The format provides a general structure that is common to a large class of logics. The proof theory and semantics of all logics in the class are defined by the standard format. The same holds for most of the metatheory, as will be shown in the next chapter. From this point on, adaptive logic is used in the restricted sense of a logic in standard format.

The notion of a dynamic proof requires careful analysis. This is provided in Section 4.7. A further intuitive clarification of dynamic proofs is offered in terms of their game-theoretic interpretation in Section 4.9. The puzzle caused by the combination of dynamic proofs with a static semantics is solved in Section 4.10 where a dynamic semantics is articulated.

4.1 Introduction

Adaptive logics were defined in a loose way in Chapter 1. That vague description led to a large and growing variety of specific systems. An essential task for a logician is to study the meta-theoretical properties of logical systems. There are two reasons for this. First, if one does not know these properties, one cannot claim to understand the logic. Next, only in view of these properties may one argue that a logic is adequate with respect to a certain purpose.

As the set of systems grew, it dawned that many adaptive logics might have a common structure. Studying this structure would clarify the specificity of adaptive logics. Doing so might moreover result in a simplification of the whole enterprise. Indeed, it was hoped from the outset that some meta-theoretic properties of adaptive logics would solely depend on the common structure and hence might be proved without reference to the specific system.

This led to the search for the common structure, which was labelled the *standard format* for adaptive logics. The first steps in that direction were taken in [Bat01] and [Bat04]. A list of theorems and proofs was more systematically presented in [Bat07b].

The search led to impressive results, as we shall see in this chapter and the next. The standard format may be seen as a function with three arguments. By specifying the arguments one obtains an adaptive logic, viz. its proof theory and its semantics and one obtains most of the logic's metatheory: the soundness and completeness of the proof theory with respect to the semantics and a plenitude of other properties. All this will be shown in Chapter 5. Specific adaptive logics will not even be mentioned, except as illustrative examples. However, the standard format will also be referred to in other chapters, for example in Chapter 10 where computational aspects will be discussed.

Once the standard format was devised, it turned out that many adaptive logics were formulated in this format. Some adaptive logics were first described in a rather different way, but could easily be phrased in standard format. For other adaptive logics, and for some newly tackled defeasible reasoning forms, a formulation in standard format was only possible at the expense of a translation—we shall see examples in Chapter 9. So the present chapter and the next may be seen as the study of the common features of a wide class of adaptive logics.

That a standard format with such properties can be formulated, is a major advantage. Whenever we come to a new adaptive logic that can be put in standard format, there is no need to further specify its proof theory and semantics. Moreover, and this is central, there is no need to go through all the difficult metatheoretic work in order to show that the logic has the required properties. The standard format also functions as a 'discovery recipe'. When, for example in scientific methodology, one comes across an inference relation that qualifies for a precise formulation as an adaptive logic, the task of finding this adaptive logic reduces to specifying the three elements. As we shall see, there are clear heuristic means to locate the elements.

The standard format may be seen as a precise definition. The reader may wonder why I did not present this definition from the beginning of the book, why I started with a loose definition of an adaptive logic. There is a good reason for this. The adaptive logic approach is still under development; it is not a good idea to tie it up too strictly. It is very well possible that the standard format may need to be modified or generalized later. It is also possible that a different, more general 'standard format' will be found, which would apply to more adaptive logics in the sense that it imposes less requirements on them. During the last ten years, especially under the influence of young logicians in close contact with philosophers of science and other philosophers, the number and variety of adaptive logics has constantly been growing and more and more domains were explored. The present version of the standard format is apparently sufficient for nearly all of these. Still, there is no warrant that the general idea of an adaptive logic has been fully explored. So it seems better to define the notion of an adaptive logic in an intuitive way and to consider the standard format as the common structure of a broad class of adaptive logics.

This and the next chapter do not concern specific adaptive logics, but a common structure of such logics. Each of them is defined over a specific language, which will be denoted by the variable $\mathcal{L}$. In many cases $\mathcal{L}$ will be $\mathcal{L}_s$, but it may also be $\mathcal{L}_M$ or another language. I shall stick to the convention that the set of formulas of $\mathcal{L}$ is countable. Whatever $\mathcal{L}$, its extension with the classical symbols will be denoted by $\mathcal{L}_+$. Note that the set of formulas of the extended language is also countable. More details on the extension are presented in Section 4.3.

Some further clarification seems desirable. If the adaptive logic is corrective, its lower limit logic is weaker than **CL** and the use of the classical logical symbols is obvious. Some may wonder whether the symbols have any use when the adaptive logic is ampliative. For one thing, I shall need them in order to state the inference rules, the marking definitions, the semantics, and much of the metatheory in general terms. Moreover, the presence of classical disjunction will enable me to express that the person constructing the proof realizes that a *Dab*-formula was derived.

It is often said that a logic defines the meaning of its logical symbols. The topic is a touchy one in the context of an adaptive logic. A first possible stand is that the meaning of the logical symbols is defined by the standard of deduction—in our case **CL** for conventional reasons. But this is odd. In the context of $\mathbf{CLuN}^m$, for example, the meaning of the standard negation would then be classical. So if some *Dab*-formulas are derivable from the premise set, they are actually nonsensical, but the adaptive logic still enables one to handle the nonsense in a coherent way. A second possible stand is to see the meaning of the standard logical symbols as defined by the lower limit logic, which always will be a logic that has static proofs, and to see the meaning of the classical logical symbols as defined by classical logic, which actually is included in the lower limit logic. A third possible stand is to see, in the context of the adaptive logic, the meaning of the standard logical symbols as defined by the adaptive logic. So these symbols have a *variable* meaning. Some occurrences of the symbol may have the classical meaning whereas others have a different meaning. Thus, in the context of $\mathbf{CLuN}^m$, some occurrences of negation have a classical meaning whereas other have a paraconsistent meaning—the occurrence of negation in $\neg A$ is classical if $\neg A$ has the same force as $\check{\neg} A$, for example because it enables one to adaptively derive B from $A \vee B$. Whichever stand one takes, it is possible to tell a coherent story on what goes on in an adaptive logic. So I shall not defend any of the stands in this book.

In devising the standard format, the central aim was obviously unification within the adaptive framework. In this context, it is essential to remember the central aim of the adaptive framework, which is to provide a formal and strict control of dynamic reasoning. In this sense, the dynamic proof theory is more essential than the semantics. The semantics has a clear function, which was clarified in Section 1.6, and it is essential that adaptive logics have a semantics. However, the dynamic proof theory has the clearly different function to explicate actual reasoning. This is the point at which control is required. The control will proceed as expected: in terms of conditions and marks. In actual reasoning, conditions may be forgotten and marks (revoking previously drawn conclusions) may be defective. This is inevitable in view of the complexity of the dynamic inferences. It presumably is also the reason why it took so many centuries to arrive at a decent explication. With all this in mind, let us go.

4.2 The Definition

The aim of adaptive logics is to explicate reasoning forms that proceed in terms of a certain language $\mathcal{L}$ (with $\mathcal{W}$ as its set of closed formulas). The explication, however, proceeds in terms of $\mathcal{L}_+$, which was introduced before in this chapter and has $\mathcal{W}_+$ as its set of closed formulas. beter uitwerken?

An adaptive logic in standard format, $\mathbf{AL}\colon \wp(\mathcal{W}_+) \rightarrow \wp(\mathcal{W}_+)$ is defined by a triple:

(1) A *lower limit logic* **LLL**: a logic that has static proofs.
(2) A *set of abnormalities* $\Omega \subseteq \mathcal{W}_+$: a set of formulas characterized by a (possibly restricted) logical form F; or a union of such sets. oud
(3) An *adaptive strategy*: Reliability or Minimal Abnormality.

These three elements will provide the required control to handle defeasible logics in a decent way. As we shall see when we come to the proof theory, the lower limit logic and the set of abnormalities jointly define the rules of the proofs, whereas the set of abnormalities and the strategy jointly define the marking definition. As we shall see when we come to the semantics, the lower limit logic provides the models of the premises, whereas the set of abnormalities and the strategy jointly define the criterion that selects the adaptive models of the premise set from its lower limit models.

Let us start with the lower limit logic, which is supposed to be defined over a language $\mathcal{L}$. By Theorems 1.5.3–1.5.8, **LLL** is reflexive, transitive, monotonic, uniform, and compact, and there is a positive test for it. I shall always suppose that **LLL** has a characteristic semantics. Still, if no characteristic semantics is given for **LLL**, the adaptive logic has no semantics but the proof theory still works fine. Of course a ready-made semantics may always be devised in view of Suszko's [Sus77] demonstration that every logic has a two-valued semantics. One simply turns every S-rule into a property of the models; the semantics thus obtained might be ugly and might fail to offer any insights not contained in the S-rules, but it is obviously characteristic. If **LLL** is given by a semantics and not by a set of S-rules, the adaptive logic will have no proof theory, but its semantics still works fine. In this case one may devise a proof theory directly from the semantics—replacing $\vdash$ by $\vDash$ in the generic rules RC and RU from Section 4.4 below.[1]

Throughout this book, the lower limit logic **LLL** will be taken to contain **CL** (in $\mathcal{L}_+$) as described in Section 4.3.

Typical for adaptive logics is that their consequence set extends the **LLL**-consequence set by presupposing that 'as many' members of Ω are false as the premise set permits. The logical form F that characterizes the set of abnormalities Ω may be restricted. This means that the metavariables that occur in the logical form may be required to denote formally specific entities. Let us consider an example. Some inconsistency-adaptive logics have $\{\exists(A \wedge \neg A) \mid A \in \mathcal{F}_s^p\}$ as their set of abnormalities, in which $\mathcal{F}_s^p$ is the set of *primitive* formulas—see Section 7.3 for examples and details. Compare this to the adaptive logics $\mathbf{CLuN}^r$ and $\mathbf{CLuN}^m$, presented in Chapter 2, that have $\{\exists(A \wedge \neg A) \mid A \in \mathcal{F}_s\}$ as their set of abnormalities. We have seen some other restrictions on Ω in Chapter 3.

An adaptive logic was described as a function $\mathbf{AL}\colon \wp(W) \to \wp(W)$. So the premises and conclusions belong to the closed formulas of a given language $\mathcal{L}$. It may be felt as odd that the set of abnormalities, Ω, comprises formulas that belong to $\mathcal{W}_+$ and hence may contain classical symbols. The reason for this is pragmatic in nature. By allowing sets of abnormalities $\Omega \subset \mathcal{W}_+$, a much larger number of sensible logics will be allowed by the standard format. These logics are very close to the logics we have met in Chapter 2. Many of them form natural counterparts to the logics described there. Moreover, allowing for $\Omega \subset \mathcal{W}_+$ does not in any way harm the proofs. We shall see that *Dab*-formulas have to be expressed by *classical* disjunctions anyway and *Dab*-formulas are derived in the proofs. So there is no harm in the fact that further classical symbols occur within abnormalities that make up the disjuncts of *Dab*-formulas.

It is important to differentiate between the question whether a logic complies

[1] Several proof theories may obviously define the same consequence relation. Similarly, as we saw before, several semantic systems may be equivalent.

with the definition of an adaptive logic in standard format, and the question whether such a logic is sensible. As I shall show in the next chapter, there are a number of border cases that are not sensible from a technical point of view because they are useless complications. I shall briefly review the most important cases in the next three paragraphs. The existence of such border cases, however, is not a reason to complicate the above definition. If the standard definition guarantees certain properties only for logics that fulfil some further conditions, I shall explicitly state so in the relevant theorems.

If Ω is empty, the adaptive logic **AL** is simply its lower limit **LLL** in disguise; all abnormalities are false because there are none. Moreover, the upper limit logic is also identical to the lower limit—the upper limit logic will be defined in general below. If Ω is the set of all closed formulas of the considered language, then again **AL** is identical to **LLL**. This is a trivial consequence of Theorems 5.3.1 and 5.3.3, which are proved in the next chapter.

If the logical form F, which characterizes the set of abnormalities, is not **LLL**-contingent, then either $\vdash_{\mathbf{LLL}} \mathsf{F}$ or $\vdash_{\mathbf{LLL}} \check{\neg}\mathsf{F}$. In the former case, every member of Ω is a **LLL**-consequence of every premise set. This has as an effect that **AL** is identical to **LLL**. In the latter case all members of Ω lead to triviality on **LLL**. I refer to Section 5.9 for details. There, I also consider the case in which some formulas of the form F are not **LLL**-contingent.

In the two previous chapters, I already introduced the expression $Dab(\Delta)$. Here is the right place to give it its general meaning: in the expression $Dab(\Delta)$, Δ will always be a finite subset of Ω, and $Dab(\Delta)$ will denote the *classical* disjunction of the members of Δ—the disjuncts may be defined to occur in a certain order or not, all such disjunctions being logically equivalent anyway.[2] If Δ is a singleton, $Dab(\Delta)$ is an abnormality (a member of Ω) and no classical disjunction occurs. If $\Delta = \emptyset$, $Dab(\Delta)$ is the empty string and $A \check{\vee} Dab(\Delta)$ is A. The reason why $Dab(\Delta)$ denotes the *classical* disjunction of the members of Δ is extensively discussed in Section 4.9.3.

The need for a strategy was already illustrated in the two previous chapters. For many premise sets Γ, lower limit logics **LLL**, and sets of abnormalities, $Dab(\Delta)$ is **LLL**-derivable from Γ whereas no member of Δ is. The strategy determines what it means to interpret the premises 'as normally as possible' in such cases. Two basic strategies serve this purpose: Reliability and Minimal Abnormality. Their precise meaning is defined by the marking definition at the syntactic level and by the model selection mechanism at the semantic level—both are presented in subsequent sections.

For some adaptive logics and for some selected premise sets, Reliability and Minimal Abnormality come to the same. Where this is the case, the proof theory and the semantics may be phrased in a much less sophisticated way than in general. This is why the label the Simple Strategy is often used for such cases. The matter will be discussed in Section 6.1.1. This finishes the explanation on the standard format.

An adaptive logic **AL** can now be described in a different way. The **AL**-consequences of Γ are all the formulas that can be derived from Γ by **LLL** *and* by relying on the supposition that "the members of Ω are false in as far as Γ permits them to be false". This expression is ambiguous, but the strategy

[2]To be more precise, equivalent with respect to **LLL** extended with the classical logical symbols.

disambiguates it.

The lower limit logic **LLL** and the set of abnormalities Ω jointly determine a so-called *upper limit logic* **ULL**. Syntactically **ULL** is obtained by adding to **LLL** a S-rule that connects abnormality to triviality. Put differently, the upper limit logic **ULL** is exactly like the lower limit logic, except that it trivializes abnormalities. In other words:

Definition 4.2.1 $\Gamma \vdash_{\mathbf{ULL}} A$ *iff* $\Gamma \cup \{\check{\neg} B \mid B \in \Omega\} \vdash_{\mathbf{LLL}} A$.

Semantically, the upper limit logic of an adaptive logic may be characterized similarly, viz. in an abstract way (whatever the structure of the models). A semantics for a logic **L** can be seen as a set of models and this set determines the semantic consequence relation.

Definition 4.2.2 *An* **LLL***-model is an* **ULL***-model iff it verifies no member of* Ω.

Note that some premise sets may not have any **ULL**-models and hence are **ULL**-trivial. In Section 4.6, some properties of **ULL** will be proven.

If F, the form that characterizes Ω, is **LLL**-contingent, but some formulas of this form are **LLL**-theorems, there are no **ULL**-models and $Cn_{\mathbf{ULL}}(\Gamma)$ is trivial for all Γ.[3] For this reason, it is more elegant in such cases to require, in the definition of Ω, that the abnormalities are not **LLL**-theorems. So instead of $\Omega = \{\mathsf{F} \mid \ldots\}$, in which F is a logical form and the dots impose a condition on F, we would define $\Omega = \{\mathsf{F} \mid \nvdash_{\mathbf{LLL}} \mathsf{F}; \ldots\}$. An example, taken from Section 9.2, is $\Omega = \{\neg\Diamond A \mid \nvdash_{\mathbf{S5}} \neg\Diamond A; A \in \mathcal{F}_s\}$, in which a specific predicative version of **S5** is the lower limit logic. This set of abnormalities comprises all formulas of the form $\neg\Diamond A$ that are not **S5**-theorems. As a result the upper limit logic is **Triv**, obtained by adding the axiom $\Diamond A \supset A$ to **S5** or even to **T**.

While writing the first paper on adaptive logics, I was convinced that an adaptive logic is defined by a lower limit logic **LLL**, an upper limit logic **ULL**, and a strategy—the adaptive logic would then try to approach the upper limit logic from the lower limit logic in view of the strategy. This was a mistake and it was only after a variety of adaptive logics was available that the mistake became clear. There may be many adaptive logics between a given **LLL** and a given **ULL**; their differences are determined by the set of abnormalities Ω. So a **LLL** and a **ULL** do not define the set of abnormalities. However, once **LLL** and Ω are fixed, so is **ULL**—so **ULL** is a side effect of **LLL** and Ω. It is nevertheless interesting to consider **ULL** because it is, in a clear and clarifying sense, an upper limit. As I shall prove in the next chapter, every adaptive consequence set coincides with the upper limit consequence set, $Cn_{\mathbf{AL}}(\Gamma) = Cn_{\mathbf{ULL}}(\Gamma)$, whenever Γ is a *normal* premise set, viz. a premise set that does not require any abnormality to be true. For this reason the complication described in the previous paragraph is worth being mentioned. If, for example, Ω comprises some **LLL**-theorems, these are required to be true by every premise set, whence there are no normal premise sets. By 'filtering out' (see Section 5.9) the **LLL**-theorems from the set of abnormalities, the adaptive logic remains the same, viz. assigns

[3]No language is specified in "$Cn_{\mathbf{ULL}}(\Gamma)$ is trivial". Here and in all similar cases, I mean that both readings of the statement hold true: $Cn^{\mathcal{L}}_{\mathbf{ULL}}(\Gamma)$ is $\mathcal{L}$-trivial and $Cn^{\mathcal{L}_+}_{\mathbf{ULL}}(\Gamma)$ is $\mathcal{L}_+$-trivial.

the same consequences to the same premise sets, but there are normal premise sets. So abnormalities that are **LLL**-theorems are not 'genuine' abnormalities. By retaining only the 'genuine' abnormalities in Ω, one is able to separate the *normal* premise sets from the abnormal ones.

In certain circumstances (and for some logicians always) it is possible to consider a logic as the standard of deduction. If the standard of deduction is the upper limit logic of an adaptive logic, the adaptive logic is *corrective*. If the standard of deduction is the lower limit logic of an adaptive logic, the adaptive logic is called *ampliative*. The standard of deduction has been discussed in Chapter 1. There I also promised not to take a stand on the matter and to adopt a pragmatic and conventional position in the present book, which is to take **CL** as the standard of deduction.

Let us consider some examples of adaptive logics. The inconsistency-adaptive $\mathbf{CLuN}^m$ from Chapter 2 is defined by:

(1) *lower limit logic*: **CLuN**
(2) *set of abnormalities*: $\Omega = \{\exists(A \wedge \neg A) \mid A \in \mathcal{F}_s\}$
(3) *adaptive strategy*: Minimal Abnormality

In view of generality, it would be wiser to use classical symbols even in the abnormalities. In this case one defines $\Omega = \{\check{\exists}(A \mathbin{\check{\wedge}} \neg A) \mid A \in \mathcal{F}_s\}$. This would especially make matters more transparent where different corrective adaptive logics operate together. The disadvantage is that this would make the outlook of this book rather weird. In general, I shall use the standard symbols *within* abnormalities when these have the same meaning as the classical symbols. The major exception will be Chapter 8, where combinations of logics and of sets of abnormalities frequently occur.

The inconsistency-adaptive $\mathbf{CLuN}^r$, also presented in Chapter 2, is defined by the same elements, except that Reliability is its strategy. The upper limit logic of both adaptive logics is **CL**, syntactically obtained by extending **CLuN** with the axiom[4] $(A \wedge \neg A) \supset B$ and semantically obtained by restricting the set of **CLuN**-models to those that verify no inconsistency. If **CL** is considered to be the standard of deduction (as I conventionally do), $\mathbf{CLuN}^m$ and $\mathbf{CLuN}^r$ are corrective adaptive logics. If a theory that was intended to be consistent and was given **CL** as its underlying logic turns out to be inconsistent, one wants to interpret it 'as normally as possible' in order to forge a consistent replacement for it *by reasoning from it.*

The (ampliative) logic of inductive generalization: $\mathbf{IL}^m$ from Chapter 3 is defined by:

(1) *lower limit logic*: **CL**
(2) *set of abnormalities*: $\Omega = \{\exists A \wedge \exists\neg A \mid A \in \mathcal{F}_s^\circ\}$, in which $\mathcal{F}_s^\circ$ is the set of formulas that contain no individual constants and no quantifiers (the purely functional formulas)
(3) *adaptive strategy*: Minimal Abnormality

The upper limit logic is **UCL**, obtained syntactically by extending **CL** with the axiom $\exists\alpha A(\alpha) \supset \forall\alpha A(\alpha)$, which reduces non-uniformity to triviality, and obtained semantically by restricting the set of **CL**-models to the uniform **CL**-models. Uniformity is obviously an idea taken from [Car52]. In all **UCL**-models $v(\pi^r) \in \{\emptyset, D^{(r)}\}$: the extension of a predicate of rank r is either empty or universal (the set of all r-tuples of members of the domain). Needless to say,

[4]The axiom is contextually equivalent to $\exists(A \wedge \neg A) \supset B$.

applying **UCL** to the actual world results in triviality because not all objects have the same properties, viz. the world is not (fully) uniform. The $\mathbf{IL}^m$-consequences of our observational data contain the generalizations that would hold in the world if it were as uniform as is compatible with our observational data.

The set referred to in the last sentence is not as easily obtained as one might be tempted to think. If the data comprise Pa, Qa, Rb, $\neg Qb$, Pc and Rc, neither $\forall x(Px \supset Qx)$ nor $\forall x(Rx \supset \neg Qx)$ are derivable because they are not jointly compatible with Pc and Rc. I refer to Chapter 3 for the remarkable properties of the logic of inductive generalization and for the way in which it may be combined with adaptive logics handling background knowledge.

As a final example, consider the (ampliative) adaptive logic of plausibility $\mathbf{T}^m$:

(1) *lower limit logic*: **T** (a specific predicative version of this logic, for example the one obtained by extending **K** from Section 3.6 by requiring that R is reflexive).
(2) *set of abnormalities*: $\Omega = \{\Diamond A \wedge \neg A \mid A \in \mathcal{F}_s^p\}$—recall that $\mathcal{F}_s^p$ is the set of primitive formulas of the standard language.
(3) *adaptive strategy*: Minimal Abnormality.

The upper limit logic is again **Triv**. Intuitively, the premises of the form $\Diamond A$ may be read as stating that A is plausible.[5] The adaptive logic $\mathbf{T}^m$ interprets the premises in such a way that plausible formulas are true 'in as far as the premises permit'. More often than not, one wants plausibility to come in degrees. This is not a difficult problem. We have seen examples already in Section 3.6 and the full solution comes in Chapter 6.

In the last two examples, the axiom added to the lower limit in order to obtain the upper limit is contextually equivalent to the axiom that connects abnormalities to triviality. Adding the restriction that occurs in the respective definitions of Ω is superfluous.

4.3 Excursion on the Classical Symbols

The extension with the classical symbols deserves some careful attention. Both technical matters and philosophical matters have to be cleared up.

Technical issues At first sight, the extension seems a simple matter. One obtains $\mathcal{L}_+$ by adding the classical symbols to $\mathcal{L}$, adds to **LLL** the S-rules governing the classical logical symbols, and adds clauses governing the classical symbols to the **LLL**-semantics. There are, however, two ways to do all this.

The classical symbols may be *intertwined* with the symbols of $\mathcal{L}$. This means that the rules (or theorems) holding for the formulas of $\mathcal{L}$ are also taken to be valid for the formulas of $\mathcal{L}_+$. Take **CLuN** as an example. If the classical symbols are intertwined, $\check{\neg} A \vee \neg\check{\neg} A$ is a theorem because $A \vee \neg A$ is a theorem.

The classical symbols may also be *superimposed* on $\mathcal{W}$, the formulas of $\mathcal{L}$. This means that the S-rules of **LLL** are restricted to $\mathcal{W}$, to which they applied before the classical symbols were added, whereas the S-rules for the classical

[5] On the present approach, all **CL**-consequences of A are also plausible, but this can be avoided, for example as in [BH01].

symbols are defined for all formulas of $\mathcal{L}_+$. Semantically, the clauses for the logical symbols of $\mathcal{L}$ are restricted to $\mathcal{W}$, while the clauses for the classical symbols are unrestricted. Applying all this to **CLuN**, $A \vee \neg A$ is a theorem, and so is $\neg A \check{\vee} \check{\neg}\neg A$ because it has the form $A \check{\vee} \check{\neg} A$, but $\check{\neg} A \vee \neg\check{\neg} A$ is not a theorem. Although other constructions are possible, the easiest way to realize superposition, rather than intertwining, is by restricting the formation rules. First $\mathcal{W}$ is defined. Next $\mathcal{W}_+$ is defined by specifying $\mathcal{W} \subset \mathcal{W}_+$ and by introducing the classical symbols: if $A \in \mathcal{W}_+$, then $\check{\neg} A \in \mathcal{W}_+$—and similarly for the other classical symbols.

Intertwining the classical symbols may cause turmoil. It is impossible to describe in the general and abstract case what the intertwining will look like, because one does not know what might be the structure of the **LLL**-semantics. This is an annoying complication for general chapters like the present one, especially as different sorts of intertwining may be possible if the models are complex. Superimposing, to the contrary, is easy to realize because it may be realized independently of the internal structure of the models of the **LLL**-semantics. It is sufficient to give any semantics such properties as $M \Vdash \check{\neg} A$ iff $M \nVdash A$, $M \Vdash A \check{\supset} B$ iff $M \nVdash A$ or $M \Vdash B$, etc.

The official position of this book will be that the classical symbols are *superimposed*. Of course, intertwining causes no harm and may be easier to understand for some readers.

In applications of adaptive logics, the premises and conclusion will always belong to the native language of the lower limit logic and the classical symbols will only have an auxiliary function, as in the rule RC from Section 4.4. They will simplify the object-level proofs and sometimes make the semantics more transparent. This will become clear in the present chapter, for example where I introduce the generic rules of the adaptive proof theory. Moreover, the presence of the classical logical symbols will greatly simplify metalinguistic proofs. This will especially become clear in Chapter 5. Actually, many theorems do not contain a reference to classical symbols (in the statement of the theorem). Most of these can be proved without referring to classical symbols, but the proofs become more longwinded.[6]

There are some application contexts, especially where an adaptive logic characterizes a defeasible logic that was presented in the literature in a different format, in which two variants of some logical symbols are required. Thus, one of the adaptive characterizations of the Rescher–Manor consequence relations—see Section 9.7—requires the presence of a classical and a paraconsistent negation, viz. the negation of **CLuN**. This has caused some confusion in the past. I think the best way to handle it is as follows. We consider a version of **CLuN** that is defined over the standard language $\mathcal{L}_s$ extended with a classical negation which will be named by the symbol $\sim$. This will be the native language for those applications and the symbol $\sim$ obviously needs to be intertwined. This language will then be extended with the classical symbols, $\check{\neg}$, $\check{\vee}$, and so on. Of course, $\sim$ and $\check{\neg}$ have exactly the same meaning. Still, it is useful to distinguish between them in view of their function.

At this point, I need to say a word on the trivial model. Some philosophers adduce arguments for a semantics that comprises the trivial model—see for

[6]Basically, the expression $M \Vdash \check{\neg} A$ should be replaced by $M \nVdash A$. If classical negation occurs in a subformula of A, the matter may become more tedious.

example [Pri87]. The trivial model is the model that verifies all closed formulas of the language, in other words a model M for which $\{A \mid M \Vdash A\}$ is the set of all formulas of the language. The whole point is: trivial with respect to which language? Consider the **CLuN**-semantics defined for the language $\mathcal{L}_s$ as in Section 2.2. This semantics obviously comprises a trivial model: the one in which $v(A) = 1$ for all $A \in \mathcal{W}_{\mathcal{O}}$—the formulas and pseudo-formulas comprising the standard symbols as well as pseudo-constants. As soon as one moves to $\mathcal{L}_S$, however, the presence of the trivial model would be a hindrance and is redundant anyway. Let me first say why it is redundant. Removing the trivial model from a semantics does not change the semantic consequence relation—this holds whatever the language is. Indeed, let S_2 be obtained by removing the trivial model from S_1. If all S_2-models of Γ verify A, then so do all S_1 models of Γ because the trivial model verifies every formula. If all S_1-models of Γ verify A, then so do all S_2 models of Γ, except that, if Γ has only trivial S_1-models then it has no S_2-models; but then "all S_2-models of Γ verify A" holds vacuously. Next, the presence of the trivial model would be a hindrance in the case of the extended language—the one that comprises the classical logical symbols. It would require a more tiresome formulation of the semantics. For example, the clause "$v_M(\check{\neg}A) = 1$ iff $v_M(A) = 0$" would need to be modified to "$v_M(\check{\neg}A) = 1$ iff $v_M(A) = 0$ or M is trivial". Still, let us remember the following lemma for the record.

Lemma 4.3.1 *Where $A \in \mathcal{W}$, $\Gamma \vDash_{\mathbf{L}} A$ iff A is verified by all non-$\mathcal{W}$-trivial* **L***-models of Γ. (Trivial Model Lemma)*

Philosophical issues Priest has argued, in [Pri87], that a logic should not rule out that all statements are true. As such, this is not a very convincing argument. In the same sense, a logic should not rule out that nothing exists, but most predicative logics do. Logics devoid of presuppositions do not exist. Whether the trivial model should be present in a semantics depends on one's presuppositions. If one thinks to have philosophical reasons to reject classical negation *in all contexts*, then one should not use classical negation, whence the trivial model is harmless. If one considers classical negation to be sensible in some contexts, then in those contexts one should not have a model that is trivial with respect to $\mathcal{W}_+$ although there may still be a model that is trivial with respect to $\mathcal{W}$. Note that this model, which is trivial with respect to $\mathcal{W}$, may be extended into a model for $\mathcal{L}_+$. In the extended model $v_M(\check{\neg}A) = 0$ for all $A \in \mathcal{W}_s$. Even in $\mathcal{L}$, the trivial model will be redundant, as was explained before. But it need not be a hindrance at all. The **CLuN**-model that is trivial with respect to $\mathcal{W}_s$, for example, agrees with the most natural rendering of the semantic clauses. It would be silly to remove it and it would be hard to argue, given the meaning of the logical symbols, that it does not represent a possible state of the world. In that sense Priest is right.

I have presented arguments for extending a standard language with the classical logical symbols. These referred to handiness and simplicity. Moreover, being a contextualist and (provisionally) convinced that the logical systems dealt with are consistent (even if some of them may handle inconsistent situations), I decided to use a metalanguage that is classical throughout and I decided to push part of this metalanguage into the object language by superimposing the classical logical symbols. As was mentioned in Section 2.5, dialetheists consider classical negation a nonsensical operator. Does this mean that the approach followed in this book reduces adaptive logics to rubbish in their eyes? Relying on the fact that the inconsistency-adaptive consequence set of a *consistent* Γ is identical to the **CL**-consequence set of Γ—see the first statement in Item 1 of Theorem 5.6.7—Priest has argued in [Pri87], that reasoning in terms of classical logic may be seen as reasoning in terms of an inconsistency-adaptive logic while forgetting about the conditions.[7] This he calls the classical recapture. If the logical systems in this book are consistent, the conclusions drawn about them by **CL** and by, say $\mathbf{CLuN}^m$, are identical. If the logics turn out to be inconsistent,[8] their metatheory is still non-trivial on the inconsistency-adaptive logic. So the correct (but more complicated) reading of the classical metatheoretic proofs in this book should read as phrased in terms of an inconsistency-adaptive logic. Obviously, part of this metatheory should be revoked if the logics turn out to be inconsistent, but let us postpone that problem until it arises. reeds zwart boek nazien

Some people consider my approach as conceptually confusing because I identify logics defined with respect to their native language $\mathcal{L}$ with logics defined with respect to $\mathcal{L}_+$, which extends $\mathcal{L}$ with superimposed classical connectives. So let me state my position in this respect as clearly as possible. On the one hand, the aim of adaptive logics is to explicate reasoning forms that concern premises and candidate conclusions that belong to a certain native language $\mathcal{L}$. This is the reason why I require that the premises and conclusion belong to $\mathcal{W}$. The explication, however, proceeds in $\mathcal{L}_+$. This is the reason why I do not refrain from having the classical logical symbols in abnormalities, or even in certain theorems. What is essential for this construction, is that, for any logic **L**, whether adaptive or not, the transition from $\mathcal{L}$ to $\mathcal{L}_+$ is conservative. What this means is that, if $\Gamma \subseteq \mathcal{W}$, then $Cn^{\mathcal{L}}_{\mathbf{L}}(\Gamma) = Cn^{\mathcal{L}_+}_{\mathbf{L}}(\Gamma) \cap \mathcal{W}$. Does this identity hold? It does if the logics are consistent. If they are not, one has to appeal to the classical recapture.

4.4 Proofs

The dynamics of the proofs is controlled by the *conditions* (finite subsets of Ω) that are attached to lines and by the marking definition. While lines are added to a proof by applying the rules of inference, the marking definition determines for every stage of the proof which lines are 'in' and which are 'out'—stages were defined in Definition 1.5.1. The rules of inference are determined by the lower limit logic **LLL** and by the set of abnormalities Ω, whereas the marking definition is determined by Ω and by the strategy. So the lines that occur (marked or unmarked) in a proof are independent of the strategy.

A line of an annotated proof consists of a line number, a formula, a justification, and a condition. The presence of the latter distinguishes dynamic proofs from usual proofs. The justification consists of a (possibly empty) list of line numbers (from which the formula is derived) and of the name of a rule.

As remarked before, the rules determine which lines (consisting of the four

[7] If the premises are consistent, the sequence of formulas (second elements of the lines of annotated proofs) are identical for both logics—no line is marked in the adaptive proof. If the premise set is inconsistent, the sequences are obviously different—see the Blindness strategy in Section 6.1.2.

[8] A logic **L** would be inconsistent if, for some Γ and A, $\Gamma \vdash_{\mathbf{L}} A$ as well as $\Gamma \nvdash_{\mathbf{L}} A$.

aforementioned elements) may be added to a given proof. The only effect of the marking definition is that, at every stage of the proof, certain lines are marked whereas others are unmarked. Whether a line is marked depends only on the condition of the line and on the minimal *Dab*-formulas—see below—that have been derived in the proof; this holds for all marking definitions. Whether the marks are considered as parts of the annotation is obviously a conventional matter.

I shall discuss the notion of an adaptive proof below, but first present the rules of inference and the marking definitions. The rules of inference reduce to three generic rules. Where

$$A \qquad \Delta$$

abbreviates that A occurs in the proof on the condition Δ, the generic rules are:

Prem If $A \in \Gamma$:

$$\begin{array}{ll} \ldots & \ldots \\ \hline A & \emptyset \end{array}$$

RU If $A_1, \ldots, A_n \vdash_{\mathbf{LLL}} B$:

$$\begin{array}{ll} A_1 & \Delta_1 \\ \ldots & \ldots \\ A_n & \Delta_n \\ \hline B & \Delta_1 \cup \ldots \cup \Delta_n \end{array}$$

RC If $A_1, \ldots, A_n \vdash_{\mathbf{LLL}} B \check{\vee} Dab(\Theta)$:

$$\begin{array}{ll} A_1 & \Delta_1 \\ \ldots & \ldots \\ A_n & \Delta_n \\ \hline B & \Delta_1 \cup \ldots \cup \Delta_n \cup \Theta \end{array}$$

These rules were already used in the two previous chapters but are presented here in their general form (independent of the specific lower limit logic).

There is an important relation between adaptive proofs and lower limit proofs. An **AL**-proof from Γ can be seen as a **LLL**-proof in disguise; in other words, an **AL**-proof is just like a **LLL**-proof except that abnormalities that are disjuncts of the formula of a line may be moved to the condition of that line (and that other lines have an empty condition). The following lemma states the precise way in which an **AL**-proof from Γ can be seen as a **LLL**-proof in disguise. The lemma is important and will play an essential role in characterizing dynamic proofs in Section 4.7.

Lemma 4.4.1 *An* **AL**-*proof from* Γ *contains a line on which* A *is derived on the condition* Δ *iff* $\Gamma \vdash_{\mathbf{LLL}} A \check{\vee} Dab(\Delta)$. *(Conditions Lemma)*

Proof. ⇒ Let us start with a proof from Γ consisting of zero lines. The Lemma obviously obtains vacuously for such a proof. Supposing that the Lemma holds up to line n of the proof, I show that it holds for line $n+1$. Let B be the formula of line $n+1$. The line has been added by application of Prem, RU, or RC.

Case 1: Prem. So the formula of line $n+1$ is B for some $B \in \Gamma$ and its condition is $\emptyset$. As **LLL** is reflexive, $\Gamma \vdash_{\mathbf{LLL}} B$ (and remember that $B \check{\vee} Dab(\emptyset)$ is B).

Case 2: RU. So there are $A_1, \ldots, A_n$ that were derived on previous lines on the conditions $\Delta_1, \ldots, \Delta_n$ respectively, $A_1, \ldots, A_n \vdash_{\mathbf{LLL}} B$, and the condition of line $n+1$ is $\Delta_1 \cup \ldots \cup \Delta_n$. By the induction step, $\Gamma \vdash_{\mathbf{LLL}} A_1 \check{\vee} Dab(\Delta_1)$, $\ldots$ $\Gamma \vdash_{\mathbf{LLL}} A_n \check{\vee} Dab(\Delta_n)$, and hence, by **CL**-properties $\Gamma \vdash_{\mathbf{LLL}} B \check{\vee} Dab(\Delta_1 \cup \ldots \cup \Delta_n)$.

Case 3: RC. Wholly analogous to Case 2, except that $A_1, \ldots, A_n \vdash_{\mathbf{LLL}} B \check{\vee} Dab(\Theta)$, the condition of line $n+1$ is $\Delta_1 \cup \ldots \cup \Delta_n \cup \Theta$, and $\Gamma \vdash_{\mathbf{LLL}} B \check{\vee} Dab(\Delta_1 \cup \ldots \cup \Delta_n \cup \Theta)$.

$\Leftarrow$ In view of the compactness of **LLL**, there is a **LLL**-proof of $A \check{\vee} Dab(\Delta)$ from Γ. So there is a **AL**-proof from Γ, obtained by applications of Prem and RU, in which $A \check{\vee} Dab(\Delta)$ is derived on the condition $\emptyset$. By applying RC to the last step, one obtains a proof from Γ in which A is derived on the condition Δ. ■

This lemma leads immediately to the derivable rule RD. If A is derived at a line on the condition Δ and $\check{\neg} A$ is derived at a line on the condition Θ, $Dab(\Delta \cup \Theta)$ is unconditionally derivable.[9]

RD		
	A	Δ
	$\check{\neg} A$	Θ
	$Dab(\Delta \cup \Theta)$	$\emptyset$

There is a more general variant of RD. It is useful in combined adaptive logics and will be illustrated in Chapters 6 and 8.

RD		
	A	$\Delta \cup \Delta'$
	$\check{\neg} A$	$\Theta \cup \Theta'$
	$Dab(\Delta \cup \Theta)$	$\Delta' \cup \Theta'$

RD is derivable in all adaptive logics in standard format. Its derivability follows from Lemma 4.4.1 and of the meaning of the classical symbols; nothing is supposed on the nature of the abnormalities.

Trying to derive classical contradictions is useful, from a heuristic point of view, in all adaptive logics. It enables one to apply RD and hence to obtain *Dab*-formulas. The so obtained *Dab*-formulas are directly relevant to the proof at the stage. Indeed, all its disjuncts occur in conditions of at least one line, whence the derived *Dab*-formula will cause some lines to be marked.

While RD is a useful side-effect of Lemma 4.4.1, the lemma itself has a broader importance. It shows that adaptive proofs are lower limit proofs in disguise. This is an important insight. Adaptive proofs are half-way houses. They are extremely useful in that they allow one to push into the condition the abnormalities that are (classical) disjuncts of formulas. This enables one to formulate a handy marking definition for lines in view of their conditions and of the the minimal *Dab*-formulas that occur in the proof. Stating the marking definition in terms of the **LLL**-proofs would be less transparent and would lead to an awkward proof format. Indeed, one would still have to remember that the derivation of A from $A \check{\vee} Dab(\Delta)$ relies on the proviso that $Dab(\Delta)$ fulfils a certain condition. This condition heavily varies with the strategy. Yet, one

[9] Indeed, $A \check{\vee} Dab(\Delta), \check{\neg} A \check{\vee} Dab(\Theta) \vdash_{\mathbf{LLL}} Dab(\Delta \cup \Theta)$ holds in view of the superimposed classical symbols.

would have to note the proviso, in one way or other, in the justification of the line.

Apart from this direct relation between an **AL**-proof and a **LLL**-proof, there is a direct relation between an **AL**-proof and an **ULL**-proof. Deleting the conditions in an **AL**-proof results in an **ULL**-proof. Indeed, Definition 4.2.1 and Lemma 4.4.1 give us:

Lemma 4.4.2 *An* **AL**-*proof from* Γ *contains a line on which* A *is derived on the condition* Δ *iff* $\Gamma \vdash_{\mathbf{ULL}} A$.

Let us turn to the marking definitions. These require some preparation. We shall say that $Dab(\Delta)$ is a *minimal Dab-formula* at stage s of the proof iff it is the formula of a line that has $\emptyset$ as its condition and no $Dab(\Delta')$ with $\Delta' \subset \Delta$ is the formula of a line that has $\emptyset$ as its condition.

A *choice set* of $\Sigma = \{\Delta_1, \Delta_2, \ldots\}$ is a set that contains one element out of each member of Σ. A *minimal choice set* of Σ is a choice set of Σ of which no proper subset is a choice set of Σ. Where $Dab(\Delta_1)$, ..., $Dab(\Delta_n)$ are the minimal *Dab*-formulas at stage s, $U_s(\Gamma) = \Delta_1 \cup \ldots \cup \Delta_n$ and $\Phi_s(\Gamma)$ is the set of minimal choice sets of $\{\Delta_1, \ldots, \Delta_n\}$.

Definition 4.4.1 *Marking for Reliability: Line* i *is marked at stage* s *iff, where* Δ *is its condition,* $\Delta \cap U_s(\Gamma) \neq \emptyset$.

Intuitively, $U_s(\Gamma) = \Delta_1 \cup \ldots \cup \Delta_n$ is the set of all abnormalities that are unreliable with respect to Γ at stage s of the proof. Each of them is a disjunct of a minimal *Dab*-formula. As far as we know from the proof, the premises state that one of the disjuncts is true, but fail to specify which disjunct is true. Note that i is unmarked iff $\Delta \subseteq \Omega - U_s(\Gamma)$.

Definition 4.4.2 *Marking for Minimal Abnormality: Line* i *is marked at stage* s *iff, where* A *is derived on the condition* Δ *at line* i*, (i) no* $\varphi \in \Phi_s(\Gamma)$ *is such that* $\varphi \cap \Delta = \emptyset$*, or (ii) for some* $\varphi \in \Phi_s(\Gamma)$*, there is no line at which* A *is derived on a condition* Θ *for which* $\varphi \cap \Theta = \emptyset$.

As I said before, the following reads more easily: where A is derived on the condition Δ on line i, line i is *unmarked* at stage s iff (i) there is a $\varphi \in \Phi_s(\Gamma)$ for which $\Delta \subseteq \Omega - \varphi$ and (ii) for every $\varphi \in \Phi_s(\Gamma)$, there is a line at which A is derived on a condition Θ for which $\Theta \subseteq \Omega - \varphi$.

The idea behind the definition derives from the semantics—see Section 4.5. *If* the minimal *Dab*-formulas at stage s are indeed the minimal *Dab*-consequences of Γ—see again Section 4.5—then A *is* derivable iff it is true in every model of Γ that verifies one of the members of $\Phi_s(\Gamma)$.

Proofs can be made more effective (at the predicative level) by a slight modification. The idea is that, in the marking definitions, references to the conditions of lines are replaced by the Ω-closure of the conditions, where the Ω-closure of a set Σ is $Cn_{\mathbf{LLL}}(\Sigma) \cap \Omega$. Thus a line of a proof from Γ will be marked for Reliability if its condition comprises $\exists x(Px \wedge \neg Px)$ and $\exists y(Py \wedge \neg Py) \in U(\Gamma)$. An alternative is to write all *Dab*-formulas and conditions in some standard form, for example by letting them contain only alphabetically first variants of abnormalities, like $\exists x(Px \wedge \neg Px)$ and unlike $\exists y(Py \wedge \neg Py)$.

Marks may come and go. So the rules of inference combined with the marking definitions determine an unstable notion of derivability, viz. derivability at a stage:

Definition 4.4.3 *A is* derived *from* Γ at stage *s of the proof iff A is the formula of a line that is unmarked at stage s.*

We obviously also want a different, stable, kind of derivability: *final derivability.* Intuitively, A is finally derived at line i in an **AL**-proof from Γ iff A is the formula of line i, line i is unmarked, and the proof is stable with respect to line i. The latter phrase means that line i will not be marked in any extension of the proof. For some **AL**, Γ, and A, only an infinite proof from Γ in which A is the formula of a line i is stable with respect to line i. A simple example is the $\mathbf{CLuN}^r$-proof of p from $\{p \vee q, \neg q, (q \wedge \neg q) \vee (r_i \wedge \neg r_i), (q \wedge \neg q) \supset (r_i \wedge \neg r_i)\}_{(i \in \{0,1,\ldots\})}$. Only after $r_i \wedge \neg r_i$ was derived for all $i \in \mathbb{N}$ does the proof become stable.

Needless to say, the existence of an infinite proof is not established by producing the proof but by reasoning in the metalanguage. More importantly, it is more attractive to define final derivability in a different but extensionally equivalent way, as it was defined from the very beginning, viz. as follows.

Definition 4.4.4 *A is* finally derived *from* Γ *on line i of a proof at finite stage s iff (i) A is the second element of line i, (ii) line i is not marked at stage s, and (iii) every extension of the proof in which line i is marked may be further extended in such a way that line i is unmarked.*

Definition 4.4.5 $\Gamma \vdash_{\mathbf{AL}} A$ *(A is* finally **AL**-derivable *from* Γ*) iff A is finally derived on a line of a proof from* Γ*.*

These definitions are adequate in that they warrant the existence of an **AL**-proof that (i) contains an unmarked line l at which A is derived from Γ and (ii) is stable with respect to line l. That this is so is proven as Theorem 5.4.1 in the next chapter.

Definition 4.4.4 requires s to be finite. This means that it refers to a finite sequence of finite lists of lines. Still, the definition also refers to extensions of the stage, and these may be infinite. And indeed, if the strategy is Minimal Abnormality, the definition has to refer to infinite extensions in order to be adequate.[10] This will be illustrated by the premise set Γ_4 in the Many Turns dialogue in Subsection 4.9.2.

For Reliability, however, Definition 4.4.5 is still adequate if only finite extensions are considered in Definition 4.4.4. This will be proven as Theorem 5.4.3 in the next chapter. Whether the extensions are finite or infinite, clause (iii) of Definition 4.4.4 will have to be established by a reasoning in the metalanguage.

From now I shall use $\mathbf{AL}^r$ as a variable for adaptive logics that have Reliability as their strategy and $\mathbf{AL}^m$ as a variable for adaptive logics that have Minimal Abnormality as their strategy. So $\Gamma \vdash_{\mathbf{AL}^r} A$ (A is *finally* $\mathbf{AL}^r$*-derivable* from Γ) iff A is finally derived on a line of a proof from Γ in which lines are marked according to the Reliability strategy. Similarly, $\Gamma \vdash_{\mathbf{AL}^m} A$ (A is *finally* $\mathbf{AL}^m$*-derivable* from Γ) iff A is finally derived on a line of a proof from Γ in which lines are marked according to the Minimal Abnormality strategy.

An important insight, which was mentioned for a specific case in Chapter 2, is the following. While **ULL** extends **LLL** by validating some further *rules*, viz. S-rules, **AL** extends **LLL** by validating some *applications* of those rules.

[10] If the definition did not refer to infinite extensions, it would not be possible to show that $\Gamma \vdash_{\mathbf{AL}} A$ (i) guarantees that there is an **AL**-proof from Γ that is stable to a line at which A is derived and (ii) is entailed by $\Gamma \vDash_{\mathbf{AL}} A$ (completeness).

Before leaving the matter, let me add a more general comment. Adaptive logics are logics, viz. functions $\mathbf{AL}\colon \wp(\mathcal{W}) \to \wp(\mathcal{W})$. So the derivability relation is denoted by $\vdash_{\mathbf{AL}}$ and $Cn^{\mathcal{L}}_{\mathbf{AL}}(\Gamma)$ denotes the set $\{A \mid \Gamma \vdash_{\mathbf{AL}} A; A \in \mathcal{W}\}$. In Section 1.5, we have seen that there are two ways in which one may define that A is a theorem of **AL**, $\vdash_{\mathbf{AL}} A$: (i) $\emptyset \vdash_{\mathbf{AL}} A$ or (ii) $\Gamma \vdash_{\mathbf{AL}} A$ for all Γ. For adaptive logics (as for most logics that have dynamic proofs), the definitions are not coextensive. Thus, where **AL** is an adaptive logic in standard format, the first definition provides the theorems of the upper limit logic, $\emptyset \vdash_{\mathbf{AL}} A$ iff $\vdash_{\mathbf{ULL}} A$, whereas the second definition provides the theorems of the lower limit, $\Gamma \vdash_{\mathbf{AL}} A$ holds for all Γ iff $\vdash_{\mathbf{LLL}} A$.

4.5 Semantics

The adaptive semantics selects some **LLL**-models of Γ as **AL**-models *of* Γ. The selection depends on Ω and on the strategy. First we need some technicalities. Let $Dab(\Delta)$ be a *minimal Dab-consequence of* Γ iff $\Gamma \vDash_{\mathbf{LLL}} Dab(\Delta)$ and, for all $\Delta' \subset \Delta$, $\Gamma \nvDash_{\mathbf{LLL}} Dab(\Delta')$. Where $Dab(\Delta_1)$, $Dab(\Delta_2)$, ... are the minimal *Dab*-consequences of Γ, $U(\Gamma) = \Delta_1 \cup \Delta_2 \cup \ldots$. Where M is a **LLL**-model, $Ab(M) = \{A \in \Omega \mid M \Vdash A\}$.

Definition 4.5.1 *A* **LLL**-*model* M *of* Γ *is* reliable *iff* $Ab(M) \subseteq U(\Gamma)$.

Definition 4.5.2 $\Gamma \vDash_{\mathbf{AL}^r} A$ *iff* A *is verified by all reliable models of* Γ.

Minimal Abnormality is even simpler from a semantic point of view.

Definition 4.5.3 *A* **LLL**-*model* M *of* Γ *is* minimally abnormal *iff there is no* **LLL**-*model* M' *of* Γ *such that* $Ab(M') \subset Ab(M)$.

Definition 4.5.4 $\Gamma \vDash_{\mathbf{AL}^m} A$ *iff* A *is verified by all minimally abnormal models of* Γ.

Let $\mathcal{M}^{\mathbf{LLL}}_{\Gamma}$ be the set of **LLL**-models of Γ, $\mathcal{M}^{\mathbf{ULL}}_{\Gamma}$ the set of **ULL**-models of Γ, $\mathcal{M}^{m}_{\Gamma}$ the set of $\mathbf{AL}^m$-models (minimal abnormal models) of Γ, and $\mathcal{M}^{r}_{\Gamma}$ the set of $\mathbf{AL}^r$-models (reliable models) of Γ. If $M \in \mathcal{M}^{\mathbf{LLL}}_{\Gamma}$, the definitions warrant that $M \in \mathcal{M}^{r}_{\Gamma}$ iff M verifies no other abnormalities than those that are unreliable with respect to Γ; the definitions warrant that $M \in \mathcal{M}^{m}_{\Gamma}$ iff no other **LLL**-model of Γ is (set theoretically) less abnormal than M.

In Section 4.4 I introduced $\Phi_s(\Gamma)$. It is useful to introduce already here a closely related notion: $\Phi(\Gamma)$. Where $Dab(\Delta_1)$, ..., $Dab(\Delta_n)$ are the minimal *Dab-consequences* of Γ, $\Phi(\Gamma)$ is the set of minimal choice sets of $\{\Delta_1, \ldots, \Delta_n\}$—with $\Phi(\Gamma) = \{\emptyset\}$ if no *Dab*-formula is **LLL**-derivable from Γ. This notion is clarifying with respect to the proof theory as well as with respect to the semantics. If all minimal *Dab*-consequences of Γ would be derived at stage s of a proof, then $\Phi_s(\Gamma)$ would be identical to $\Phi(\Gamma)$—of course, for some Γ, the number of *Dab*-consequences of Γ may be infinite. So, for any stage s, $\Phi_s(\Gamma)$ may be seen as an estimate of $\Phi(\Gamma)$; the marks at stage s may be considered to provide a better approximation of final derivability to the extent that $\Phi_s(\Gamma)$ is a better estimate of $\Phi(\Gamma)$. Relevant for the semantics is that I shall prove, in the next chapter, that a **LLL**-model M of Γ is a minimally abnormal model of Γ iff $Ab(M) \in \Phi(\Gamma)$.

4.6 The Upper Limit Logic

Some properties of the upper limit logic are spelled out in this section. Let $\Omega^{\check{\neg}} =_{df} \{\check{\neg}A \mid A \in \Omega\}$. Definition 4.2.2 gives us at once:

Lemma 4.6.1 $\Gamma \vDash_{\mathbf{ULL}} A$ *iff* A *is verified by the* **LLL**-*models of* Γ *that verify no member of* Ω.

Moreover, Definition 4.2.1 and Lemma 4.6.1 give us:

Theorem 4.6.1 $\Gamma \vdash_{\mathbf{ULL}} A$ *iff* $\Gamma \vDash_{\mathbf{ULL}} A$. *(Soundness and Completeness of* **ULL**.*)*

Premise sets that do not require any abnormalities to be true, deserve a special name:

Definition 4.6.1 *A premise set* Γ *is* normal *with respect to an adaptive logic* **AL** *iff there is no* $Dab(\Delta)$ *for which* $\Gamma \vdash_{\mathbf{LLL}} Dab(\Delta)$. Γ *is abnormal iff it is not normal.*

Lemma 4.6.2 Γ *is normal with respect to an adaptive logic* **AL** *iff* Γ *has* **ULL**-*models.*

Proof. $\Rightarrow$ Suppose that Γ is normal with respect to logic **AL**. It follows that $\Gamma \cup \Omega^{\check{\neg}}$ is $\check{\neg}$-consistent and hence has **LLL**-models each of which falsify all members of Ω. But then Γ has **ULL**-models.

$\Leftarrow$ Suppose that Γ has **ULL**-models. As none of these models verifies a member of Ω, there is no $Dab(\Delta)$ for which $\Gamma \vDash_{\mathbf{LLL}} Dab(\Delta)$. By the soundness and completeness of **LLL** with respect to its semantics, there is no $Dab(\Delta)$ for which $\Gamma \vdash_{\mathbf{LLL}} Dab(\Delta)$. So Γ is normal with respect to **AL** in view of Definition 4.6.1. ■

The following theorem is extremely important. It shows a way in which the set of abnormalities connects the upper limit logic to the lower limit logic. This theorem is the 'motor' for the adaptive logic. By applying **AL**, we try to get as close to **ULL** as possible—close with respect to Ω that is. Theorem 4.6.2 informs us that, for every **ULL**-consequence of Γ, there is (at least one) $\Delta \subseteq \Omega$ for which $A \check{\vee} Dab(\Delta)$ is a **LLL**-consequence of Γ. If Γ allows one to consider all members of Δ as false—the precise meaning of this depends on the strategy—then A is also an **AL**-consequence of Γ.

Theorem 4.6.2 $\Gamma \vdash_{\mathbf{ULL}} A$ *iff there is a finite* $\Delta \subseteq \Omega$ *such that* $\Gamma \vdash_{\mathbf{LLL}} A \check{\vee} Dab(\Delta)$. *(Derivability Adjustment Theorem)*

Proof. The following five statements are equivalent—the first follows from the last by Theorem 4.6.1 and Lemma 4.6.1:

$$\Gamma \vdash_{\mathbf{ULL}} A$$

In view of Definition 4.2.1:

$$\Gamma \cup \Omega^{\check{\neg}} \vdash_{\mathbf{LLL}} A$$

As **LLL** is compact:

$$\Gamma' \cup \Delta^{\check{\neg}} \vdash_{\mathbf{LLL}} A \text{ for a finite } \Gamma' \subseteq \Gamma \text{ and a finite } \Delta \subseteq \Omega$$

As **LLL** contains **CL**:

$$\Gamma' \vdash_{\mathbf{LLL}} A \check{\vee} Dab(\Delta) \text{ for those } \Gamma' \text{ and } \Delta$$

As **LLL** is monotonic:

$$\Gamma \vdash_{\mathbf{LLL}} A \check{\vee} Dab(\Delta) \text{ for a finite } \Delta \subseteq \Omega\,.$$

■

Definition 4.2.1 gives us:

Theorem 4.6.3 **ULL** *has static proofs.*

Corollary 4.6.1 **ULL** *is Reflexive, Transitive, Monotonic, Uniform and Compact, and there is a positive test for it.*

Remember that we extended **LLL** so as to contain **CL** in $\mathcal{L}_S$. So **ULL** also contains **CL** in $\mathcal{L}_S$.

The upper limit logic is axiomatized by adding to **LLL** the axiom schema $\check{\neg}\mathsf{F}$, or an equivalent one, with the restriction that pertains to the logical form characterizing Ω. Often adding the restriction to the axiom is superfluous. Indeed, for most sets of abnormalities characterized by a restricted form F, it holds that, for every A of the form F there is a finite $\Delta \in \Omega$ for which $A \vdash_{\mathbf{LLL}} Dab(\Delta)$. In this case (and in some others as well) no **ULL**-model verifies any formula of the unrestricted form F, even if a restriction occurs in the definition of Ω. It also is not very elegant that classical negation occurs in the axiom that brings one from **LLL** to **ULL**. Often there are equivalent axioms that belong to $\mathcal{W}$. Thus adding $(A \wedge \neg A) \supset B$ to **CLuN** gives one **CL**.

4.7 Dynamic Proofs

The aim of the present section is to provide the theoretical backing for dynamic proofs. The central task is to control the dynamics—this is the unusual feature. I shall describe dynamic proofs in such a way that they minimally depart from static proofs and I shall make sure that the points at which they depart are transparent. In the present section, dynamic proofs are described in general. The specificity of adaptive proofs is the topic of the next section.

The central elements of annotated dynamic proofs are rules, lines, lists of lines, and a marking definition. The central differences with static proofs are that lines comprise a condition and the presence of the marking definition. As the rules of dynamic proofs are more general than those of static proofs, I shall simply call them rules. A *rule* is a metalinguistic expression of the form $\Upsilon/\mathsf{A}{:}\Pi$, read as "to derive A on the condition Π from Υ", in which A is a metalinguistic formula and Υ and Π are recursive sets of metalinguistic formulas. A rule specifies that from formulas of a certain form another formula of a corresponding form may be derived on a condition, which is a set of formulas of a further form. A rule is *finitary* iff Υ is a finite set.[11]

As for static proofs, I shall suppose that a premise rule is always present. However, I shall not require in the present general setting, that the premise rule enables one to introduce premises on an empty condition. This requires some clarification. The S-rule Prem from Section 1.5, may be adjusted to a *rule* and the most literal way to do so is to phrase it as "If $A \in \Gamma$, then $\emptyset/A{:}\emptyset$." Let us use the name Prem for this *rule* (as well as for the S-rule of static proofs). As said in the previous paragraph, I shall also allow for other premise rules, which require that premises are introduced in the proof on a condition. We shall meet such rules when we come, in Section 9.9 to the 'direct proof theory' for Rescher–Manor logics from Section 9.7, a premise is introduced on the condition that it is consistent. Such direct proofs are dynamic proofs. The underlying logic is not adaptive, however, but it can be characterized by an adaptive logic under a translation.

A *line* of a dynamic annotated proof will be a quadruple comprising a line number, a formula, a justification, and a condition. The first three elements are as for static proofs, except that the justification now contains a rule instead of a S-rule. As before R_l will denote the rule applied to add the line, and N_l is the set of lines to which the rule is applied. The condition is a (finite) set of formulas.

The application of a rule deserves some attention. The idea is that inference carries over conditions. If (formulas of the form of) all members of Υ are the formulas of lines in a list, and Π' is the union of the conditions of those lines, then the application of the rule $\Upsilon/\mathsf{A}{:}\Pi$ leads to adding a line that has A as its formula and $\Pi \cup \Pi'$ as its condition.

As for S-rules, a *restriction* may be attached to a rule, provided that it can be decided whether the restriction is fulfilled by inspecting the list of lines to which the application of the rule belongs.

In this section, $\mathcal{R}$ will denote a set of rules (not of S-rules unless otherwise specified) and "line" will mean a line of a dynamic annotated proof. Given a set $\mathcal{R}$ of rules and a list L of lines, a line l of L is $\mathcal{R}$-*correct* iff (i) it is the result of the application of the rule $R_l \in \mathcal{R}$ to the formulas and conditions of the members of N_l and (ii) all members of N_l precede l in the list.

Definition 4.7.1 *A* marking definition *determines, for every stage of a dynamic proof, whether a line i of the stage is marked or unmarked. The definition proceeds in terms of a requirement that connects the formula-condition couple of line i to the set of formula-condition couples of the other lines of the stage.* NOG LOSSER?

Marks may come and go, as we have seen in previous chapters. The operation of the marking definition is very different from that of the rules. The person constructing the proof decides which rule is applied at a certain point and hence which line is added to the previous stage. The marking definition determines all by itself which lines are marked and which are unmarked at a stage of the proof. Note that rules may be applied to unmarked lines as well as to marked

[11] For all logics we shall consider, the condition will also be a finite set. This, however, is not required for the rule to be finitary. A finitary rule is one that may be applied at a finite point in a proof.

ones.[12]

Let us proceed to logics that have dynamic proofs. I shall make a possibly unexpected move in this connection. In view of the previous paragraph, it is possible to consider the effect of the rules independently of the effect of the marking definition. This means that one can consider the lists of lines that follow each other as a result of the application of rules without taking any account of the marks. A chain of such lists differs only from the static proofs from Section 1.5 in that the lines have a condition. Apart from that, however, the similarity is striking. This is why I shall introduce *static proofs* determined by a set of *rules* $\mathcal{R}$ (and not by the marking definition). Note, however, that these static proofs are not proofs of formulas but of formulas on a condition. After having defined static $\mathcal{R}$-proofs, I shall move on to dynamic $\mathcal{R}$-proofs properly by considering the effects of the marking definition. I first literally repeat Definitions 1.5.1–1.5.3, which now refer to the modified definition of a $\mathcal{R}$-correct line.

Definition 4.7.2 *A $\mathcal{R}$-stage from (the premise set) Γ is a list of $\mathcal{R}$-correct lines.*

Definition 4.7.3 *Where L and L' are $\mathcal{R}$-stages from Γ, L' is an* extension *of L iff all elements that occur in L occur in the same order in L'.*

Definition 4.7.4 *A static $\mathcal{R}$-*proof from Γ *is a chain of $\mathcal{R}$-stages from Γ, the first element of which is the empty list and all other elements of which are extensions of their predecessors.*

Corresponding to Definitions 1.5.4 and 1.5.5 are the two following ones. The only difference lies in the reference to the condition.

Definition 4.7.5 *A static $\mathcal{R}$-*proof of $A{:}\Delta$ from Γ *is a static $\mathcal{R}$-proof from Γ in which, from a certain stage on, there is a line that has A as its formula and Δ as its condition.*

In view of Definition 4.7.4, Definition 4.7.5 comes to: a static $\mathcal{R}$-*proof of* $A{:}\Delta$ *from* Γ is a proof from Γ in which a line of a stage has A as its formula and Δ as its condition.

Definition 4.7.6 $\Gamma \vdash_{\mathcal{R}} A{:}\Delta$ ($A{:}\Delta$ is $\mathcal{R}$-derivable from Γ) *iff there is a static $\mathcal{R}$-proof of $A{:}\Delta$ from Γ.*

Just like S-rules, nearly every rule $\Upsilon/\mathsf{A}{:}\Pi$ has applications to sets of formulas with a lower cardinality than that of Υ. Often, Π is then also reduced to a set with a lower cardinality. In this sense every infinitary rule R generates a recursive set of finitary rules, say $\text{fin}(R)$. The proof of the following theorem is wholly analogous to the proof of Theorem 1.5.1.

Theorem 4.7.1 *If $\mathcal{R}$ is a recursive set of rules, then there is a recursive set $\mathcal{R}'$ of* finitary *rules such that $\Gamma \vdash_{\mathcal{R}'} A{:}\Delta$ iff $\Gamma \vdash_{\mathcal{R}} A{:}\Delta$.*

Let M refer to a marking definition. By a $\mathcal{R}$-M-proof (from some Γ), I shall mean a static $\mathcal{R}$-proof (from Γ) to which the marking definition M was applied.

[12]If stage s is obtained by adding line l as the result of applying rule R_l to lines N_l, and some of these lines are marked, then line l will nearly always be marked at stage s. Still it may be sensible to add line l in view of its being unmarked at a later stage.

Definition 4.7.7 *A is* $\mathcal{R}$-M-derived from Γ at a stage s *iff, for some* Δ, s *is a stage of a* $\mathcal{R}$-M*-proof from* Γ *and* A *is the formula of an unmarked line of* s.

Definition 4.7.8 *A* $\mathcal{R}$-M*-proof from* Γ *is* stable with respect to line i *from a stage* s *on iff (i) line* i *occurs in* s *and (ii) if line* i *is marked, respectively unmarked, at stage* s*, then it is marked, respectively unmarked, in all extensions of* s.

Definition 4.7.9 $\Gamma \vdash^M_{\mathcal{R}} A$ *(A is* $\mathcal{R}$-M-derivable from Γ*) iff* A *is the formula of an unmarked line* i *of a stage of an* $\mathcal{R}$-M*-proof from* Γ *and the proof is stable with respect to line* i.

Definition 4.7.10 *A logic* **L** *is* defined by *a recursive set* $\mathcal{R}$ *of rules and a marking definition iff* $\Gamma \vdash_{\mathbf{L}} A$ *iff* $\Gamma \vdash^M_{\mathcal{R}} A$.

Definition 4.7.11 *A logic* **L** has dynamic proofs *iff it is defined by a recursive set* $\mathcal{R}$ *of rules and a marking definition, and has no static proofs.*

verborgen: stijl static pr.

The proof of the following theorem is wholly similar to the proof of Theorem 1.5.7.

Theorem 4.7.2 *If* **L** *is defined by a recursive set* $\mathcal{R}$ *of rules and a marking definition, and* $\Gamma \vdash_{\mathbf{L}} A{:}\Delta$*, then there is a static* $\mathcal{R}$*-proof of* $A{:}\Delta$ *from* Γ *in which* A *is the formula and* Δ *the condition of the* last *line of the* last *stage. (Finiteness of Conditional Derivation)*

Definition 4.7.12 *Where* $\mathbf{L}_1$ *has dynamic proofs and* $\mathbf{L}_2$ *has static proofs,* $\mathbf{L}_1$ S-agrees with $\mathbf{L}_2$ *iff there is a function* $f\colon \mathcal{W}_+ \times \wp(\mathcal{W}_+) \to \mathcal{W}_+$ *such that* $\Gamma \vdash_{\mathbf{L}_1} A{:}\Delta$ *iff* $\Gamma \vdash_{\mathbf{L}_2} f(A, \Delta)$.

The "S-agrees" abbreviates "statically agrees": with respect to its static proofs, $\mathbf{L}_1$ corresponds to $\mathbf{L}_2$ except that some formulas are written in different places of the line. The definition refers to $\mathcal{W}_+$ in view of the subsequent paragraph. For some adaptive logics, actually for most inconsistency-adaptive logics, referring to $\mathcal{W}$ is sufficient and dialetheists will consider this as the only viable alternative—see also Section 4.9.3. Purists will want to relativize S-agreement to a certain language.

Lemma 4.4.1 gives us the following corollary. Indeed, let $f(A, \Delta) = A \check{\vee} Dab(\Delta)$—remember that $A \check{\vee} Dab(\emptyset)$ is A—and consider an **AL**-proof from Γ. Replacing every line at which A is derived on the condition Δ by $f(A, \Delta)$ (and fixing the justification) results in a **LLL**-proof from Γ.

Corollary 4.7.1 *Every adaptive logic* **AL** *S-agrees with its lower limit logic* **LLL**.

It seems wise to comment again on premise rules at this point. In the third 3rd OK?
paragraph of this section, I allowed for premise rules that introduce premises on a non-empty condition. If $\mathbf{L}_1$ has dynamic proofs and Prem is not one of its rules (but a conditional premise rule is), then it is unlikely that $\mathbf{L}_1$ will S-agree with a logic $\mathbf{L}_2$ that has static proofs. The reason for this is that $\mathbf{L}_2$ necessarily has the S-rule Prem. So it is unlikely that a function f will do the job required by Definition 4.7.12. I shall return to the point in Section 9.9. In the case

of adaptive logics, the rule Prem functions as the natural counterpart for the S-rule Prem.

Dynamic proofs were introduced in a very general way in the present section. In the next section, I comment on the specific properties of the dynamic proofs of adaptive logics. The present section is important, however, because we shall need the general notion of dynamic proofs in Section 9.9 where 'direct proofs' are discussed. Still, the general approach to dynamic proofs is not very clarifying on the essential point, which is the control of the dynamics of the proofs. This will change drastically in the next section.

4.8 Adaptive Dynamic Proofs . . .

The dynamic proofs of adaptive logics are special cases of the dynamic proofs described in the previous section. As a result, they have a set of nice properties. I briefly review these features in this section and in the next. Part of these features derive from the presence of the rule Prem. However, two further specifications are equally important.

A first and essential specification concerns the marking definition. In the case of adaptive logics, this proceeds in terms of the condition of the considered line, (for Minimal Abnormality) the other conditions on which the formula of the line has been derived at the stage, and the minimal *Dab*-formulas derived at the stage. The definition of a minimal *Dab*-formula involves that the formula was derived on the empty condition. This means that the minimal *Dab*-formulas are derived by the lower limit logic, which forms a warrant against circularity.

The second specification concerns the conditions and their meaning. The conditions are finite sets—this is itself a fundamental difference with the general approach described in the previous section. Moreover, that A:Δ is derived expresses that A is true if all members of Δ are false, in other words, that the disjunction of A and the members of Δ is derivable. So, as follows from the proof of Lemma 4.4.1, every line of an adaptive proof can be rewritten as a line of the **LLL**-proof, replacing A:Δ by $A \check{\vee} Dab(\Delta)$. So every adaptive logic **AL** S-agrees with its lower limit logic and its static proofs are **LLL**-proofs in disguise.

This is not a minor point. Adaptive consequence sets have a very high degree of complexity, which makes it understandable that many centuries of logic did not contribute anything to defeasible reasoning. So it is extremely important that the proofs are *simple*, viz. static proofs in disguise, that the marking definition is decidable, and that the criteria for final derivability—see Section 10.2—offer a simple heuristic means, notwithstanding the absence of a positive test. The absence of a positive test forces one to act, more often than not, on the basis of the incomplete analysis provided by a proof at a stage. Final derivability is in general far too complex for humans to reach, but derivability at a stage should be as simple as possible—a point first clearly made in [Ver09].

The two specifications are responsible for the simplicity of the proofs. Moreover, and as promised, they guarantee a maximal control of the dynamics of the proofs.

The two specifications have also a different effect. The general Definition 4.7.10 of final derivability refers essentially to a proof that is stable with respect to a line at which the conclusion was derived. As I already remarked in

Section 4.4, there are Γ and A for which only infinite proofs from Γ are stable with respect to a line at which A was derived (on a condition). By the two specifications, it is possible to show that Definitions 4.7.10 and 4.4.4 agree—see Lemma 5.4.1 in the next chapter. This means that final derivability can be defined with reference to a finite proof. Obviously, one still needs a reasoning in the metalanguage about all possible extensions of the proof, but the proof itself can be written down.

Allow me to mention another consequence of the specifications: static proofs are limit cases of the dynamic proofs of adaptive logics. So this means that dynamic proofs are natural generalizations of static proofs. Actually, logics that have static proofs may be obtained from logics that have dynamic proofs in four different ways. First way: a logic having static proofs is obtained by removing the conditional rule RC. As a result, all lines have an empty condition, no line will ever be marked, and no dynamics occurs. The resulting logic is **LLL**. Second way: a logic having static proofs is obtained by defining the set of abnormalities in such a way that the conditional rule RC never applies. An example is where $\Omega = \emptyset$. So no line will ever be marked and no dynamics occurs. In this case the resulting static logic is again **LLL**.

There is a third way: a logic having static proofs is obtained by adopting the Blindness strategy. This strategy will be described in Section 6.1.2. It is extremely simple. It comes to: never mark any line. The result is obviously **ULL**. Fourth way: a logic having static proofs is obtained by replacing the conditional rule RC by a variant that, instead of adding Δ to the condition, adds $\emptyset$. This comes to presupposing that abnormalities are false and may be be disregarded. As a result, all conditions are again empty, all dynamics is removed, and the resulting logic is **ULL**.

4.9 . . . and Dialogues

One of the advantages of Definition 4.4.4 is that it has attractive game-theoretic or dialogic interpretations. These will form the topic of the rest of this section. Illustrating the dialogues will enable me to present some more exotic features of adaptive logics.

The dialogues I want to propose are somewhat unusual. In the most usual dialogues a winning strategy for the proponent leads to the proof of a theorem. However, adaptive logics have no theorems of their own. If theorems are defined by $\emptyset \vdash A$, then the theorems of the adaptive logic, for example $\mathbf{CLuN}^r$, are identical to those of its upper limit logic, in the example **CL**. If theorems are defined by "for all Γ, $\Gamma \vdash A$", then the theorems of the adaptive logic are identical to those of its lower limit logic, in the example **CLuN**—obviously all theorems of the lower limit logic are theorems of the upper limit logic. It follows that one cannot define the adaptive consequence relation in terms of theorems, but that dialogues for the consequence relation should be devised. So one will have to adjust the description of a dialogue from, for example, [Rahar] or [RK05], and there will be a few peculiarities. It is not difficult to define such dialogues, for example for $\mathbf{CLuN}^r$ and $\mathbf{CLuN}^m$ (after first devising them for **CLuN**). Tableau methods presented in [BM00b] and [BM01b] form a good start. The tableau methods may even be simplified by extending the language with classical negation, whence there is no need for signed formulas.

to which I return briefly in Section CONCLUSION

Let us now turn to the unusual dialogues I announced. The proponent claims that $\Gamma \vdash_{\mathbf{AL}} A$ and the opponent denies this. We let the proponent and opponent construct a proof together, giving each a specific task. The proponent starts. She has to produce an adaptive proof from Γ that contains a line l at which A is derived. If, at the end of the dialogue, l is unmarked, the proponent wins; otherwise the opponent wins.

This kind of dialogue is completely silly if the logic has static proofs. Indeed, the opponent has no specific role to play: no contribution to the proof forms a means to attack the derivability of the conclusion from the premises. The situation is dramatically different for logics that have dynamic proofs. If the conclusion A is not derivable from the premises Γ by the lower limit logic, then the proponent can only derive it on a non-empty condition. We have seen that the resulting proof does not constitute a demonstration of $\Gamma \vdash_{\mathbf{AL}} A$. Actually, no proof forms such a demonstration. So it seems natural to construct a demonstration of $\Gamma \vdash_{\mathbf{AL}} A$ as a dialogue between a proponent, who tries to show that A is finally derivable but has to defend herself against moves of the opponent. Let me first comment on the natural character of the approach.

First a comparison. Every logician is acquainted with the situation in which he or she tries to find out whether a formal system has a certain property. If one is convinced that the property holds, one will attempt to prove so. If one does not find the proof, this very fact will undermine the conviction. At some point one will become convinced that the property does not hold and one will try to find a counterexample—often insights from the failing proof will indicate in which direction to look for a counterexample. If, in turn, one fails to produce a counterexample, this may induce one to look again for a proof, etc. The alternating phases may be seen as a dialogue between a proponent and an opponent.

Let us now look more closely at adaptive logics. The idea is that abnormalities are presupposed to be false unless and until proven otherwise. So two different aims should be realized in a well-directed proof: to establish the conclusion on some condition and to establish that the condition is safe—in the case of Reliability, this means that no member of the condition is unreliable; in the case of Minimal Abnormality, it means that the condition does not overlap with a minimal choice set of all *Dab*-consequences of the premises *and* that, for each such minimal choice set φ, the conclusion can be derived on a condition that does not overlap with φ. So it is indeed natural to see this as a dialogue in which the proponent first establishes the conclusion on some condition, next the opponent tries to show that the condition is unsafe, next the proponent tries to reestablish the safety of the condition, and so on. Several variant dialogues are possible, even for the same strategy. They will be considered in some detail below.

Although no dynamic proof will establish that a conclusion is finally derived from a premise set, the metalevel reasoning that is required next to the proof can be seen in dialogic terms: the conclusion is finally derivable iff the proponent can uphold it against every possible attack.

It seems to me that this is at the heart of all forms of defeasible reasoning: that one establishes a conclusion on some condition and that the condition can be maintained in the face of every possible attack.

Given the differences between the two strategies, I shall first consider the dialogues for one strategy, and next for the other. Four dialogues will be presented and they have the same names for both strategies.

4.9.1 Reliability

Stability with respect to a line In this type of dialogue, the proponent first establishes the conclusion on some condition on an unmarked line, say line l, of a (finite or infinite) proof. Next, the opponent may extend the proof. The opponent wins if he produces an extension in which line l is marked; otherwise the proponent wins. The proponent has a winning strategy iff she can produce a proof that warrants her winning.

This approach is all right, but requires that the proponent sometimes starts off with an infinite proof—this can only mean that she starts off describing such a proof. I shall show this by way of an example. Consider the premise set $\Gamma_1 = \{p \vee q, {\sim}q, (q \wedge \neg q) \vee (r_i \wedge \neg r_i), (q \wedge \neg q) \supset (r_i \wedge \neg r_i) \mid i \in \mathbb{N}\}$ and let the proponent aim at establishing $\Gamma_1 \vdash_{\mathbf{CLuN}^r} p$. Consider a finite proof, produced by the proponent, that starts off with

1	$p \vee q$	Prem	$\emptyset$
2	${\sim}q$	Prem	$\emptyset$
3	p	1, 2; RC	$\{q \wedge \neg q\}$

and moreover contains a (forcibly finite) number of triples of lines of the following form

j	$(q \wedge \neg q) \vee (r_i \wedge \neg r_i)$	Prem	$\emptyset$
$j+1$	$(q \wedge \neg q) \supset (r_i \wedge \neg r_i)$	Prem	$\emptyset$
$j+2$	$r_i \wedge \neg r_i$	j, j+1; RU	$\emptyset$

Clearly, line 3 is unmarked in this proof. However, if the opponent extends the proof with the lines

k	$(q \wedge \neg q) \vee (r_l \wedge \neg r_l)$	Prem	$\emptyset$
$k+1$	$(q \wedge \neg q) \check{\vee} (r_l \wedge \neg r_l)$	k; RU	$\emptyset$

for a r_l that does not yet occur in the proof, then line 3 is marked.[13] So the proponent loses. Of course, she should have a winning strategy, because $\Gamma_1 \vdash_{\mathbf{CLuN}^r} p$. And indeed there is one, but it requires that the proponent introduces all premises and all connected lines $j+1$ and $j+2$, which means that she should produce an infinite proof in her first move. This is not handy. Infinite proofs cannot be produced, but should be handled by a metalevel reasoning. It would be more attractive if at least the first move in the dialogue would be a proof that can actually be produced. Also, not much dialogue is involved in this kind of game. The outcome fully depends on the first move of the proponent. She has a winning strategy iff she is able to produce, as her first step, a proof that is stable with respect to an unmarked line at which the conclusion is derived.

Incidentally, some readers might balk at the artificiality of the premise set Γ_1. It is indeed hard to imagine real life applications in which the depicted complication would arise. Nevertheless, in describing logics, one should consider all possible complications, whether they are artificial or not.

[13] Line 3 is marked at stage $k+1$ of the proof, not at stage k. This follows from the convention that "*Dab*-formula" strictly refers to a *classical* disjunction of abnormalities. Showing that the formula of line k is equivalent to a *Dab*-formula requires a deductive step, which here is taken by the opponent.

Many turns This kind of dialogue is definitely more fascinating than the previous one. In her first move, the proponent produces a finite proof that contains an unmarked line, say line l, in which the conclusion is derived on a condition Δ. Next, the opponent tries to show that Δ is unreliable by producing a finite extension of the proof. If the opponent's move is successful, line l is marked at the last stage of the extended proof. The proponent reacts by trying to finitely extend the proof in such a way that line l is unmarked. And so on. The proponent has a winning strategy iff she is able to produce the required initial proof and to answer every move of the opponent, viz. iff she is able to extend every new extension in such a way that line l is unmarked.

To illustrate the matter, consider again Γ_1 from the previous dialogue and let the logic be $\mathbf{CLuN}^r$. Suppose that the proponent starts as follows:

1	$p \vee q$	Prem	$\emptyset$
2	$\sim q$	Prem	$\emptyset$
3	p	1, 2; RC	$\{q \wedge \neg q\}$

The opponent may reply, for example, by the following extension:

4	$(q \wedge \neg q) \vee (r_1 \wedge \neg r_1)$	Prem	$\emptyset$
5	$(q \wedge \neg q)\, \check{\vee}\, (r_1 \wedge \neg r_1)$	4; RU	$\emptyset$

Note that line 3 is marked at stage 5 of the proof. The proponent will answer by extending the proof as follows:

6	$(q \wedge \neg q) \supset (r_1 \wedge \neg r_1)$	Prem	$\emptyset$
7	$r_1 \wedge \neg r_1$	6, 4; RU	$\emptyset$

As $U_7(\Gamma_1) = \{r_1 \wedge \neg r_1\}$, line 3 is unmarked at stage 7 of the proof. Of course, the opponent may still attack, and will perhaps attack more forcibly:

8	$(q \wedge \neg q) \vee (r_2 \wedge \neg r_2)$	Prem	$\emptyset$
9	$(q \wedge \neg q)\, \check{\vee}\, (r_2 \wedge \neg r_2)$	8; RU	$\emptyset$
$\vdots$			
24	$(q \wedge \neg q) \vee (r_{10} \wedge \neg r_{10})$	Prem	$\emptyset$
25	$(q \wedge \neg q)\, \check{\vee}\, (r_{10} \wedge \neg r_{10})$	24; RU	$\emptyset$

Line 3 is now again marked, but the proponent will reply by

26	$(q \wedge \neg q) \supset (r_2 \wedge \neg r_2)$	Prem	$\emptyset$
27	$r_2 \wedge \neg r_2$	26, 8; RU	$\emptyset$
$\vdots$			
42	$(q \wedge \neg q) \supset (r_{10} \wedge \neg r_{10})$	Prem	$\emptyset$
43	$r_{10} \wedge \neg r_{10}$	42, 24; RU	$\emptyset$

after which line 3 is unmarked. So the proponent defended herself adequately.

The premise set is a dull one, but it candidly illustrates that, in some cases, every attack can be answered successfully, whereas a successful attack is possible after any finite number of defenses—actually infinitely many different attacks are possible at any finite point.

Several comments are appropriate. First, the proponent is able to answer every move of the opponent in the example. So the proponent has a winning strategy in the dialogue for $\Gamma_1 \vdash_{\mathbf{CLuN}^r} p$. Next, only after an infinite sequence of attacks and defenses will the opponent's means to attack have been exhausted. Finally, one may introduce a convention to terminate the dialogue after finitely many turns, provided the proponent is allowed to make the final move. The proponent and opponent may make a deal about this either beforehand or during the dialogue, the choice may be with one of them, or whatever one likes. Especially if the premise set is less orderly and hence more interesting, it seems attractive to let the dialogue go on until the opponent gives up. This will enable the opponent to try out different lines of attack in view of the premises. All that is essential is that the proponent is given a defense after every attack.

POP The name of this dialogue abbreviates proponent-opponent-proponent. The proponent starts by producing a finite proof in which the conclusion is derived on a condition Δ on an unmarked line, say line l. Next, the opponent tries to show that Δ is unreliable by producing a finite extension. If the opponent is successful, line l is marked at the last stage of the extended proof. The proponent reacts by trying to finitely extend the proof in such a way that line l is unmarked. The dialogue is the optimized simplification of the many turns dialogue type—one attack and one reply. The proponent wins the dialogue if line l is unmarked after her reaction. The proponent has a winning strategy iff she is able to win, whatever be the opponent's attack.

Calling premises This is a type of dialogue in which the opponent makes the last move. The proponent starts by producing a finite proof that contains an unmarked line, say l, in which the conclusion is derived from the premises on a condition Δ. At this point, the opponent does not extend the proof, but chooses a finite set Γ' of premises, which he will use in his attack. Next, the proponent may finitely extend her proof, introducing whatever premises she wants. Finally, the opponent attacks, viz. extends the extension, introducing as premises *only* members of Γ' and deriving *only* formulas from the premises he himself introduced. Typical for this dialogue type is that the proponent should foresee which attacks the opponent may produce from the called premises.

The proponent wins the dialogue if line l remains unmarked after the opponent's attack; otherwise the opponent wins. The proponent has a winning strategy iff she can proceed in such a way that she wins the dialogue. This means that she is able to produce a proof in which the conclusion is derived, say at line l, and that, whatever finite $\Gamma' \subseteq \Gamma$ the opponent chooses, she can produce an extension of her proof that the opponent cannot extend in such a way that line l is marked, provided that the opponent only derives formulas that follow from Γ'. This type of dialogue illustrates that the proponent can defend herself by finitely many moves against all *Dab*-formulas that are derivable from any finite $\Gamma' \subseteq \Gamma$ *just in case* the conclusion is finally derivable from Γ.

More dialogue types may be possible, but those described before are sufficient to make the point I was trying to make. I still have to prove that the dialogues are adequate. Let a dialogue of each of these types be called a dialogue for $\Gamma \vdash_{\mathbf{AL}^r} A$, in which Γ is the premise set and A is the conclusion. The proof of the following theorem is rather simple because the dialogues 'interpret' definitions given before in view of theorems proved in the next chapter.[14]

[14]It is not very orthodox to refer to 'future' theorems, but it would be too awkward to

Theorem 4.9.1 *For the four dialogue types described holds: the proponent has a winning strategy in the dialogue for* $\Gamma \vdash_{\mathbf{AL}^r} A$ *iff* $\Gamma \vdash_{\mathbf{AL}^r} A$.

Proof. Let us start with *POP*. $\Rightarrow$ Suppose that the proponent derives A on the condition Δ, say at line l, and that $\Gamma \nvdash_{\mathbf{AL}^r} A$. In view of Theorem 5.3.1, it follows that there is a minimal *Dab*-consequence of Γ, say $Dab(\Theta)$, for which $\Delta \cap \Theta \neq \emptyset$. By the compactness of **LLL**, $Dab(\Theta)$ is **LLL**-derivable from a finite $\Gamma' \subseteq \Gamma$. So the opponent will introduce the members of Γ' and derive $Dab(\Theta)$, at which point line l is marked. As there is no way to extend the proof in such a way that line l is unmarked, the proponent has no winning strategy. $\Leftarrow$ Obvious in view of Definitions 4.4.4 and 4.4.5 and Theorem 5.4.3.

Many turns. $\Rightarrow$ As for POP, except that it is sufficient for the opponent to derive $Dab(\Theta)$ in one of his attacks. $\Leftarrow$ In view of Definitions 4.4.4 and 4.4.5 and Theorem 5.4.3, the proponent has a successful reply after every attack.

Stability with respect to a line. This is an obvious consequence of the proof for POP in view of Theorem 5.4.1.

Calling premises. $\Rightarrow$ As for POP, except that the opponent calls the members of Γ'. $\Leftarrow$ Suppose that $\Gamma \vdash_{\mathbf{AL}^r} A$. In view of Theorem 5.3.1, there is a finite $\Delta \subset \Omega$ for which $\Gamma \vdash_{\mathbf{LLL}} A \check{\vee} Dab(\Delta)$ and $\Delta \cap U(\Gamma) = \emptyset$. So the proponent derives A on the condition Δ, say on line l. Where $\Gamma' \subseteq \Gamma$ is a finite set, let $Dab(\Theta_1), \ldots, Dab(\Theta_n)$ be the minimal *Dab*-consequences of Γ'—every finite Γ' has finitely many minimal *Dab*-consequences. As $\Delta \cap U(\Gamma) = \emptyset$, there is, for every Θ_i $(1 \leq i \leq n)$ for which $\Theta_i \cap \Delta \neq \emptyset$, a $\Theta'_i \subset \Theta_i$ for which $Dab(\Theta'_i)$ is a minimal *Dab*-consequence of Γ and $\Delta \cap \Theta'_i = \emptyset$. So it is sufficient that the proponent derives these finitely many $Dab(\Theta'_i)$—this requires only a finite proof in view of the compactness of **LLL**—in order to warrant that line l will not be marked if the proof is extended by consequences of Γ'. ■

To prove the adequacy of the Calling Premises dialogue, it is essential that the opponent cannot rely in his extension on premises introduced by the proponent. This is related to the fact that most adaptive logics are not compact. The following example illustrates the lack of compactness of $\mathbf{CLuN}^r$. Let $\Gamma_2 = \{((p \vee q) \wedge \neg q) \wedge ((q \wedge \neg q) \vee (r_1 \wedge \neg r_1))\} \cup \{((q \wedge \neg q) \supset (r_i \wedge \neg r_i)) \wedge ((q \wedge \neg q) \vee (r_{i+1} \wedge \neg r_{i+1})) \mid i \in \mathbb{N}\}$. All **CLuN**-models of Γ_2 verify $r_i \wedge \neg r_i$ for all $i \in \mathbb{N}$, and some verify no other abnormality. So $U(\Gamma_2) = \{r_i \wedge \neg r_i \mid i \in \mathbb{N}\}$, whence $\Gamma_2 \vdash_{\mathbf{CLuN}^r} p$. However, there is no finite $\Gamma' \subset \Gamma_2$ for which $\Gamma' \vdash_{\mathbf{CLuN}^r} p$. The example also illustrates that, if the opponent were allowed to rely on premises introduced by the proponent, then he would be able to win the dialogue for $\Gamma_2 \vdash_{\mathbf{CLuN}^r} p$, even if he chose $\Gamma' = \emptyset$. But of course the proponent should have a winning strategy because $\Gamma_2 \vdash_{\mathbf{CLuN}^r} p$.

The last comment on the dialogues for Reliability concerns decidability. Although the proponent has a winning strategy or does not have one, for each specific Γ and A, it is very well possible that we are unable to find out which of the two is the case. This is related to the undecidability at the predicative level. The absence of a positive test for (predicative) final derivability has the effect that a concrete dialogue will not constitute a demonstration that A is derivable from Γ. Only establishing that the proponent has a winning strategy will do so and it is only possible to establish this by a reasoning at the metalevel. All this

postpone the proofs of the present theorems to the next chapter. Needless to say, no circularity is involved in the proofs.

is unavoidable in the case of defeasible reasoning, unless one artificially restricts it to decidable fragments.[15]

4.9.2 Minimal Abnormality

Let us now move on to the Minimal Abnormality strategy. In general, Minimal Abnormality requires more complex proofs than Reliability. For some Γ and A, A can only be derived on an unmarked line if A is derived on several conditions (and hence on several lines). So the proponent has not only to derive *Dab*-formulas in order to show that some of the *Dab*-formulas in the opponent's attack are not minimal. Often, the proponent should also derive the intended conclusion on several conditions. She has to do so in order to show that, for each $\varphi \in \Phi_s(\Gamma)$, the conclusion is derivable on a condition Δ for which $\Delta \cap \varphi = \emptyset$. Note, however, that, in order for the proponent to win any of the dialogues, the conclusion should be derived at a line of the original proof stage and *this* line should be unmarked at the end of the dialogue. verborgen

Stability with respect to a line This dialogue is identical to its namesake for Reliability. And so is the inconvenience: in some cases the only winning strategy for the proponent requires that she produces an infinite proof in her first move. The dialogue for $\Gamma_1 \vdash_{\mathbf{CLuN}^m} p$ illustrates this.

Many turns This dialogue is identical to its namesake for Reliability, except that not all restrictions on the finiteness of the proof and its extensions can be upheld. Actually, several complications should be considered.

Let $\Gamma_3 = \{(p_i \wedge \neg p_i) \vee (p_j \wedge \neg p_j) \mid i \neq j; i, j \in \mathbb{N}\} \cup \{q \vee (p_i \wedge \neg p_i) \mid i \in \mathbb{N}\}$. As $\Phi(\Gamma_3) = \{\{p_i \wedge \neg p_i \mid i \in \mathbb{N}\} - \{p_j \wedge \neg p_j\} \mid j \in \mathbb{N}\}$, it is easily seen (in view of Theorem 5.3.3) that $\Gamma_3 \vdash_{\mathbf{CLuN}^m} q$ (because q can be derived on the condition $\{p_j \wedge \neg p_j\}$ for every $j \in \mathbb{N}$). This seems to work fine with a finite proof and finite extensions. The proponent starts off with, for example, the proof

1	$q \vee (p_1 \wedge \neg p_1)$	Prem	$\emptyset$
2	q	1; RC	$\{p_1 \wedge \neg p_1\}$

after which the opponent offers a finite reply, an extension of 1–2 in which line 2 is marked. There are infinitely many such extensions. All that is required for line 2 to be marked is that, where s is the last stage of the extension, there is a $\varphi \in \Phi_s(\Gamma_3)$ for which $p_1 \wedge \neg p_1 \in \varphi$. A simple example is the extension:

3	$(p_0 \wedge \neg p_0) \vee (p_1 \wedge \neg p_1)$	Prem	$\emptyset$
4	$(p_0 \wedge \neg p_0) \check{\vee} (p_1 \wedge \neg p_1)$	Prem	$\emptyset$

Consider such an extension and let it count l lines. As the extension is finite, at most finitely many letters p_i occur in it. So the proponent can simply pick a p_i that does not occur in the extension and add the lines:

$l+1$	$q \vee (p_i \wedge \neg p_i)$	Prem	$\emptyset$
$l+2$	q	$l+1$; RC	$\{p_i \wedge \neg p_i\}$

[15] If the premises and conclusion belong to a **CL**-decidable fragment of the language and the premise set is finite, then it is decidable whether the proponent has a winning strategy. This follows from a forthcoming result on the embedding of (full predicative) **CLuN** into **CL**—for the result on the propositional case see [BDCK99].

As p_i does not occur up to line l, $p_i \wedge \neg p_i$ is not a member of any $\varphi \in \Phi_{l+2}(\Gamma_3)$ and hence line $l+2$ is unmarked. Moreover, as some $\varphi \in \Phi_{l+2}(\Gamma_3)$ are bound not to contain $p_1 \wedge \neg p_1$, line 2 is unmarked. So all seems well: $\Gamma_3 \vdash_{\mathbf{CLuN}^m} q$ and the proponent has a reply to every attack of the opponent on 1–2.

However, consider $\Gamma_4 = \{(p_i \wedge \neg p_i) \vee (p_j \wedge \neg p_j) \mid i \neq j; i, j \in \mathbb{N}\} \cup \{q \vee (p_i \wedge \neg p_i) \mid i \in \mathbb{N} - \{0\}\}$—so $\Gamma_4 = \Gamma_3 - \{q \vee (p_0 \wedge \neg p_0)\}$. As $\Phi(\Gamma_4) = \{\{p_i \wedge \neg p_i \mid i \in \mathbb{N}\} - \{p_j \wedge \neg p_j\} \mid j \in \mathbb{N}\}$, we now have (in view of Theorem 5.3.3) that $\Gamma_4 \nvdash_{\mathbf{CLuN}^m} q$ (because q cannot be derived on the condition $\{p_0 \wedge \neg p_0\}$). The only point at which the proponent turns out to lose the game is after all *Dab*-formulas of the form $(p_i \wedge \neg p_i) \check{\vee} (p_j \wedge \neg p_j)$ have been derived. As there is no line on which q is derived on the condition $\{p_0 \wedge \neg p_0\}$, all lines on which q is derived are marked at this stage, call it s. Indeed, every condition on which q has been derived, overlaps with $\{\{p_i \wedge \neg p_i \mid i \in \mathbb{N}\} - \{p_0 \wedge \neg p_0\}\} \in \Phi_s(\Gamma_4) = \Phi(\Gamma_4)$.

As was shown in the previous paragraphs, this dialogue type requires infinity in some way or other. Either we have to allow that the opponent attacks (finitely many times) by an infinite extension and that the proponent defends by infinite extensions, or we have to allow the game to go on infinitely. The need to refer to infinite proofs at a stage will return in the other types of dialogue. The first move of the proponent, to the contrary, may be required to be finite.

Some people will not like dialogues that require infinite lists of formulas. Yet, the requirement is unavoidable for characterizing final derivability on the Minimal Abnormality strategy. Note that this is not too bad. Even for usual dialogues, the question is not who wins the game, but whether the proponent has a winning strategy. In order to show this, one may need to refer to infinitely many possible dialogues even in the case of **CL**. Each of these is finite, whereas the dialogues considered in this paper are infinite if Minimal Abnormality is the strategy. Of course, if it can be demonstrated that the proponent has a winning strategy, then this metalinguistic demonstration is finite.

POP The long discussion of the previous dialogue type gives us at once the insights required for describing this type. The dialogue is identical to its namesake for Reliability, except that the extension of the proof and the extension of the extension should be allowed to be infinite. As was remarked before, the existence or absence of a winning strategy for the proponent has to be established at the metalevel anyway.

Calling premises This dialogue type is identical to that for Reliability, except that the opponent is allowed to delineate an infinite set of premises and that, after this, the proponent is allowed to produce an infinite extension of her proof.

The proof of the following theorem proceeds as the proof of Theorem 4.9.1. There is one difference. The task of the proponent in a defense is double. First, for some minimal *Dab*-formulas $Dab(\Theta)$ derived by the opponent, she should derive a $Dab(\Theta')$ with $\Theta' \subset \Theta$.[16] Next, she should derive the conclusion on a set of conditions. By doing so, she should produce a stage s in which the following situation holds: for every $\varphi \in \Phi_s(\Gamma)$, the conclusion should be derived

[16]In the case of the Calling Premises dialogue, the proponent should do this for all *Dab*-formulas $Dab(\Theta)$ that are **LLL**-derivable from Γ'.

on a condition Δ for which $\Delta \cap \varphi = \emptyset$.[17] Note that she is able to do so, for each dialogue type, just in case $\Gamma \vdash_{\mathbf{AL}^m} A$.

Theorem 4.9.2 *For the four dialogue types described holds: the proponent has a winning strategy in the dialogue for* $\Gamma \vdash_{\mathbf{AL}^m} A$ *iff* $\Gamma \vdash_{\mathbf{AL}^m} A$.

Given the absence of a positive test (in general), the computational complexity of adaptive logics is greater than that of classical (predicative) logic and it is greater for Minimal Abnormality than for Reliability. This does not prevent one, however, from describing dialogue types and to show them adequate.

4.9.3 Excursion: Classical Disjunctions in *Dab*-formulas

Whenever the disjunction (or a disjunction) of the standard language has the same meaning as classical disjunction, one may introduce the convention that standard disjunctions of abnormalities count as *Dab*-formulas—let us call them *Dab*-formulas-by-convention. It is important to realize, however, that this is a convention and not the official definition. Let me illustrate this by a premise set that is a slight complication of Γ_3 from page 137.

Let $\Gamma_5 = \{(p_i \wedge \neg p_i) \vee (p_j \wedge \neg p_j) \mid i, j \in \mathbb{N}; i \neq j\} \cup \{(\bigwedge\{(p_j \wedge \neg p_j) \vee (p_k \wedge \neg p_k) \mid j, k \in \{0, \ldots, i+2\}; j < k\}) \supset (q \vee (p_i \wedge \neg p_i)) \mid i \in \mathbb{N}\}$. Incidentally, the shortest member of the second 'part' of the premise set is $(((p_0 \wedge \neg p_0) \vee (p_1 \wedge \neg p_1)) \wedge ((p_0 \wedge \neg p_0) \vee (p_2 \wedge \neg p_2)) \wedge ((p_1 \wedge \neg p_1) \vee (p_2 \wedge \neg p_2))) \supset (q \vee (p_0 \wedge \neg p_0))$. Note that $\Phi(\Gamma_5) = \Phi(\Gamma_3) = \{\{p_i \wedge \neg p_i \mid i \in \mathbb{N}\} - \{p_j \wedge \neg p_j\} \mid j \in \mathbb{N}\}$. So every $\varphi \in \Phi(\Gamma_5)$ iff φ comprises all but one $p_i \wedge \neg p_i$. Moreover, in a proof from Γ_5, q is derivable on the condition $\{p_i \wedge \neg p_i\}$ for every $i \in \mathbb{N}$. So the proponent should have a winning strategy in this case.

The premises are prepared in such a way that, in order to derive q on some condition $\{p_i \wedge \neg p_i\}$, say on line l, one has to introduce first a number of premises that are *Dab*-formulas-by-convention. Moreover, if the convention is followed, these *Dab*-formulas cause line l to be marked in *every* finite proof. In other words, the proponent would have to produce, as her first move, an infinite stage of a proof. So, while the convention simplifies matters in a number of cases, it drastically complicates them in this respect.

By not following the convention, the proponent can produce a *finite* proof from Γ_5 in which q is derived on some condition but that does not contain *any* *Dab*-formula—she needs to introduce, for example, $(p_0 \wedge \neg p_0) \vee (p_1 \wedge \neg p_1)$, but this is not a *Dab*-formula. It is up to the opponent to derive the *Dab*-formula $(p_0 \wedge \neg p_0) \check{\vee} (p_1 \wedge \neg p_1)$. And this is sensible from a philosophical point of view. All the proponent should do in her first move is to derive the conclusion, say at line l, on a condition that will allow her to win the dialogue. Deriving *Dab*-formulas that cause line l to be marked is the task of the opponent. Only after the opponent has done so and taking the opponent's specific reaction into account, the proponent has the task to derive more *Dab*-formulas that cause the line to be unmarked again.

I still have to show that the convention does its job. To this end I prove the following lemmas.

[17] In the case of the Calling Premises dialogue, no *Dab*-formulas derivable from Γ' should make this condition false.

Lemma 4.9.1 *If* $\Gamma \vdash_{\mathbf{LLL}} A \check{\vee} Dab(\Delta)$ *and there is a* $\varphi \in \Phi(\Gamma)$ *such that* $\Delta \cap \varphi = \emptyset$*, then there is a finite* $\mathbf{AL}^m$*-proof from* Γ *in which* A *is derived on the condition* Δ *at an unmarked line. (Finite Stage Lemma – Minimal Abnormality)*

Proof. Suppose that the antecedent is true. As **LLL** is compact, $A \check{\vee} Dab(\Delta)$ is **LLL**-derivable from a finite $\Gamma' \subseteq \Gamma$. Consider a sequence of, say n, lines, all having the empty condition, such that each member of Γ' is derived by the premise rule at one of the lines. As all premises belong to $\mathcal{W}$, the only *Dab*-formulas that can occur at (this) stage n are abnormalities, viz. have the form $Dab(\Theta)$ for Θ a singleton. Let Σ be the set of all these abnormalities and note that (i) $\Phi_n(\Gamma) = \{\Sigma\}$ and (ii) $\Sigma \subseteq \varphi$ for every $\varphi \in \Phi(\Gamma)$.

Derive A on the condition Δ at line $n + 1$ by RC—this is possible because $\Gamma' \vdash_{\mathbf{LLL}} A \check{\vee} Dab(\Delta)$. *Case 1:* $\Delta = \emptyset$. So line $n + 1$ is unmarked. *Case 2:* $\Delta \neq \emptyset$. So $\Phi_{n+1}(\Gamma) = \Phi_n(\Gamma) = \{\Sigma\}$. As, in view of the supposition, $\Delta \cap \Sigma = \emptyset$, line $n + 1$ is unmarked at stage $n + 1$ of the proof. ■

Lemma 4.9.2 *If* $\Gamma \vdash_{\mathbf{LLL}} A \check{\vee} Dab(\Delta)$ *and* $\Delta \cap U(\Gamma) = \emptyset$*, then there is a finite* $\mathbf{AL}^r$*-proof from* Γ *in which* A *is derived on the condition* Δ *at an unmarked line. (Finite Stage Lemma – Reliability)*

The proof proceeds exactly as the one of Lemma 4.9.1, except that this time, in the first paragraph, (i) $U_n(\Gamma) = \Sigma$ and (ii) $\Sigma \subseteq U(\Gamma)$ and that, in the second paragraph, $U_{n+1}(\Gamma) = U_n(\Gamma) = \Sigma$.

4.10 Dynamic Semantics

The semantics of adaptive logics defines a semantic consequence relation that is sound and complete with respect to final derivability—soundness and completeness proofs follow in the next chapter. Final derivability refers (in a precise way) to a stable stage of proofs, a stage at which the dynamics has come to an end. So the adaptive semantics is not in any way related to the most typical aspect of adaptive proof theory, viz. its dynamics.

There was a time when some people were puzzled by this. As there is nothing dynamic about the semantic consequence relation, so they reasoned, it is determined 'beforehand' which formulas are finally derivable from a given premise set. That being so, whence the need for dynamics at the level of the proofs? And there is another puzzle. If the dynamics of the proofs is real, one should be able to describe a corresponding dynamics at the level of the semantics. A third question might be phrased as follows: What is the relation between the insight provided by a stage of a proof and the derivability relation?

There is something I should straighten out before continuing. That it is determined 'beforehand' which formulas are finally derivable from a given premise set has no impact on the static or dynamic character of the proofs. That determination is simply a consequence of the fact that adaptive logics are logics, viz. map every premise set to a consequence set. The dynamics of the proof derives from the complexity of adaptive consequence relations. It is typical for consequence relations for which there is no positive test. So the dynamics is not related to definability but to computability.

The block semantics, which I describe below,[18] will provide a semantic counterpart to the dynamics of the proofs. The block semantics also enables one to make sense of the insight in the premises that is provided by a proof at a stage and of the question what happens to this insight as the proof proceeds to the next stage. In dynamic proofs, a growing insight in the premises may lead to a typically dynamic behaviour of the set of derived formulas: some unconditionally derived formulas may be gained, some conditionally derived formulas may be gained, but some conditionally derived formulas may also be lost.

Of course static proofs also provide an insight in the premises and this insight may grow as the proof proceeds. So the block semantics is not typical for logics that have static proofs. This is why I shall start by describing the block semantics for **CL**.

Not all uses of the block semantics can be discussed here. I shall confine the discussion to what is typical for adaptive logics and refer to [Bat95] for conceptual change and several other aspects.

Intuitively, a *block* is an unanalysed formula. Blocks containing the same formula may occur at different places in a proof, without being identified. This is why I shall characterize blocks by a number and a formula, and write them officially as, for example, $[\![14, (p \wedge \neg q) \supset r]\!]$, in shorthand as $[\![(p \wedge \neg q) \supset r]\!]^{14}$. Combining blocks by logical terms and parentheses, we obtain block formulas.

Consider an annotated proof. The block analysis of this proof is determined by the *discriminations* and *identifications* the author of the proof has *minimally* made in the formulas of the proof in order to construct the proof (according to the annotation). These are two different operations. It is one thing to see that a formula has the form $A \supset B$ and it is another thing to see that the antecedent A is identical to another formula that occurs in the proof. In order to apply MP, one needs to see both. As a result of the block analysis, the proof is turned into a block proof.

An example of a **CL**-proof will clarify the matter. I do not first present the original **CL**-proof; it is easy enough to reconstruct that: delete the double brackets and the superscripted numbers. At stage 2 of the proof the block analysis may look as follows.

1	$[\![(p \supset \neg q) \supset (p \wedge (\neg r \vee \neg p))]\!]^1$	Prem
2	$[\![p \supset \neg q]\!]^2$	Prem

I write "may look as follows" because the block numbers are obviously arbitrary; all that is required is that they are different.

Two formulas were introduced. As far as the block analysis is concerned, they are just two different blocks. Indeed, there is no need to have seen any structure in the formulas in order to introduce them. This changes drastically at the next stage, where Modus Ponens is applied:

1	$[\![p \supset \neg q]\!]^2 \supset [\![p \wedge (\neg r \vee \neg p)]\!]^3$	Prem
2	$[\![p \supset \neg q]\!]^2$	Prem
3	$[\![p \wedge (\neg r \vee \neg p)]\!]^3$	1, 2; Modus Ponens

In order to derive 3 from 1 and 2, the person constructing the proof must have seen that 1 is an implicative formula—here called a discrimination—that the

[18] The block semantics was presented in [Bat95] and [Bat98]. The presentation following in the text is slightly different—I hope it is even more transparent.

antecedent of 1 is identical to 2, and that 3 is identical to the consequent of 1—two identifications. It is not required that this person has any idea of the contents of blocks 2 and 3.

If the proof is continued by deriving p from 3 by Simplification (to derive A from $A \wedge B$; to derive B from $A \wedge B$), then the block analysis of the result will read:

1	$[\![p \supset \neg q]\!]^2 \supset ([\![p]\!]^4 \wedge [\![\neg r \vee \neg p]\!]^5)$	Prem
2	$[\![p \supset \neg q]\!]^2$	Prem
3	$[\![p]\!]^4 \wedge [\![\neg r \vee \neg p]\!]^5$	1, 2; Modus Ponens
4	$[\![p]\!]^4$	3; Simplification

Note that block 3 is replaced everywhere in the proof by the conjunction of blocks 4 and 5, so not only at line 3 but also at line 1.

There is no need for the author of the proof to have seen that p occurs in blocks 2 and 5. Even if this is seen at a later stage, these occurrences need not be identified with the occurrence of p in block 4. They may appear, say, as $[\![p]\!]^{17}$ and $[\![p]\!]^{22}$. By now it should be clear what I mean by the block analysis of a static proof.

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Let us move on to the *block language* $b\mathcal{L}_s$ for the standard language $\mathcal{L}_s$. For every $A \in \mathcal{W}_s$, $b\mathcal{S}$ contains a denumerable set of blocks $[\![A]\!]^i$; these function just like sentential letters, whence the name of their set. For every $\beta \in \mathcal{C}$, $b\mathcal{C}$ contains a denumerable set of blocks $[\![\beta]\!]^i$, which function like individual constants; for every $\alpha \in \mathcal{V}$, $b\mathcal{V}$ contains a denumerable set of blocks $[\![\alpha]\!]^i$, which function like individual variables. The definition of functional blocks is hardly more complicated. For any open formula A, we replace every occurrence of a free variable by a dash. Thus $Pxa \supset Qx$ becomes $P{-}a \supset Q{-}$ and $Pxa \supset Qy$ becomes the same expression. Where r is the number of dashes in such an expression B, $b\mathcal{P}^r$ contains denumerably many blocks $[\![B]\!]^i$, which function like predicates of rank r. The set of open and closed block formulas, $b\mathcal{F}$ and $b\mathcal{W}$ respectively, are defined in the same way as $\mathcal{F}$ and $\mathcal{W}$ are defined, except that $\mathcal{S}$ is replaced by $b\mathcal{S}$, $\mathcal{P}$ by $b\mathcal{P}^r$, and so on. All blocks should have different numbers, but this is not a problem as there are only denumerably many blocks.

To see the functioning at the predicative level, consider the transition from $\forall x Px$ to Pa. This will be analysed as the transition from $\forall [\![x]\!]^{31}[\![P-]\!]^{41}[\![x]\!]^{31}$ to $[\![P-]\!]^{41}[\![a]\!]^{21}$—the block numbers are of course arbitrary. The block $[\![P-]\!]^{41}$ indicates that its contents is a 'predicative' block of rank 1. The transition from $\forall x(Px \supset Qx)$ to $Pa \supset Qa$ will be analysed as the transition from block formula $\forall [\![x]\!]^{31}[\![P- \supset Q-]\!]^{42}[\![x]\!]^{31}[\![x]\!]^{31}$ to block formula $[\![P-]\!]^{42}[\![a]\!]^{21}[\![a]\!]^{21}$. The variable block occurs twice in the first expression because $[\![P- \supset Q-]\!]^{42}$ is a block of rank 2 and the variable x occurs twice in the original formula. Similarly for the constant block in the second expression. The transition from $a = b$ and $Pa \supset Pa$ to $Pa \supset Pb$ will be analysed as the transition from $[\![a]\!]^{211} = [\![b]\!]^{22}$ and $[\![Pa \supset P-]\!]^{46}[\![a]\!]^{211}$ to $[\![Pa \supset P-]\!]^{46}[\![b]\!]^{22}$. Obviously $[\![Pa \supset P-]\!]^{46}$ is a predicate block of rank 1; in order to replace the second occurrence of a in $Pa \supset Pa$ by b, there is no need to see that a occurs twice.

Incidentally, blocks like $[\![p]\!]^{22}$, $[\![P-]\!]^{41}$, and $[\![b]\!]^{22}$ may be considered as fully analysed—no deeper analysis is possible. Note that two such blocks may have the same contents without being identified.

The block semantics is simply the usual semantics applied to the block language. The semantic consequence relation is also defined as usual. So what is the big deal? Consider the block semantics for **CL** as an example. We know that **CL**-proofs are sound with respect to the **CL**-semantics. This means that every formula derived in a proof is a semantic consequence of the premise set. The premise set is available before the proof starts. So it is, at this point, just a set of sentential blocks. With this in mind, let us concentrate on the above example of a proof.

At stage 2 of the proof, two premises were introduced, the sentential blocks 1 and 2, and these are obviously true if the premises are true. At stage 3 of the proof, block 1 is analysed as "block 2 implies block 3"—$[\![1]\!]$ is analysed as $[\![2]\!] \supset [\![3]\!]$ for short—and block 3 is derived by modus ponens. What is the semantic counterpart of this? The major change is that $[\![2]\!] \supset [\![3]\!]$ is required to have the same truth value as $[\![1]\!]$. As $[\![1]\!]$ is true in all models of the premises—it is a premise—$[\![2]\!] \supset [\![3]\!]$ is now required to be true in all models of the premises: in all of them either $[\![2]\!]$ is false or $[\![3]\!]$ is true. But then $[\![3]\!]$ is true in all those models because $[\![2]\!]$ is a premise. A similar situation arises at stage 4 of the proof: $[\![4]\!] \wedge [\![5]\!]$ is required to have the same truth value as $[\![3]\!]$. As $[\![3]\!]$ was true in all models of the premises, so is $[\![4]\!] \wedge [\![5]\!]$ and hence also $[\![4]\!]$, which is the 'formula' of line 4.

The models of the block premises verify all block premises but may assign any values to other blocks. These models are also the block models of stage 1 and of stage 2 of the proof. At stage 3, $[\![1]\!]$ is analysed as $[\![2]\!] \supset [\![3]\!]$. So all block models of stage 3 verify $[\![3]\!]$. Similarly for stage 4 of the proof. These examples involve analysis (discriminations), but also identification. If, at stage 3, the author of the proof only sees that $[\![1]\!]$ is an implication, but does not *identify* the antecedent with $[\![2]\!]$, then he or she could not have derived $[\![3]\!]$. Indeed, $[\![3]\!]$ does not follow, for example, from $[\![2]\!]$ and $[\![6]\!] \supset [\![3]\!]$. Obviously, this analysis may be sensible even if no identification takes place. For example, from $[\![6]\!] \supset [\![3]\!]$ follows $\neg[\![3]\!] \supset \neg[\![6]\!]$. And it is easy enough to devise examples where only identification takes place: several blocks with different numbers are given the same number.

So, as the example proof and its block analysis (discriminations and identifications) proceed, more block formulas are required to have a definite value. This means that the number of models is *lowered*. This was the first point I wanted to make. As the proof proceeds, the number of models shrink. So the block models corresponding to the stages of the proof may be seen as explicating the insights in the premises that grow as the proof becomes longer.

Do the insights in the premises grow whenever one passes to the next stage? Of course not. Here is an example.

5	$[\![p]\!]^4 \vee [\![s]\!]^7$	4; Addition

No discriminations or identifications took place. So no insight in the premises is gained. This is probably the reason why many beginning logic students quarrel about applications of Addition that serve no purpose for analysing other formulas.

This brings us to the next point: the distinction between *informative* steps and uninformative steps of a proof. A step is informative iff the models of the stage to which the step leads are a proper subset of the models of the previous stage. So step 5 is uninformative. Nothing is known about the truth-value of the new block, $[\![s]\!]^7$, and $[\![p]\!]^4 \vee [\![A]\!]^i$ is already true, for all A and i, in all models of stage 4 of the proof. That the models of stages 5 and 4 are identical is obviously related to the fact that no further discriminations or identifications were made in the transition to stage 5. This does not mean that step 5 is useless. For example, it may lead to an analysing step in the presence of the premise $t \vee \neg(p \vee s)$.[19] The step may also be useful because $p \vee s$ was the goal of the proof, by which I mean that one was trying to show that $p \vee s$ is derivable from the premises. So the step may be useful. Nevertheless, it is uninformative.

Incidentally, the distinction between informative moves and uninformative ones solves the logical omniscience[20] puzzle and related puzzles, which derive from the idea that the contents of the conclusion of a deductive inference should be included in the contents of its premises. This is so only if the contents of a statement is measured in an absolute way, viz. in terms of the set of models of the premises. By measuring this information in terms of the block models of the premises as determined by a certain stage of a proof, one measures insight in the premises, and it is readily seen that this insight may increase as the proof proceeds. The thesis that one cannot learn anything by means of deduction is counterintuitive and is now seen to be simply wrong.

Let me phrase this in a way that is more generous towards the traditional view. One may distinguish between the amount of information that is contained in a set of formulas and the amount of information that has been extricated from a premise set. The information contained in a set of formulas is the information that *can* be extricated. The models of a premise set measure the amount of information contained in it; the block models corresponding to a stage of a proof from the premises measure the amount of information that has been extricated from the premise set at the stage.

Let us turn to adaptive logics. Phrasing the matter in terms of the standard format would make the discussion rather abstract and difficult. So let me consider a specific logic, viz. $\mathbf{CLuN}^r$. Once the situation is clear for the special case, the generalization to the standard format is easily made by the reader.

A first matter to be considered is the introduction of elements in the condition. These need to have the form $\exists(A \wedge \neg A)$. So it is clear which discriminations and identifications are required in order to perform the move. Here are some examples. Consider, as a first example, the transition from $\neg p$ on the condition Δ and $q \vee p$ on the condition Θ to q on the condition $\Delta \cup \Theta \cup \{p \wedge \neg p\}$—by $b\Delta$ I denote what became of Δ as the result of the block analysis.

$\neg[\![p]\!]^i$	$b\Delta$
$[\![q]\!]^j \vee [\![p]\!]^i$	$b\Theta$
$[\![q]\!]^j$	$b\Delta \cup b\Theta \cup \{[\![p]\!]^i \wedge \neg[\![p]\!]^i\}$

Here is a predicative example: the transition from $\forall x(Px \supset Qx)$ on the condition Δ and $\neg Qa$ on the condition Θ to $\neg Pa$ on the condition $\Delta \cup \Theta \cup \{Qa \wedge \neg Qa\}$.

$\forall[\![x]\!]^i([\![P-]\!]^j[\![x]\!]^i \supset [\![Q-]\!]^k[\![x]\!]^i)$	$b\Delta$
$\neg[\![Q-]\!]^k[\![a]\!]^l$	$b\Theta$
$\neg[\![P-]\!]^j[\![a]\!]^l$	$b\Delta \cup b\Theta \cup \{[\![Q-]\!]^k[\![a]\!]^l \wedge \neg[\![Q-]\!]^k[\![a]\!]^l\}$

Handling *Dab*-formulas requires different considerations. Remember that *Dab*-formulas are always derived (so not premises) because they are classical

[19] The step may be avoided in some versions of **CL**, but not in others.

[20] The connection with logical omniscience is obvious: if it were available, there would be no need for dynamic proofs.

disjunctions of abnormalities—see also Subsection 4.9.3. Consider the following proof fragment, in which A is any formula and B and C are abnormalities.

8	A	...	$\Delta \cup \{B\}$
9	$B \check{\vee} C$	...	$\emptyset$

If $B \check{\vee} C$ is the only *Dab*-formula that occurs at the stage, line 8 is marked in view of the marking definition (for Reliability or for Minimal Abnormality). So the block analysis of the proof must reveal that $B \check{\vee} C$ is a *Dab*-formula. In other words, if a *Dab*-formula occurs in the proof, its block analysis must reveal that it is a classical disjunction of abnormalities.

This is not all that is required. In order for the markings to be correct, the relevant abnormalities must also be identified. In the above example, B must be turned into the same block formula on lines 8 and 9. One way to realize this is that, as soon as a *Dab*-formula occurs in a block proof, the blocks making up the disjuncts of the block formula are identified with the blocks that occur in conditions and have the same content *and* the blocks that occur in a *Dab*-formula and have the same content are identified. To see that the latter is required, consider the situation where a block formula A is derived on the condition $[\![p]\!]^1 \wedge \neg[\![p]\!]^1$ as well as on the condition $[\![p]\!]^2 \wedge \neg[\![p]\!]^2$ and that the only *Dab*-formula in the proof is $([\![p]\!]^1 \wedge \neg[\![p]\!]^1) \vee ([\![p]\!]^2 \wedge \neg[\![p]\!]^2)$. If $[\![p]\!]^1$ and $[\![p]\!]^2$ are not identified, the two lines at which A is derived are unmarked on the Minimal Abnormality strategy, which is obviously wrong.

All this is a consequence of the nature of dynamic proofs. The author of the proof is still allowed to extend the proof with any correct lines she wants to add. However, in order to find out which formulas are derived at a certain stage and which are not, she is required to apply the marking definition. In terms of the block analysis, this means that she is required to check which formulas are *Dab*-formulas and that she is required to identify, whenever this is possible, the blocks that make up *Dab*-formulas with each other was well as with the blocks that occur in conditions.[21] So, while the rules still function exactly as for static proofs, the marking definition imposes a further requirement. The insights provided by the proof do not only depend on the author of the proof, viz. the applied rules, but also on what the proof teaches us about derivability at a stage. In semantic terms, this corresponds to checking which models of the premises are eliminated by the transition to the new stage of the proof.

As for static proofs, we distinguish between informative moves and uninformative ones. The criterion is also the same: a move is informative iff it decreases the number of block models of the premises. The only difference lies in the special role played by the inference of minimal *Dab*-formulas. These not only increase the insight in the premises in a direct way, in that we learn that this specific *Dab*-formula follows from the premises, but also in an indirect way because they have consequences for the derivability of conditional formulas.

This reveals an interesting aspect of the dynamic proofs of adaptive logics. As a proof proceeds, the insight in the premises may increase and *never* decreases. The insight increases if the steps are informative, which means that the number of models decreases. Moreover, if an adaptive proof consists of informative steps, the insight converges towards a limit—the set of consequences of a premise set is denumerable.

[21] The abnormalities stem from formulas occurring at earlier lines of the proof. So unifying the block numbers will also have an effect at those lines.

This claim needs qualification. Which set of models decreases? Obviously the set of lower limit models. Let us consider an extremely simple **CLuN**^{r}-proof.

1	$[\![p]\!]^1 \vee [\![q]\!]^2$	Prem	$\emptyset$
2	$\neg[\![q]\!]^2$	Prem	$\emptyset$
3	$[\![r \wedge q]\!]^3$	Prem	$\emptyset$
4	$[\![p]\!]^1$	1, 2; RC	$\{[\![q]\!]^2 \wedge \neg[\![q]\!]^2\}$

At stage 4 of this proof p (or $[\![p]\!]^1$) is derived from the premises. Of course, the proof may be continued.

1	$[\![p]\!]^1 \vee [\![q]\!]^2$	Prem	$\emptyset$	
2	$\neg[\![q]\!]^2$	Prem	$\emptyset$	
3	$[\![r]\!]^4 \wedge [\![q]\!]^2$	Prem	$\emptyset$	
4	$[\![p]\!]^1$	1, 2; RC	$\{[\![q]\!]^2 \wedge \neg[\![q]\!]^2\}$	$\checkmark^5$
5	$[\![q]\!]^2 \wedge \neg[\![q]\!]^2$	2, 3; RU	$\emptyset$	

At stage 5, p is not derived from the premises.

The set of lower limit models, here **CLuN**-models, of the premise set decreases as the proof proceeds from one stage to the next by an informative step and remains the same if the proof proceeds by a non-informative step—the lower limit models are obtained, in agreement with Theorem 4.4.1, by reading a line at which a formula A is derived on the condition Δ as a line with formula $A \check{\vee} Dab(\Delta)$–all this obviously turned into block formulas.

The adaptive models, here **CLuN**^{r}-models, of the premise set need not decrease or remain the same. Actually a discontinuity may occur, as the example illustrates. Thus the **CLuN**^{r}-models of the premises as determined by stage 4 of the proof are all consistent (verify no abnormality) whereas the **CLuN**^{r}-models determined by stage 5 are all inconsistent. So no **CLuN**-model is a **CLuN**^{r}-model of both stages. In this sense, the adaptive block semantics nicely explicates the insight in final derivability provided by the proof at a stage.

This point cannot be sufficiently stressed. On the one hand, the lower limit logic explicates the insight in the premises. This never decreases and increases with every informative step. The insight provided by a stage of the proof provides an 'estimation' of the final consequence set, viz. the adaptive consequence set. This estimation does not and cannot converge. It cannot converge as a result of the computational complexity of the final consequence set. Phrased in terms closer to our inferential experience, the further analysis of the premises (and the introduction of further premises in the proof) may lead to the derivation of 'shorter' Dab-formulas: in a proof in which $Dab(\Delta)$ was already derived a $Dab(\Delta')$ is derived such that $\Delta' \subset \Delta$. As an effect of the resulting insight, marked lines may be unmarked. The further analysis of the premises and the introduction of further premises in the proof may also lead to the derivation of 'new' Dab-formulas: in a proof, in which the derived minimal Dab-formulas are $Dab(\Delta_1), \ldots, Dab(\Delta_n)$, a $Dab(\Theta)$ is derived for which Θ is neither a subset nor a superset of any Δ_i $(1 \leq i \leq n)$. As an effect of the resulting insight, unmarked lines may be marked. The result of all this is that, while the insights in the premises, as provided by the proof, converge, the estimations of the final consequence set, made on the basis of the insights provided by subsequent stages, may display discontinuities.

In some cases, derivability at a stage is the best we may obtain. The matter is further discussed in Chapter 10, where we shall see that there are procedures that form criteria for final derivability. Their basic aim is to guide the derivation in order to obtain conclusive insights with respect to the final derivability of a certain formula from a premise set. We shall see that the procedures do an excellent job. Nevertheless, they cannot always be conclusive because of the absence of a positive test. Where they are not conclusive, there is at least some comfort in the knowledge that the adaptive proofs provide an increasing insight

The dynamic proofs of adaptive logics have a further strength. They warrant, under a very simple criterion for sensibility, that the estimate of final derivability is correct *if* the insight provided by the present stage of the proof is correct. This is most easily seen by embedding the block language into the standard language. I take the latter to be $\mathcal{L}_s$ as I did throughout this section. The required translation function is independent of the logic—in the examples **CL**, **CLuN**, and **CLuN**^{m}.

1. Where $[\![A]\!]^i \in b\mathcal{S}$, $\text{tr}([\![A]\!]^i) = p_i$.
2. Where $[\![\alpha]\!]^i \in b\mathcal{C}$, $\text{tr}([\![\alpha]\!]^i) = a_i$.
3. Where $[\![\alpha]\!]^i \in b\mathcal{V}$, $\text{tr}([\![\alpha]\!]^i) = x_i$.
4. Where $[\![A]\!]^i \in b\mathcal{P}^r$, $\text{tr}([\![A]\!]^i) = P^r_i$.

We extend the function to block formulas by identifying their translation with the formulas obtained by translating the blocks that occur in them and we extend the function to sets of block formulas by identifying their translation with the set of the translations of their elements.

Where **L** is a logic over $\mathcal{L}_s$, **bL** the corresponding block logic over $b\mathcal{L}_s$, Γ a set of block formulas and A a block formula, the following fact is obvious.

Fact 4.10.1 $\Gamma \vdash_{\mathbf{bL}} A$ *iff* $\text{tr}(\Gamma) \vdash_{\mathbf{L}} \text{tr}(A)$ *and* $\Gamma \vDash_{\mathbf{bL}} A$ *iff* $\text{tr}(\Gamma) \vDash_{\mathbf{L}} \text{tr}(A)$.

Now we come to a really fascinating fact, but it requires some introduction. Suppose that a block proof contains the formulas $[\![p]\!]^1 \vee ([\![q]\!]^2 \wedge \neg[\![q]\!]^2)$ and $\neg[\![p]\!]^1 \wedge [\![r]\!]^3$, that both occur on the empty condition, but that no *Dab*-formula occurs in the proof. We shall say that this proof is *Dab*-incomplete. The Dab-formula $([\![p]\!]^1 \wedge \neg[\![p]\!]^1 \vee ([\![q]\!]^2 \wedge \neg[\![q]\!]^2)$ is obviously derivable and does not occur in the proof. Note that this *Dab*-formula would not be derivable if, in the example, $\neg[\![p]\!]^1 \wedge [\![r]\!]^3$ is replaced by $\neg[\![p]\!]^4 \wedge [\![r]\!]^3$. In general a block proof at a stage is said to be *Dab-complete* iff all *Dab*-formulas that are derivable on the present block analysis are derived in the proof. The fascinating and obvious fact is the following.

Fact 4.10.2 *If a block proof at stage s is Dab-complete, then all block formulas derived at stage s are finally derived on the block analysis defined by s.*

What I mean is this. Consider a *Dab*-complete block proof at a stage. Translate every formula and condition that occurs in the proof by means of the translation function tr.[22] The translations of the block formulas of unmarked

[22]Blocks that occur in previous stages of the proof may be disregarded. Alternatively, where a block A is identified with a block formula B, one may add $\text{tr}(A) \stackrel{\vee}{\equiv} \text{tr}(B)$ to the translated premises.

lines are finally derivable from the translations of the block premise set. It is instructive to consider the translation of stages 4 and 5 of the last example proof.

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4.11 Modifications to Adaptive Logics from Previous Chapters

From what we learned in the present chapter, we can phrase those logics in their official form, viz. in standard format. In Chapter 10, we shall see the importance of this.

Most of the logics were formulated for $\mathcal{L}_s$. We upgrade this to $\mathcal{L}_S$, still requiring that the the premises and conclusion are members of $\mathcal{W}_s$. On the road to the conclusion, classical negation may be used, as suggested in Section 2.5, but this is not necessary. The language schema $\mathcal{L}_m$ of the logics $\mathbf{K}^r$ and $\mathbf{K}^m$ will be upgraded to $\mathcal{L}_M$, but the premises and conclusion will belong to $\mathcal{W}_m$. Note, however, that these are combined logics, which will be dealt with in Chapter 6.

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The only classical symbols that officially have to occur in proofs are classical disjunctions; they need to occur in *Dab*-formulas to indicate that the person constructing the proof realizes that a *Dab*-formula is derived. As stated in Section 4.2, abnormalities will as much as possible be phrased in terms of the standard symbols, mainly for aesthetic reasons.

All the rest can be safely left to the reader. In the considered chapters, the lower limit logic, the set of abnormalities, and the strategy are clearly indicated.