Chapter 5

Metatheory of The Standard Format

In this chapter many properties of adaptive logics in standard format are stated and proven. All theorems proven apply to all the adaptive logics. That is the use of the standard format anyway.
Some adaptive logics are border cases, for example those that reduce to Tarski logics and have static proofs. If a property does not hold for all adaptive logics in standard format, I shall prove a theorem that involves a restriction. Thus, not all adaptive logics are non-monotonic, but those for which $Cn_{\mathbf{LLL}}(\Gamma) \subset Cn_{\mathbf{AL}}(\Gamma)$ are.
The order in which theorems are proven is not related to their importance, but to the fact that the proof of some theorems becomes more perspicuous if other theorems are proved first.
If you are reading this book out of interest for the use and nature of adaptive logics, you may skip the metatheoretic proofs. Still it is advisable to carefully consider the theorems themselves because they reveal the central properties of adaptive logics in a clear and concise way.

Notwithstanding all the work that was done on defeasible reasoning forms, many logicians mistrust the enterprise. There are indeed many differences with the simple and straightforward situation that obtains in Tarski logics. Defeasible logics have unusual properties and their proofs—many approaches to defeasible logics never even defined any proofs—are dynamic and so unusual. So it is extremely important to show, by strict and usual metatheoretic means, that adaptive logics have also many nice properties, comforting properties one might say, and that the same holds for dynamic proofs. This is what I shall do in the present chapter. Some readers may think that I am overdoing the matter. They underestimate the depth of the misunderstandings as well as the hard hostility of some traditional logicians.

As in the previous chapter, **AL** is a variable for adaptive logics whereas $\mathbf{AL}^m$ and $\mathbf{AL}^r$ are variables for adaptive logics that have respectively Reliability and Minimal Abnormality as their strategy. As before, $\Gamma^{\check{\neg}}$ denotes $\{\check{\neg}A \mid A \in \Gamma\}$, and analogously for other sets superscripted with $\check{\neg}$.

5.1 Some Facts about Choice Sets

The present section is meant to list some facts about minimal choice sets and to clarify the relations between the minimal *Dab*-consequences of Γ and $\Phi(\Gamma)$.

Remember that a choice set of $\Sigma = \{\Delta_1, \Delta_2, \ldots\}$ contains a member of every $\Delta_i \in \Sigma$ and that a minimal choice set of Σ is a choice set of Σ that is not a proper superset of a choice set of Σ. Let us begin with an obvious fact.

Fact 5.1.1 *If Σ is a set of sets and $\Delta \in \Sigma$, then, for every minimal choice set φ of Σ, $\varphi \cap \Delta \neq \emptyset$.*

Not every Σ has minimal choice sets. If, for example, $\Delta_i = \{i, i+1, i+2, \ldots\}$ for every i, then Σ does not have minimal choice sets—with thanks to Christian Straßer for the example. However, we are here only interested in Σ for which all Δ_i are finite sets.

Fact 5.1.2 *If Σ is a set of finite sets, then Σ has minimal choice sets.*

Let me first show this. Call φ *minimal with respect to* a $\Delta \in \Sigma$ iff no choice set φ' of Σ is such that $\varphi' \subset \varphi$ and $\varphi - \varphi' \subseteq \Delta$. As Δ is finite, there are, for every choice set φ of Σ at most finitely many different choice sets φ' of Σ for which $\varphi' \subset \varphi$ and $\varphi - \varphi' \subseteq \Delta$. So either φ or one of these φ' is a minimal choice set of Σ. In other words, a choice set of Σ that is minimal with respect to Δ. Moreover, as Σ is denumerable, there is a choice set of Σ that is minimal with respect to every $\Delta \in \Sigma$. Suppose that φ is minimal with respect to every $\Delta \in \Sigma$ but is not a minimal choice set of Σ. So there is a choice set φ' of Σ such that $\varphi' \subset \varphi$. It follows that there is a $\Delta_i \in \Sigma$ such that $\varphi' \cap \Delta_i \subset \varphi \cap \Delta_i$. If there are more such Δ_i, chose one and let $\varphi'' = \bigcup\{\varphi \cap (\Delta_j - \Delta_i) \mid j \in \{1, 2, \ldots\}\} \cup (\varphi' \cap \Delta_i)$.[1] Obviously φ'' is choice set of Σ, $\varphi' \subset \varphi'' \subset \varphi$ and $\varphi - \varphi'' \subseteq \Delta$. So φ is not minimal with respect to Δ_i, which contradicts the supposition.

Fact 5.1.3 *If Σ is a set of sets and φ is a minimal choice set of Σ, then, for every $A \in \varphi$, there is a $\Delta \in \Sigma$ for which $\varphi \cap \Delta = \{A\}$.*[2]

If $A \in \varphi$ were not the only member of at least one $\Delta \in \Sigma$, then $\varphi - \{A\}$ would still contain a member of every $\Delta \in \Sigma$ in view of Fact 5.1.1, whence φ would not be a minimal choice set of Σ.

It does not follow from Fact 5.1.3 that a minimal choice set of Σ comprises exactly one element of every member of Σ. Let A, B, C, and D be different formulas (or other entities) and let $\Sigma_1 = \{A, B, C\}, \{A, D\}, \{B, D\}\}$. The minimal choice sets of Σ_1 are $\{A, B\}$, $\{A, D\}$, $\{B, D\}$, and $\{C, D\}$. Three of these contain two elements of a member of Σ_1.

Fact 5.1.4 *If Σ and Σ' are sets of sets, then for every choice set φ of $\Sigma \cup \Sigma'$ there is a choice set ψ of Σ for which $\psi \subseteq \varphi$.*

[1] If $j = i$, then $\varphi \cap (\Delta_j - \Delta_i) = \emptyset$.

[2] A related and obvious fact, which I shall not need, reads: If Σ is a set of sets, φ is a choice set of Σ, and, for every $A \in \varphi$, there is a $\Delta \in \Sigma$ for which $\varphi \cap \Delta = \{A\}$, then, φ is a minimal choice set of Σ.

As this obviously holds for all choice sets of $\Sigma \cup \Sigma'$, it also holds for the minimal ones. Let φ be a minimal choice set of $\Sigma \cup \Sigma'$ and let ψ be a choice set of Σ for which $\psi \subseteq \varphi$. If ψ is not a minimal choice set of Σ, then there is a minimal choice set ψ' of Σ for which $\psi' \subset \psi$, whence $\psi' \subseteq \varphi$. So the following fact obtains:

Fact 5.1.5 *If Σ and Σ' are sets of sets, then, for every minimal choice set φ of $\Sigma \cup \Sigma'$, there is a minimal choice set ψ of Σ for which $\psi \subseteq \varphi$.*

Suppose that every member of Σ' is a superset of a member of Σ and let φ be a minimal choice set of $\Sigma \cup \Sigma'$. In view of Fact 5.1.5, there is a minimal choice set ψ of Σ for which $\psi \subseteq \varphi$. As ψ contains an element of every member of Σ, it contains an element of every member of $\Sigma \cup \Sigma'$ by the supposition. So as φ is a minimal choice set of $\Sigma \cup \Sigma'$, $\varphi = \psi$. In other words, $\Sigma \cup \Sigma'$ and Σ have the same minimal choice sets, as is stated in the following fact.

Fact 5.1.6 *If Σ and Σ' are sets of sets and for every $\Theta \in \Sigma'$ there is a $\Delta \in \Sigma$ for which $\Delta \subseteq \Theta$, then φ is a minimal choice set of $\Sigma \cup \Sigma'$ iff φ is a minimal choice set of Σ.*

So if Σ comprises every Δ for which $Dab(\Delta)$ is a minimal Dab-consequence of Γ and Σ' comprises (any number of) Δ for which $Dab(\Delta)$ is a Dab-consequence of Γ, then the set of minimal choice sets of $\Sigma \cup \Sigma'$ is identical to the set of minimal choice sets of Σ. In other words, it does not make any difference whether $\Phi(\Gamma)$ is defined in terms of the *minimal* Dab-consequences of Γ or in terms of all Dab-consequences of Γ.

Suppose that $\varphi_1, \varphi_2, \ldots$ are the minimal choice sets of Σ. The minimal choice sets of $\Sigma \cup \{\{A_1, \ldots, A_n\}\}$ are the minimal sets among $\varphi_1 \cup \{A_1\}, \ldots, \varphi_1 \cup \{A_n\}, \varphi_2 \cup \{A_1\}, \ldots$. If $\{A_1, \ldots, A_n\} \cap \varphi_k = \emptyset$, then $\varphi_k \cup \{A_1\}, \ldots, \varphi_k \cup \{A_n\}$ are minimal. Actually, there are only two cases in which a $\varphi_k \cup \{A_i\}$ $(1 \leq i \leq n)$ is not a minimal choice set of $\Sigma \cup \{\{A_1, \ldots, A_n\}\}$.

The first case is that some $A_j \in \varphi_k$ $(1 \leq j \leq n)$ whereas $A_i \notin \varphi_k$. Then $\varphi_k \cup \{A_j\} \subset \varphi_k \cup \{A_i\}$ and hence $\varphi_k \cup \{A_i\}$ is not a minimal choice set of $\Sigma \cup \{\{A_1, \ldots, A_n\}\}$. The second case is where both φ_k and φ_j are minimal choice sets of Σ and $\varphi_j \cup \{A_i\} \subset \varphi_k \cup \{A_i\}$. This is only possible if $A_i \in \varphi_j$, $A_i \notin \varphi_k$, and $\varphi_j - \{A_i\} \subset \varphi_k$.[3]

Suppose now that φ_k is a minimal choice set of Σ and no $\varphi_k \cup \{A_i\}$ $(1 \leq i \leq n)$ is a minimal choice set of $\Sigma \cup \{\{A_1, \ldots, A_n\}\}$. It follows that, for every A_i, $A_i \notin \varphi_k$ and there is a minimal choice set φ_j of Σ for which $A_i \in \varphi_j$ and $\varphi_j - \{A_i\} \subseteq \varphi_k$.

This result may be easily generalized to the case in which the minimal choice sets of Σ are compared to the minimal choice sets of $\Sigma \cup \{\Delta\}$. This gives us:

Fact 5.1.7 *If Σ is a set of sets, ψ is a minimal choice set of Σ, and there is no minimal choice set φ of $\Sigma \cup \{\Delta\}$ for which $\psi \subseteq \varphi$, then $\Delta \cap \psi = \emptyset$ and, for every $A \in \Delta$, there is a minimal choice set ψ' of Σ, for which $A \in \psi'$, and $\psi' - \{A\} \supset \psi$.*

Let us consider some simple illustrations. Let $\Sigma = \{\{A, B, C\}, \{B, D\}, \{C, E\}\}$. The minimal choice sets of Σ are $\{A, D, E\}$, $\{B, C\}$, $\{B, E\}$, and $\{C, D\}$. Where

[3]Note that it is not possible that $\varphi_j - \{A_i\} = \varphi_k$.

$\Delta = \{B\}$, the minimal choice sets of $\Sigma \cup \{\Delta\}$ are $\{B, C\}$ and $\{B, E\}$. None of these extends either $\{A, D, E\}$ or $\{C, D\}$ and the reader can easily check that Fact 5.1.7 obtains. Another example is where $\Sigma = \{\{A, B, C\}, \{D, E\}\}$, $\psi = \{C, E\}$ and $\Delta = \{B, D\}$. This is left to the reader.

Let Σ and Σ' be sets of sets. Suppose that, for every minimal choice set ψ of Σ and for every $\Delta \in \Sigma'$, there is a minimal choice set φ of $\Sigma \cup \{\Delta\}$ such that $\psi \subseteq \varphi$. It is easily seen that, for every minimal choice set ψ of Σ, there is a minimal choice set φ of $\Sigma \cup \Sigma'$ such that $\psi \subseteq \varphi$. This is phrased in a more useful way as the following fact.

Fact 5.1.8 *If Σ and Σ' are sets of sets, ψ is a minimal choice set of Σ, and there is no minimal choice set φ of $\Sigma \cup \Sigma'$ for which $\psi \subseteq \varphi$, then there is a $\Delta \in \Sigma'$ such that $\Delta \cap \psi = \emptyset$ and, for every $A \in \Delta$, there is a minimal choice set ψ' of Σ, for which $A \in \psi'$ and $\psi' - \{A\} \subset \psi$.*

Let $\Sigma = \{\{B_1, \ldots, B_n\}\}$ in which $B_i \neq B_j$ for all different $i, j \in \{1, \ldots, n\}$. So the minimal choice sets of Σ are $\{B_1\}$, ..., $\{B_n\}$. Suppose that there is no minimal choice set φ of $\Sigma \cup \Sigma'$ for which $\{B_1\} \subseteq \varphi$—that is, there is no minimal choice set φ of $\Sigma \cup \Sigma'$ for which $B_1 \in \varphi$. So, by Fact 5.1.8, there is a $\Delta \in \Sigma'$ such that $\Delta \cap \{B_1\} = \emptyset$ and, for every $A \in \Delta$, there is a minimal choice set ψ' of Σ, for which $A \in \psi'$ and $\psi' - \{A\} \subset \{B_1\}$. This is obviously only possible iff $\Delta = \{\{B_2, \ldots, B_n\}\}$, in other words if $\{\{B_1, \ldots, B_n\}\}$ is not a minimal member of $\Sigma \cup \Sigma'$, where Δ_1 is a *minimal* member of Σ_1 iff $\Delta_1 \in \Sigma_1$ and there is no $\Delta_2 \in \Sigma_1$ for which $\Delta_2 \subset \Delta_1$.

Fact 5.1.9 *If Σ is a set of sets and Δ is a* minimal *member of Σ, then, for every $A \in \Delta$, there is a minimal choice set φ of Σ such that $A \in \varphi$.*

***Dab*-consequences** The choice sets that interest us in this book are those needed for describing $\Phi(\Gamma)$. So, where $Dab(\Delta_1)$, $Dab(\Delta_2)$, ... are the minimal *Dab*-consequences of Γ, we are interested in the minimal choice sets of $\Sigma = \{\Delta_1, \Delta_2, \ldots\}$. What follows is meant to provide insight, rather than to substantiate metalinguistic proofs.

Let $Dab(\Delta_1)$, $Dab(\Delta_2)$, ..., $Dab(\Theta_1)$, $Dab(\Theta_2)$, ... be all *Dab*-consequences of Γ and let $\Sigma' = \{\Theta_1, \Theta_2, \ldots\}$. An important insight provided by Fact 5.1.6 is that the set of minimal choice sets of Σ is identical to the set of minimal choice sets of $\Sigma \cup \Sigma'$. So $\Phi(\Gamma)$ may be defined from either of these sets.

The set of minimal *Dab*-consequences of Γ is in a specific sense deductively closed. More particularly, if $A \vdash_{\mathbf{LLL}} B$, then $A \check{\vee} C_1 \check{\vee} \ldots \check{\vee} C_n \vdash_{\mathbf{LLL}} B \check{\vee} C_1 \check{\vee} \ldots \check{\vee} C_n$. If $A, B, C_1, \ldots, C_n \in \Omega$, this fact may cause some confusion. It is this confusion that I try to remove below.

For the sake of definiteness, let us consider examples illustrating the behaviour of $\mathbf{CLuN}^m$. This behaviour depends on properties of the standard format and hence is the same for all adaptive logics that follow the Minimal Abnormality strategy. Consider first a premise set that has

$$((p \wedge \neg p) \wedge \neg(p \wedge \neg p)) \check{\vee} (q \wedge \neg q) \check{\vee} (r \wedge \neg r)$$

as a minimal *Dab*-consequence. In this case

$$(p \wedge \neg p) \check{\vee} (q \wedge \neg q) \check{\vee} (r \wedge \neg r)$$

is also a minimal *Dab*-consequence of Γ. One might wonder whether this situation does not cause a problem. Indeed, all models that verify $(p\wedge\neg p)\wedge\neg(p\wedge\neg p)$ also verify $p\wedge\neg p$ and need not verify either $q\wedge\neg q$ or $r\wedge\neg r$. So if the above listed *Dab*-formulas are the only minimal *Dab*-consequences of Γ, the minimal abnormal models of Γ verify respectively (i) $(p\wedge\neg p)\wedge\neg(p\wedge\neg p)$ and $p\wedge\neg p$, (ii) $q\wedge\neg q$, and (iii) $r\wedge\neg r$. But do the minimal choice sets take this into account?

They do. The minimal choice sets of $\{\{(p\wedge\neg p)\wedge\neg(p\wedge\neg p), q\wedge\neg q, r\wedge\neg r\}, \{p\wedge\neg p, q\wedge\neg q, r\wedge\neg r\}\}$ are (i) $\{(p\wedge\neg p)\wedge\neg(p\wedge\neg p), p\wedge\neg p\}$, (ii) $\{q\wedge\neg q\}$, and (iii) $\{r\wedge\neg r\}$ as required. It is instructive to check this. It is also instructive to see why it holds: obviously $\{q\wedge\neg q\}$ and $\{r\wedge\neg r\}$ are minimal choice sets. So combining one of them either with $(p\wedge\neg p)\wedge\neg(p\wedge\neg p)$ or with $p\wedge\neg p$ delivers a choice set that is not minimal. As a result, $(p\wedge\neg p)\wedge\neg(p\wedge\neg p)$ and $p\wedge\neg p$ end up in the same minimal choice set. This is precisely what we need in order to characterize the minimal abnormal models.

The same mechanism guarantees that the minimal choice sets behave as desired with respect to alphabetic variants of existentially closed abnormalities. Suppose that $\exists x(Px\wedge\neg Px)\check{\vee}\exists x(Qx\wedge\neg Qx)$ is a minimal *Dab*-consequence of a premise set Γ. Obviously $\exists x(Px\wedge\neg Px)\check{\vee}\exists y(Qy\wedge\neg Qy)$ is then also a minimal *Dab*-consequence of Γ. Clearly the models that verify $\exists x(Qx\wedge\neg Qx)$ also verify $\exists y(Qy\wedge\neg Qy)$ and *vice versa*—both formulas are **CLuN**-equivalent. Again, minimal choice sets behave exactly as desired. To see this, let us restrict our attention to variants in x and y:

$$\exists x(Px\wedge\neg Px)\check{\vee}\exists x(Qx\wedge\neg Qx)$$
$$\exists x(Px\wedge\neg Px)\check{\vee}\exists y(Qy\wedge\neg Qy)$$
$$\exists y(Py\wedge\neg Py)\check{\vee}\exists x(Qx\wedge\neg Qx)$$
$$\exists y(Py\wedge\neg Py)\check{\vee}\exists y(Qy\wedge\neg Qy)$$

The minimal choice sets of the set of sets of disjuncts of these four formulas are $\{\exists x(Px\wedge\neg Px), \exists y(Py\wedge\neg Py)\}$ and $\{\exists x(Qx\wedge\neg Qx), \exists y(Qy\wedge\neg Qy)\}$. There are four non-minimal choice sets: $\{\exists x(Px\wedge\neg Px), \exists x(Qx\wedge\neg Qx), \exists y(Qy\wedge\neg Qy)\}$, $\{\exists y(Py\wedge\neg Py), \exists x(Qx\wedge\neg Qx), \exists y(Qy\wedge\neg Qy)\}$, $\{\exists x(Qx\wedge\neg Qx), \exists x(Px\wedge\neg Px), \exists y(Py\wedge\neg Py)\}$, and $\{\exists y(Qy\wedge\neg Qy), \exists x(Px\wedge\neg Px), \exists y(Py\wedge\neg Py)\}$. This clearly reveals the underlying mechanism. If a choice set contains a formula and not its variant, then it contains both variants of the other formula, whence it is not minimal. If variants in all variables are considered, the situation remains similar: one obtains the choice set comprising every relettering of $\exists x(Px\wedge\neg Px)$ and the choice set comprising every relettering of $\exists x(Qx\wedge\neg Qx)$.

These are simple examples. However, in view of the aforementioned facts, the insights provided by the examples can easily be generalized to more 'dependencies' between disjuncts of minimal *Dab*-consequences of Γ or to their combination with further minimal *Dab*-consequences. Put a bit less accurately, entailments between abnormalities lead to a multiplication of minimal *Dab*-consequences which causes dependent abnormalities to end up in the same minimal choice sets.

The behaviour of $\Phi_s(\Gamma)$ (in the proof theory) is obviously completely different. A *Dab*-formula may be derived at a stage while the *Dab*-formulas that it entails are not. For this reason, it is better to speed up the proofs as described in the third paragraph following Definition 4.4.2.

zien of definitie niet verzet is; bedoeld is: 2e alinea voor Def.4.4.3

5.2 Strong Reassurance

Graham Priest's **LP**m from [Pri91] is an adaptive logic which is not in standard format because it selects models in terms of properties of the assignment (or interpretation) and not in terms of the formulas verified by the model—see [Bat99c] for a discussion. **LP**m has the odd property that some models are not selected because there are less abnormal models, but that none of the latter are selected either because there are still less abnormal models. So there is an infinite sequence of less and less abnormal models, which means that none of them is selected. This has a number of clear disadvantages. Consider the premise set Γ comprising, for every $n \in \{2, 3, \ldots\}$, the formula $\exists x_1 \ldots \exists x_n((Px_1 \wedge \neg Px_1) \wedge \ldots (Px_n \wedge \neg Px_n) \wedge \neg(x_1 = x_2) \wedge \ldots \wedge \neg(x_1 = x_n) \wedge \neg(x_2 = x_3) \wedge \ldots \wedge \neg(x_{n-1} = x_n))$. The premise set Γ states that there are infinitely many different objects that both have and have not property P. Moreover, Γ does not require that any objects are different of themselves. However, as is shown in [Bat00a], it is a **LP**m-consequence of Γ that a unique object is different from itself and both has and has not property P: $\exists x(\neg x = x \wedge \forall y((Py \wedge \neg Py) \supset y = x))$. The presence of this conclusion is caused by an odd property of **LP**m: the premise set has infinitely many models that require an infinity of objects to both have and not have property P, and no object is different from itself in them, but none of these models is minimally abnormal. Let me briefly show why this is so. Consider models that have the infinite domain $\{o_0, o_1, \ldots\}$. On Priest's standard convention, identity holds only between an object and itself (but an object may also be different from itself). Start with the model in which every object of the domain is P as well as $\neg P$ (and is consistent with respect to every other property). A less abnormal model is the one in which everything except for o_0 is P as well as $\neg P$; an even less abnormal model is the one in which everything except for o_0 and o_1 is P as well as $\neg P$; and so on. For each such model, there is a less abnormal one of the same sort. So there is an infinitely 'descending' sequence of models. The only minimal abnormal models are those in which exactly one object both has and has not property P *and* is different from itself. Note that the premise set indeed has such a model.

What is odd here is that infinitely many models are not selected, viz. all those in which no object is different from itself, but no selected model is less abnormal than those non-selected models because every selected model verifies another abnormality, viz. that an object is different from itself. To avoid such oddities, that a model is not selected should be justified by the presence of a selected less abnormal model. This property was labelled Strong Reassurance, Smoothness, or Stopperedness. I now prove that this property holds for adaptive logics in standard format. The property will play a role in proofs of many subsequent theorems.[4]

Theorem 5.2.1 *If $M \in \mathcal{M}_\Gamma^{\mathbf{LLL}} - \mathcal{M}_\Gamma^m$, then there is a $M' \in \mathcal{M}_\Gamma^m$ such that $Ab(M') \subset Ab(M)$. (Strong Reassurance for Minimal Abnormality.)*

Proof. The theorem holds vacuously if $\mathcal{M}_\Gamma^m = \mathcal{M}_\Gamma^{\mathbf{LLL}}$. So consider a $M \in$

[4]Just for the record: adaptive logics in standard format select all the models in the described sequence as long as no individual constants are mapped on the inconsistent objects.

$\mathcal{M}_\Gamma^{\mathbf{LLL}} - \mathcal{M}_\Gamma^m$. Let $D_1, D_2, \ldots$ be a list of all members of Ω and define:

$$\Delta_0 = \emptyset\,;$$

if $Ab(M') \subseteq Ab(M)$ for some **LLL**-model M' of $\Gamma \cup \Delta_i \cup \{\check{\neg} D_{i+1}\}$, then

$$\Delta_{i+1} = \Delta_i \cup \{\check{\neg} D_{i+1}\}\,,$$

otherwise

$$\Delta_{i+1} = \Delta_i\,;$$

finally

$$\Delta = \Delta_0 \cup \Delta_1 \cup \Delta_2 \cup \ldots$$

The theorem is established by the following three steps.

Step 1: $\Gamma \cup \Delta$ has **LLL**-models. This follows immediately from the construction of Δ and from the compactness of **LLL**.

Step 2: If M' is a model of $\Gamma \cup \Delta$, then $Ab(M') \subseteq Ab(M)$.

Suppose there is a $D_j \in \Omega$ such that $D_j \in Ab(M') - Ab(M)$. Let M'' be a model of $\Gamma \cup \Delta_{j-1}$ for which $Ab(M'') \subseteq Ab(M)$. As $D_j \notin Ab(M)$, $D_j \notin Ab(M'')$. Hence M'' is a model of $\Gamma \cup \Delta_{j-1} \cup \{\check{\neg} D_j\}$ and $Ab(M'') \subseteq Ab(M)$. So $\check{\neg} D_j \in \Delta_j \subseteq \Delta$. As M' is a model of $\Gamma \cup \Delta$, $D_j \notin Ab(M')$. But this contradicts the supposition.

Step 3: Every model of $\Gamma \cup \Delta$ is a minimal abnormal model of Γ.

Suppose that M' is a model of $\Gamma \cup \Delta$, but is not a minimal abnormal model of Γ. Hence, by Definition 4.5.3, there is a model M'' of Γ for which $Ab(M'') \subset Ab(M')$.

It follows that M'' is a model of $\Gamma \cup \Delta$. If it were not, then, as M'' is a model of Γ, there is a $\check{\neg} D_j \in \Delta$ such that M' verifies $\check{\neg} D_j$ and M'' falsifies $\check{\neg} D_j$. But then M' falsifies D_j and M'' verifies D_j, which is impossible in view of $Ab(M'') \subset Ab(M')$.

Consider any $D_j \in Ab(M') - Ab(M'') \neq \emptyset$. As M'' is a model of $\Gamma \cup \Delta_{j-1}$ that falsifies D_j, it is a model of $\Gamma \cup \Delta_{j-1} \cup \{\check{\neg} D_j\}$. As $Ab(M'') \subset Ab(M')$ and $Ab(M') \subseteq Ab(M)$, $Ab(M'') \subset Ab(M)$. It follows that $\Delta_j = \Delta_{j-1} \cup \{\check{\neg} D_j\}$ and hence that $\check{\neg} D_j \in \Delta$. But then $D_j \notin Ab(M')$. Hence, $Ab(M'') = Ab(M')$. So the supposition leads to a contradiction. ■

In order to show that Strong Reassurance also holds for adaptive logics that have Reliability as their strategy, I need two lemmas, which will also play a role in later sections. Recall that, where $Dab(\Delta_1)$, $Dab(\Delta_2)$, ... are the minimal *Dab*-consequences of Γ, $\Phi(\Gamma)$ is the set of minimal choice sets of $\{\Delta_1, \Delta_2, \ldots\}$.

If Γ has no **LLL**-models, it has no $\mathbf{AL}^m$-models. So, by the soundness and completeness of **LLL** with respect to its semantics, $\Gamma \vdash_{\mathbf{LLL}} A$ for all A. It follows that $\Phi(\Gamma) = \{\Omega\}$. The more interesting case is when Γ has models.

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Lemma 5.2.1 *If Γ has* **LLL***-models, then $\varphi \in \Phi(\Gamma)$ iff $\varphi = Ab(M)$ for some $M \in \mathcal{M}_\Gamma^m$.*

Proof. Suppose that Γ has **LLL**-models. As every **LLL**-model M of Γ verifies all minimal *Dab*-consequences of Γ, Fact 5.1.3 gives us:

(†) Every **LLL**-model M of Γ verifies the members of a $\varphi \in \Phi(\Gamma)$.

Suppose that, for some $\varphi \in \Phi(\Gamma)$, $\Gamma \cup (\Omega - \varphi)^{\check{\neg}}$ has no **LLL**-model. By the compactness of **LLL**, there is a finite $\Gamma' \subseteq \Gamma$ and a finite $\Delta \subseteq (\Omega - \varphi)$ such that $\Gamma' \cup \Delta^{\check{\neg}}$ has no **LLL**-model. But then, by **CL**-properties, $\Gamma' \vDash_{\mathbf{LLL}} Dab(\Delta)$ and, by the monotonicity of **LLL**, $\Gamma \vDash_{\mathbf{LLL}} Dab(\Delta)$, which contradicts $\Delta \subseteq (\Omega - \varphi)$. So, for every $\varphi \in \Phi(\Gamma)$, $\Gamma \cup (\Omega - \varphi)^{\check{\neg}}$ has a **LLL**-model M and, as M verifies φ in view of (†), $Ab(M) = \varphi$.

We have established that, for every $\varphi \in \Phi(\Gamma)$, there is a **LLL**-model M of Γ for which $Ab(M) = \varphi$. But then, in view of (†), every **LLL**-model M of Γ for which $Ab(M) \in \Phi(\Gamma)$ is a minimal abnormal model of Γ and no other **LLL**-model of Γ is a minimal abnormal model of Γ. ■

An immediate consequence of this lemma is the following corollary, which warrants that all minimal abnormal models of Γ are reliable models of Γ.

Corollary 5.2.1 $\bigcup \Phi(\Gamma) = U(\Gamma)$.

Lemma 5.2.2 $\mathcal{M}^{\mathbf{ULL}}_{\Gamma} \subseteq \mathcal{M}^{m}_{\Gamma} \subseteq \mathcal{M}^{r}_{\Gamma} \subseteq \mathcal{M}^{\mathbf{LLL}}_{\Gamma}$.

Proof. Immediate in view of Lemmas 4.6.1 and 5.2.1 and Definitions 4.5.1 and 4.5.3. ■

As all minimal abnormal models of Γ are reliable models of Γ, every unreliable **LLL**-model of Γ is more abnormal than some minimal abnormal model.

Theorem 5.2.2 *If $M \in \mathcal{M}^{\mathbf{LLL}}_{\Gamma} - \mathcal{M}^{r}_{\Gamma}$, then there is a $M' \in \mathcal{M}^{r}_{\Gamma}$ such that $Ab(M') \subset Ab(M)$. (Strong Reassurance for Reliability.)*

Proof. Immediate in view of Theorem 5.2.1 and Lemma 5.2.2. ■

Corollary 5.2.2 *If Γ has* **LLL***-models, Γ has* $\mathbf{AL}^m$*-models as well as* $\mathbf{AL}^r$*-models. (Reassurance.)*

Reassurance warrants that a premise set that has lower limit models, and hence is not **LLL**-trivial, also has adaptive models and hence is not **AL**-trivial. This is obviously an important property. If the transition from the lower limit logic to the adaptive logic would result in a trivial consequence set, the transition would obviously not be justifiable.

5.3 Soundness and Completeness

The soundness and completeness of adaptive logics with respect to their semantics is proved by relying on the soundness and completeness of the lower limit logics with respect to their semantics. The underlying idea is that the adaptive derivability relation and the adaptive semantic consequence relation are both characterized in terms of the lower limit.

Lemma 5.3.1 *If A is finally derived at line i of an* $\mathbf{AL}^r$*-proof from Γ, and Δ is the condition of line i, then $\Delta \cap U(\Gamma) = \emptyset$.*

Proof. Suppose that the antecedent is true but that $\Delta \cap U(\Gamma) \neq \emptyset$. Then there is a minimal *Dab*-consequence of Γ, say $Dab(\Delta')$, for which $\Delta \cap \Delta' \neq \emptyset$. So the $\mathbf{AL}^r$-proof from Γ has an extension in which $Dab(\Delta')$ is derived (on the condition $\emptyset$). But then, where s is the last stage of the extension, $\Delta' \subseteq U_s(\Gamma)$ and $\Delta \cap U_s(\Gamma) \neq \emptyset$, whence line i is marked at stage s in view of Definition 4.4.1. As $Dab(\Delta')$ is a minimal Dab-consequence of Γ, $\Delta' \subseteq U_{s'}(\Gamma)$ for all stages succeeding s. So the extension has no further extension in which line i is unmarked. In view of Definition 4.4.5, this contradicts that A is finally derived at line i of the $\mathbf{AL}^r$-proof from Γ. ∎

In the following two theorems, it is obviously possible that Δ is empty. Remember that, if that is so, $A \check{\vee} Dab(\Delta)$ is identical to A.

Theorem 5.3.1 *$\Gamma \vdash_{\mathbf{AL}^r} A$ iff there is a finite $\Delta \subset \Omega$ for which $\Gamma \vdash_{\mathbf{LLL}} A \check{\vee} Dab(\Delta)$ and $\Delta \cap U(\Gamma) = \emptyset$.*

Proof. $\Rightarrow$ Suppose that $\Gamma \vdash_{\mathbf{AL}^r} A$. So A is finally derived on line i of an $\mathbf{AL}^r$-proof from Γ. Let Δ be the condition of line i. But then $\Gamma \vdash_{\mathbf{LLL}} A \check{\vee} Dab(\Delta)$ by Lemma 4.4.1 and $\Delta \cap U(\Gamma) = \emptyset$ by Lemma 5.3.1.

$\Leftarrow$ Suppose that there is a finite $\Delta \subset \Omega$ such that $\Gamma \vdash_{\mathbf{LLL}} A \check{\vee} Dab(\Delta)$ and $\Delta \cap U(\Gamma) = \emptyset$. As **LLL** is compact, there is a finite $\mathbf{AL}^r$-proof from Γ (containing only applications of Prem and RU) in which $A \check{\vee} Dab(\Delta)$ is derived on the condition $\emptyset$. By an application of RC, a line i can be added that has A as its formula and Δ as its condition and this line is unmarked.[5]

If line i is marked in a finite extension of this proof, there are one or more $\Theta \subset \Omega$ such that $\Theta \cap \Delta \neq \emptyset$ and $Dab(\Theta)$ is derived on the condition $\emptyset$. As $\Delta \cap U(\Gamma) = \emptyset$, there is, for each such Θ, a $\Theta' \subset \Theta - \Delta$ for which $\Gamma \vdash_{\mathbf{LLL}} Dab(\Theta')$. So the extension can be further extended in such a way that, for each such Θ, $Dab(\Theta')$ occurs on the condition $\emptyset$, whence line i is unmarked. But then A is finally derived at line i in view of Definition 4.4.5. ∎

Theorem 5.3.2 *$\Gamma \vDash_{\mathbf{AL}^r} A$ iff there is a finite $\Delta \subset \Omega$ for which $\Gamma \vDash_{\mathbf{LLL}} A \check{\vee} Dab(\Delta)$ and $\Delta \cap U(\Gamma) = \emptyset$.*

Proof. $\Rightarrow$ Suppose that $\Gamma \vDash_{\mathbf{AL}^r} A$, whence all members of $\mathcal{M}^r_\Gamma$ verify A. So $\Gamma \cup (\Omega - U(\Gamma))^{\check{\neg}} \vDash_{\mathbf{LLL}} A$. As **LLL** is compact, $\Gamma' \cup \Delta^{\check{\neg}} \vDash_{\mathbf{LLL}} A$ for a finite $\Gamma' \subset \Gamma$ and a finite $\Delta \subset \Omega$. But then, by **CL**, $\Gamma' \vDash_{\mathbf{LLL}} A \check{\vee} Dab(\Delta)$. So, as **LLL** is monotonic, $\Gamma \vDash_{\mathbf{LLL}} A \check{\vee} Dab(\Delta)$.

$\Leftarrow$ Suppose there is a $\Delta \subset \Omega$ for which $\Gamma \vDash_{\mathbf{LLL}} A \check{\vee} Dab(\Delta)$ and $\Delta \cap U(\Gamma) = \emptyset$. $\Gamma \vDash_{\mathbf{AL}^r} A$ holds vacuously if $\mathcal{M}^{\mathbf{LLL}}_\Gamma = \emptyset$. So suppose that $\mathcal{M}^{\mathbf{LLL}}_\Gamma \neq \emptyset$ and that all members of $\mathcal{M}^{\mathbf{LLL}}_\Gamma$ verify $A \check{\vee} Dab(\Delta)$. By Theorem 5.2.2, $\mathcal{M}^r_\Gamma \neq \emptyset$. As $\Delta \cap U(\Gamma) = \emptyset$, all $\mathbf{AL}^r$-models of Γ falsify $Dab(\Delta)$. So all $\mathbf{AL}^r$-models of Γ verify A. ∎

As **LLL** is supposed to be sound and complete with respect to its semantics, Theorems 5.3.1 and 5.3.2 give us:

A strategy x will be said to be *adequate* iff $\mathbf{AL}^x$ is sound and complete with respect to its semantics, in other words iff $\Gamma \vdash_{\mathbf{AL}^x} A$ iff $\Gamma \vDash_{\mathbf{AL}^x} A$.

Corollary 5.3.1 *$\Gamma \vdash_{\mathbf{AL}^r} A$ iff $\Gamma \vDash_{\mathbf{AL}^r} A$. (Adequacy for Reliability.)*

Theorem 5.3.3 *$\Gamma \vdash_{\mathbf{AL}^m} A$ iff, for every $\varphi \in \Phi(\Gamma)$, there is a finite $\Delta \subset \Omega$ such that $\Delta \cap \varphi = \emptyset$ and $\Gamma \vdash_{\mathbf{LLL}} A \check{\vee} Dab(\Delta)$.*

[5] If a Dab-formula $Dab(\Theta)$ occurs in the proof, Θ is a singleton. It holds for all those Θ that $\Theta \subseteq U(\Gamma)$. So $\Delta \cap U(\Gamma) = \emptyset$ warrants that $\Delta \cap \Theta = \emptyset$.

Proof. $\Rightarrow$ Suppose that the antecedent is true. By Definitions 4.4.4 and 4.4.5 an $\mathbf{AL}^m$-proof from Γ contains a line i that has A as its formula and a $\Theta \subset \Omega$ as its condition, line i is unmarked, and every (possibly infinite) extension of the proof in which line i is marked may be further extended in such a way that line i is unmarked.

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Suppose next that the consequent is false and that all minimal *Dab*-consequences of Γ are derived in the extension of the proof. So, where s is the stage of the extension, $\Phi_s(\Gamma) = \Phi(\Gamma)$. As the consequent is false, there is a $\varphi \in \Phi_s(\Gamma)$ such that, for every $\Delta \subset \Omega$, if $\Gamma \vdash_{\mathbf{LLL}} A \check{\vee} Dab(\Delta)$, then $\Delta \cap \varphi \neq \emptyset$. So, in view of Definition 4.4.2, line i is marked in the extension of the extension, which contradicts the first supposition in view of the previous paragraph.

$\Leftarrow$ Suppose that, for every $\varphi \in \Phi(\Gamma)$, there is a $\Delta \subset \Omega$ such that $\Delta \cap \varphi = \emptyset$ and $\Gamma \vdash_{\mathbf{LLL}} A \check{\vee} Dab(\Delta)$. By Lemma 4.9.1, there is a finite proof in which A is derived at an unmarked line that has any of those Δ as its condition. Any extension of this proof can be further extended to a (possibly infinite) $\mathbf{AL}^m$-proof from Γ in which (i) every minimal *Dab*-consequences of Γ is derived on the condition $\emptyset$ and (ii) for every $\varphi \in \Phi(\Gamma)$, A is derived on a condition $\Delta \subset \Omega$ for which $\Delta \cap \varphi = \emptyset$. ■

Theorem 5.3.4 $\Gamma \vdash_{\mathbf{AL}^m} A$ *iff* $\Gamma \vDash_{\mathbf{AL}^m} A$. *(Adequacy for Minimal Abnormality.)*

Proof. Each of the following are equivalent:
(1) $\Gamma \vdash_{\mathbf{AL}^m} A$.
By Theorem 5.3.3:
(2) For every $\varphi \in \Phi(\Gamma)$, there is a $\Delta \subset \Omega$ such that $\Delta \cap \varphi = \emptyset$ and $\Gamma \vdash_{\mathbf{LLL}} A \check{\vee} Dab(\Delta)$.
By the Soundness and Completeness of **LLL**:
(3) For every $\varphi \in \Phi(\Gamma)$, there is a $\Delta \subset \Omega$ such that $\Delta \cap \varphi = \emptyset$ and $\Gamma \vDash_{\mathbf{LLL}} A \check{\vee} Dab(\Delta)$.
By Lemma 5.2.1:
(4) For every $M \in \mathcal{M}^m_\Gamma$, there is a $\Delta \subset \Omega$ such that $\Delta \cap Ab(M) = \emptyset$ and $\Gamma \vDash_{\mathbf{LLL}} A \check{\vee} Dab(\Delta)$.
By **CL**:
(4) Every $M \in \mathcal{M}^m_\Gamma$ verifies A.
By Definition 4.5.4:
(4) $\Gamma \vDash_{\mathbf{AL}^m} A$.
■

In view of Corollary 5.3.1 and Theorem 5.3.4, Corollary 5.2.2 gives us:

Corollary 5.3.2 *If* $Cn_{\mathbf{LLL}}(\Gamma)$ *is non-trivial, then* $Cn_{\mathbf{AL}^m}(\Gamma)$ *and* $Cn_{\mathbf{AL}^r}(\Gamma)$ *are non-trivial. (Syntactic Reassurance)*

As soundness and completeness was presupposed for the lower limit logic and was proved for all adaptive logics (in standard format) and for the upper limit logic, Lemma 5.2.2 gives at once another corollary.

Corollary 5.3.3 $Cn_{\mathbf{LLL}}(\Gamma) \subseteq Cn_{\mathbf{AL}^r}(\Gamma) \subseteq Cn_{\mathbf{AL}^m}(\Gamma) \subseteq Cn_{\mathbf{ULL}}(\Gamma)$.

Recall that, as the language is not specified, the expression holds for $\mathcal{L}$ as well as for $\mathcal{L}_+$, but obviously not for mixed cases.

5.4 The Adequacy of Final Derivability

In Section 4.4, that A is finally derivable from Γ (Definition 4.4.5) was defined in terms of A being finally derived in a proof from Γ (Definition 4.4.4). The latter definition requires the existence of a *finite* proof in which A is derived from Γ on an unmarked line l and that has the following property: if line l is marked in an extension of the proof, then there is a further extension (of the previous extension) in which line l is unmarked. I claimed final derivability, thus defined, is extensionally equivalent to: the existence of a (possibly infinite) proof from Γ in which A is derived at an unmarked line l' and that is stable with respect to line l' (line l' is unmarked in every extension of *this* proof). The trouble with the latter notion is that, for some A and Γ, the proof is necessarily infinite. This was the reason why I proceeded in terms of Definitions 4.4.4 and 4.4.5. Of course, I still have to show that these definitions guarantee the existence of a proof from Γ that is stable with respect to an unmarked line at which A is derived. This is what I shall establish in the present section.

Lemma 5.4.1 *If $\Gamma \vdash_{\mathbf{AL}^r} A$, then there is an $\mathbf{AL}^r$-proof from Γ in which A is derived on an unmarked line and that is stable with respect to that line.*

Proof. Suppose that $\Gamma \vdash_{\mathbf{AL}^r} A$. By Theorem 5.3.1 there is a (finite) $\Delta \subset \Omega$ for which $\Gamma \vdash_{\mathbf{LLL}} A \check{\vee} Dab(\Delta)$ and $\Delta \cap U(\Gamma) = \emptyset$. As **LLL** has static proofs, there is a finite $\mathbf{AL}^r$-proof in which $A \check{\vee} Dab(\Delta)$ is derived on the condition $\emptyset$. From this, A is derived on the condition $Dab(\Delta)$ (in one step by RC), say on line i. Let this be an $\mathbf{AL}^r$-proof at the finite stage s and call this proof p_0.

There are only countably many minimal Dab-consequences of Γ, say $Dab(\Delta_1)$, $Dab(\Delta_2), \ldots$. For each of these, there is a finite **AL**-proof, call it p_i, in which $Dab(\Delta_i)$ is derived on the condition $\emptyset$.

Consider the proof p' of which the last stage, call it s', is the concatenation $\langle \mathsf{p}_0, \mathsf{p}_1, \mathsf{p}_2, \ldots \rangle$. As all minimal Dab-consequences of Γ have been derived on the condition $\emptyset$ at stage s', $U_{s'}(\Gamma) = U(\Gamma)$. As $\Delta \cap U(\Gamma) = \emptyset$, line i is unmarked. Moreover, as all minimal Dab-consequences of Γ have been derived on the condition $\emptyset$ in s', $U_{s''}(\Gamma) = U_{s'}(\Gamma) = U(\Gamma)$ for every extension s'' of s'. So line i is unmarked in every extension s'' of s', which means that p' is stable with respect to line i. ∎

For some Γ, $\Phi(\Gamma)$ is uncountable—see Section 5.4 for an example. However, the set of Δ such that, for some $\varphi \in \Phi(\Gamma)$, $\Delta \cap \varphi = \emptyset$ and $\Gamma \vdash_{\mathbf{LLL}} A \check{\vee} Dab(\Delta)$, is a countable set—each of *these* Δ is a finite set of formulas. Moreover, for each such Δ, there is a finite proof of $A \check{\vee} Dab(\Delta)$. Let $\{\mathsf{p}'_1, \mathsf{p}'_2, \ldots\}$ be the countable set of these proofs. The proof of Lemma 5.4.2 proceeds exactly as that of Lemma 5.4.1, except that we now define p' as a proof that has as stage s' the concatenation $\langle \mathsf{p}_1, \mathsf{p}'_1, \mathsf{p}_2, \mathsf{p}'_2, \ldots \rangle$, which warrants that $\Phi_{s'}(\Gamma) = \Phi(\Gamma)$ and that, for every extension s'' of s', $\Phi_{s''}(\Gamma) = \Phi_{s'}(\Gamma) = \Phi(\Gamma)$.

Lemma 5.4.2 *If $\Gamma \vdash_{\mathbf{AL}^m} A$, then there is an $\mathbf{AL}^m$-proof from Γ in which A is derived on an unmarked line and that is stable with respect to that line.*

Whether the third element of an adaptive logic is Reliability or Minimal Abnormality, the following lemma holds.

Lemma 5.4.3 *If A is derived on an unmarked line of an* **AL**-*proof from* Γ *that is stable with respect to that line, then* $\Gamma \vdash_{\mathbf{AL}} A$

Proof. Suppose that the antecedent is true. As the unmarked line on which A is derived will not be marked in any extension of the proof, A is finally **AL**-derived in this proof. ■

Theorem 5.4.1 $\Gamma \vdash_{\mathbf{AL}} A$ *iff A is derived on an unmarked line of an* **AL**-*proof from* Γ *that is stable with respect to that line.*

Proof. Immediate in view of Lemmas 5.4.1, 5.4.2, and 5.4.3. ■

In other words, Definition 4.4.5 is adequate in that it warrants the existence of a (possibly infinite) proof from Γ that is stable with respect to an unmarked line at which A is derived.

As some people are hard to convince, let me show explicitly, although somewhat superfluously, that the proof mentioned in Definition 4.4.4 is justifiedly taken to be finite. Obviously, an **AL**-proof is *finite* iff each stage of the proof is a finite list of formulas.

Theorem 5.4.2 *If* $\Gamma \vdash_{\mathbf{AL}} A$*, then A is finally derived on a line of a finite* **AL**-*proof from* Γ.

Proof. Suppose that the antecedent is true. If the strategy is Reliability, there is a finite $\Delta \subset \Omega$ such that $\Gamma \vdash_{\mathbf{LLL}} A \check{\vee} Dab(\Delta)$ and $\Delta \cap U(\Gamma) = \emptyset$ (by Theorem 5.3.1). If the strategy is Minimal Abnormality, there is a $\Delta \subset \Omega$ such that $\Gamma \vdash_{\mathbf{LLL}} A \check{\vee} Dab(\Delta)$ and $\Delta \cap \varphi = \emptyset$ for some $\varphi \in \Phi(\Gamma)$ (by Theorem 5.3.3).

As **LLL** is compact, there is a finite $\Gamma' \subseteq \Gamma$ for which $\Gamma' \vdash_{\mathbf{LLL}} A \check{\vee} Dab(\Delta)$. So there is a finite **AL**-stage from Γ' in which occur all members of Γ' followed by a line in which A is derived on the condition Δ by application of RC. As $\Gamma \vdash_{\mathbf{AL}} A$, A is finally derived in this proof in view of definitions 4.4.1, 4.4.2, 4.4.4, and 4.4.5. ■

Maybe the last paragraph of the proof goes a bit too quick. There is a stage at which A is derived on the condition Δ, say on line l, and at which no *Dab*-formulas are derived that require line l to be marked. This is warranted by the fact that *Dab*-formulas are defined in terms of classical disjunction, which does not occur in the premises—see also Section 4.9.3. Moreover, whenever the stage is extended in such a way that line l is marked, the extension is bound to contain certain minimal *Dab*-formulas that are not minimal *Dab*-consequences of Γ. This follows from the fact that $\Delta \cap U(\Gamma) = \emptyset$, respectively $\Delta \cap \varphi = \emptyset$. So, for every *Dab*-formula a disjunct of which is a member of Δ, the *Dab*-formula resulting from deleting this disjunct is also a *Dab*-consequence of Γ. By deriving this result in the extension of the extension the original *Dab*-formula is no longer minimal (in the extension of the extension).

Minimal Abnormality is a computationally complex strategy. One of the indications for this is that Definition 4.4.4 would be inadequate if the extensions were not allowed to be infinite. Let me give an example to illustrate this. For the sake of definiteness, I shall consider the adaptive logic $\mathbf{CLuN}^m$, but the example is easily generalized to any adaptive logic that has Minimal Abnormality as its strategy by choosing a denumerable set of independent abnormalities. So, for

$\mathbf{CLuN}^m$, let $\Gamma = \{(p_i \wedge \neg p_i) \vee (p_j \wedge \neg p_j) \mid i, j \in \{1, 2, \ldots\}; i \neq j\}$, $\Delta = \{q \vee (p_i \wedge \neg p_i) \mid i \in \{1, 2, \ldots\}\}$, $\Delta' = \{q \vee (p_i \wedge \neg p_i) \mid i \in \{2, 3, \ldots\}\}$, and $\Sigma = \{p_i \wedge \neg p_i \mid i \in \{1, 2, \ldots\}\}$. For every $\mathbf{CLuN}^m$-model M of $\Gamma \cup \Delta$, $Ab(M) = \Sigma - \{p_i \wedge \neg p_i\}$ for an $i \in \{1, 2, \ldots\}$; similarly for every $\mathbf{CLuN}^m$-model M of $\Gamma \cup \Delta'$. It follows that all $\mathbf{CLuN}^m$-models of $\Gamma \cup \Delta$ verify q whereas some $\mathbf{CLuN}^m$-models of $\Gamma \cup \Delta'$ falsify q, viz. the models M for which $Ab(M) = \Sigma - \{p_1 \wedge \neg p_1\}$. The only way to ensure that q is finally derived from $\Gamma \cup \Delta$ but not from $\Gamma \cup \Delta'$ is by allowing the extensions in Definition 4.4.4 to be infinite. If q is derived on a condition $p_i \wedge \neg p_i$ ($i \in \{2, 3, \ldots\}$) at a line l in a proof from $\Gamma \cup \Delta$, then even an infinite extension of the proof can be further extended in such a way that line l is unmarked. Of course, the example is a complicated one, but it is a logicians fate to take all possibilities into account.

Fortunately, that matter is drastically simpler for the Reliability strategy. I now demonstrate that the extensions mentioned in Definition 4.4.4 may be restricted to finite extensions if Reliability is the strategy.

Theorem 5.4.3 *If the strategy is Reliability, Definitions 4.4.4 and 4.4.5 are still adequate if the extensions mentioned in Definition 4.4.4 are finite.*

Proof. Case 1: $\Gamma \vdash_{\mathbf{AL}^r} A$. Let A be finally derived on line i in an $\mathbf{AL}^r$-proof from Γ, let Δ be the condition of line i, and let s be the last stage of this proof. Consider a finite extension s' of s in which line i is marked. Stage s' counts at most finitely many minimal Dab-formulas, say $Dab(\Theta_1), \ldots, Dab(\Theta_n)$, for which $\Theta_i \cap \Delta \neq \emptyset$ $(1 \leq i \leq n)$. In view of Definitions 4.4.1, 4.4.4, and 4.4.5, there is, for each of these Θ_i, a $\Theta'_i \subset \Theta_i$ such that $\Gamma \vdash_{\mathbf{AL}} Dab(\Theta'_i){:}\emptyset$ and $\Theta'_i \cap \Delta = \emptyset$. Append the last stage of the proof of each of these $Dab(\Theta_i){:}\emptyset$ to s' and let the result be s''. Stage s'' counts finitely many lines and $\Delta \cap U_{s''}(\Gamma) = \emptyset$.

Case 2: $\Gamma \nvdash_{\mathbf{AL}^r} A$. In view of Theorem 5.3.1 it holds for all $\Delta \subset \Omega$ that $\Delta \cap U(\Gamma) \neq \emptyset$ if $\Gamma \vdash_{\mathbf{LLL}} A \check{\vee} Dab(\Delta)$. Suppose that A has been derived on the condition Δ on a line, say i, of a finite $\mathbf{AL}^r$-proof from Γ and that the last stage of this proof is s. It follows that there is a minimal Dab-consequence $Dab(\Theta)$ of Γ for which $\Theta \cap \Delta \neq \emptyset$. As $\Gamma \vdash_{\mathbf{LLL}} Dab(\Theta)$, Θ can be derived on the condition $\emptyset$ in a finite extension s' of s and there is no extension of s' in which line i is unmarked. ■

This basically shows that Reliability is drastically simpler from a computational point of view than Minimal abnormality. We shall later see the importance of this.

Before closing this section, let me deal with a further worry some readers might have: for some Γ, $\Phi(\Gamma)$ is uncountable. I shall first show that this is indeed the case. Let the logic be $\mathbf{CLuN}^m$ and let $\Gamma_6 = \{(p_i \wedge \neg p_i) \vee (p_{i+1} \wedge \neg p_{i+1}) \mid i \in \{1, 3, 5, \ldots\}\}$. It is easily seen that $\varphi \in \Phi(\Gamma_6)$ iff, for every $i \in \{1, 3, 5, \ldots\}$, φ contains either $p_i \wedge \neg p_i$ or $p_{i+1} \wedge \neg p_{i+1}$ but not both. Consider the tables in Figure 5.1. The left side table represents a (vertical) list of infinite (horizontal) lists of 0s and 1s. The vertical list is incomplete because the number of horizontal lists is known to be uncountable (by Cantor's diagonal method, which is explained in any decent logic handbook). Where a horizontal list in the left side table consists of

$$i_0, i_1, i_2, \ldots,$$

0	0	0	0	...
1	0	0	0	...
0	1	0	0	...
1	1	0	0	...
⋮				

1	3	5	7	...
2	3	5	7	...
1	3	5	7	...
2	4	5	7	...
⋮				

Figure 5.1: Uncountable $\Phi(\Gamma)$

the corresponding list in the right side table is defined by

$$(2 \times 0) + 1 + i_0, (2 \times 1) + 1 + i_1, (2 \times 2) + 1 + i_2, \ldots .$$

Two members of the right side vertical list are different from each other whenever the corresponding members of the left side list are different. So the right side vertical list is also uncountable. From every horizontal list $j_0, j_1, j_2, \ldots$ in the right side table, define a set of formulas $\{p_{j_0} \wedge \neg p_{j_0}, p_{j_1} \wedge \neg p_{j_1}, p_{j_2} \wedge \neg p_{j_2}, \ldots\}$. There are uncountably many such sets. Moreover, every such set is a member of $\Phi(\Gamma_6)$. So we have established Theorem 5.4.4.

Theorem 5.4.4 *For some premise sets $\Gamma \in \mathcal{W}_s$, $\Phi(\Gamma)$ is uncountable.*

This theorem is no reason for worry about dynamic proofs and it does not interfere with any metatheoretic proof in this book. Although $\Phi(\Gamma)$ may be uncountable, the sets $\mathcal{W}$ and $\mathcal{W}_+$ are countable. So (i) premise sets that extend Γ_6 have only countably many *Dab*-consequences, (ii) if A is a $\mathbf{CLuN}^m$-consequence of such a premise set, a countable set of Δ is sufficient to have, for each $\varphi \in \Phi(\Gamma)$, a Δ for which $A \check{\vee} Dab(\Delta)$ is **CLuN**-derivable from the premise set and $\Delta \cap \varphi = \emptyset$, and (iii) there is no need to ever refer to proofs that contain uncountably many lines.

5.5 Punctual Uniformity

This is a very short section. It is a separate section nevertheless because I got bored, over the years, to answer the 'argument' that adaptive logics are not Uniform and 'hence' are not formal logics. In the paragraphs following Corollary 1.5.3, we have seen that there is a popular concept, which I called propositional uniformity and is linked to the Uniform Substitution rule, but that the more general concept, which I called uniformity, is relative to the chosen metalanguage. Of course, every specific choice of a metalanguage defines a specific concept of uniformity.

Fact 5.5.1 *Some adaptive logics are not propositionally uniform.*

Here is a ready example for $\mathbf{CLuN}^m$. Clearly $p \vee q, \neg q \vdash_{\mathbf{CLuN}^m} p$ but $(r \wedge \neg r) \vee q, \neg q \nvdash_{\mathbf{CLuN}^m} r \wedge \neg r$, notwithstanding the fact that the latter expression is the result of systematically replacing in the former expression the sentential letter p by the formula $r \wedge \neg r$. It is instructive to consider a further example: $p \vee q, \neg q, r \vdash_{\mathbf{CLuN}^m} p$ but $p \vee q, \neg q, q \nvdash_{\mathbf{CLuN}^m} p$, notwithstanding the fact that the latter expression is the result of replacing in the former expression the sentential letter r by the formula q. This example is instructive because it shows a stronger fact.

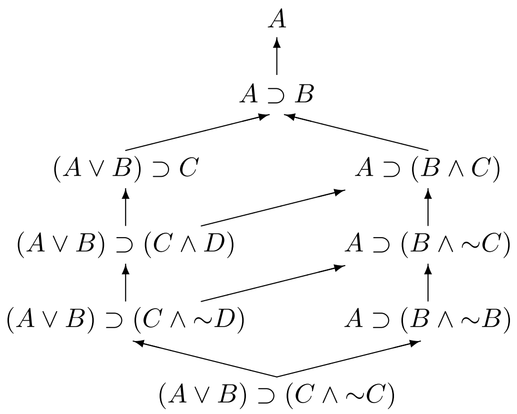

Figure 5.2: The forms of $(p \vee q) \supset (r \wedge {\sim}r)$

Fact 5.5.2 *Some adaptive logics are not uniform.*

Indeed, even the metalanguage contains metavariables for all kinds of non-logical symbols that occur in $\mathcal{L}_s$, the most specific form of $p \vee q, \neg q, r \vdash_{\mathbf{CLuN}^m} p$ is $\sigma_1 \vee \sigma_2, \neg\sigma_2, \sigma_3 \vdash_{\mathbf{CLuN}^m} \sigma_1$—the σ_i are metavariables for sentential letters—and $p \vee q, \neg q, q \vdash_{\mathbf{CLuN}^m} p$ has that form but is false. So $\sigma_1 \vee \sigma_2, \neg\sigma_2, \sigma_3 \vdash_{\mathbf{CLuN}^m} \sigma_1$ is false on the present conventions.

Does this mean that $\mathbf{CLuN}^m$ is not a formal logic? To answer this question, one should go back to the idea: a logic $\mathbf{L}$ is formal iff every true statement $A_1, \ldots, A_n \vdash_{\mathbf{L}} B$ holds true because of its form. This is clearly a sensible requirement. However, the requirement is not adequately expressed by Uniformity. Uniformity does not exhaust the formal character of logics, it is just one of the sufficient conditions for the requirement.

So let us consider the concept of a logical form. Most formulas have several logical forms. Thus $(p \vee q) \supset (r \wedge {\sim}r)$ has the nine logical forms shown in Figure 5.2—I skip those n which occur metavariables for sentential letters.[6]

Some of these forms are more specific than others. In the figure, an arrow goes from the more specific form to the next less specific form. Note that every formula has only one *most specific* form. Thus the most specific form of $(p \vee q) \supset (r \wedge {\sim}r)$ is $(A \vee B) \supset (C \wedge {\sim}C)$ (or an isomorphic formula). The formula $(p \vee p) \supset (r \wedge {\sim}r)$ has all the forms displayed in Figure 5.2, but it has moreover the form $(A \vee A) \supset (C \wedge {\sim}C)$ and this is *the* most specific form of this formula. An inference statement $A_1, \ldots, A_n \vdash_{\mathbf{L}} B$ may also have a multiplicity of logical forms, but has only one *most specific* form.

Uniformity comes to this: if the statement $A_1, \ldots, A_n \vdash_{\mathbf{L}} B$ holds true, then every statement that has the most specific form of this inference statement holds also true. So if $A_1, \ldots, A_n \vdash_{\mathbf{L}} B$ holds true holds true and $A'_1, \ldots, A'_n \vdash_{\mathbf{L}} B'$ has the same form, then $A'_1, \ldots, A'_n \vdash_{\mathbf{L}} B'$ has to hold true, even if the latter statement has also more specific forms than the former statement. This is a formal requirement, but it is neither the only possible requirement nor the only sensible one.

[6]The forms of a formula are obtained by analysis and identification—see Section 4.10.

Adaptive logics are formal logics. The easiest argument for this claim is that all rules and definitions of the proof theory refer only to logical forms and that all clauses and definitions of the semantics refer only to logical forms. This is a completely convincing argument.

Of course it is possible to define a notion similar to Uniformity and to show that it holds for adaptive logics. Let us call the notion *Punctual Uniformity.* Consider a metalanguage in which there is a specific kind of metavariables for every kind of non-logical symbols of the object language. Define, for every formula or inferential statement, a *characteristic logical form*, CF, as follows: the CF of an expression is obtained by replacing every letter by a corresponding metavariable in such a way that the same letter is replaced by the same metavariable and distinct letters are replaced by distinct metavariables. Thus the CF of $(p \vee q) \supset (r \wedge \sim r)$ is $(A \vee B) \supset (C \wedge \sim C)$ (or an isomorphic expression) and the CF of the statement $\forall x Px \vdash_{\mathbf{L}} Pa$ is $\forall \alpha \pi \alpha \vdash_{\mathbf{L}} \pi \beta$ (with $\alpha \in \mathcal{V}_s$ and $\beta \in \mathcal{C}_s$ to obtain closed formulas). A logic **L** is punctually uniform iff $A_1, \ldots, A_n \vdash_{\mathbf{L}} B$ holds true just in case every expression that has the same CF holds true. Obviously, the CF of a formula is just its most specific form.

It is obvious that adaptive logics are punctually uniform. This too is sufficient to show that they are formal logics: a statement holds true just in case every statement that has the same characteristic logical form holds true. Punctual Uniformity clearly imposes a weak requirement on logics. It is equally obvious that adaptive logics fulfil stronger formal requirements.[7] There is no need, however, to introduce such requirements in the present context. All I had to show is that adaptive logics are formal logics, and so I did.

Let me add a final comment to convince even the very stubborn. Formal logics obviously do not fulfil all formal requirements one can imagine. So formal requirements have to be justified. That $p \vee q, \neg q, r \vdash_{\mathbf{CLuN}^m} p$ whereas $p \vee q, \neg q, q \nvdash_{\mathbf{CLuN}^m} p$ is not because the first statement contains a r where the latter contains a q, but that a specific inconsistency is **CLuN**derivable from the latter premise set whereas no inconsistency is **CLuN**derivable from the former premise set. So that is a difference and actually a formal difference between the two premise sets.

The reader may want to see an example of a non-formal logic. Here are some: the formula-preferential systems formulated by Arnon Avron and Iddo Lev in [Lev00, AL01] and elsewhere, which are intended as a generalization of $\mathbf{CLuN}^m$.[8] The idea behind formula-preferential systems may be phrased as follows: where Δ is a set of formulas and **L** is a logic that has static proofs, $\Gamma \vdash_{\mathbf{L},\Delta} A$ iff A is true in all **L**-models of Γ that verify a (set-theoretically) minimal number of members of Δ—the notation $\Gamma \vdash_{\mathbf{L},\Delta} A$ is made up by me for the purpose of the present discussion. In [Mak05, p. 31], David Makinson introduces the "default-assumption consequence". Where $\Delta^{\check{\neg}} = \{\check{\neg} A \mid A \in \Delta\}$ this consequence relation comes to $\Gamma \vdash_{\mathbf{CL},\Delta^{\check{\neg}}} A$.

[7] Some of these are stronger with respect to the specified metalanguage. Others are stronger because they may be phrased in a poorer metalanguage, for example one that has only metavariables for individual constants and variables and metavariables for formulas, including mixed cases like $A(\alpha)$.

[8] As will appear in the text, the generalization consists in the fact that the set of abnormalities Ω is replaced by an arbitrary set of formulas. It is apparently impossible to characterize (let alone generalize) along these lines adaptive logics that use the Reliability strategy, or other strategies described in later chapters.

My claim then is that $\Gamma \vdash_{\mathbf{L},\Delta} A$ is not a formal logic. Recall, for a start, that $\mathbf{L}'$ is a logic iff $\mathbf{L}' \colon \wp(\mathcal{W}) \to \wp(\mathcal{W})$ where $\mathcal{W}$ is the set of closed formulas of the considered language and $\wp(\mathcal{W})$ is the power set of $\mathcal{W}$. There are only two possible ways in which one might try to see $\Gamma \vdash_{\mathbf{L},\Delta} A$ as a formal logic. (i) We consider $\Gamma \vdash_{\mathbf{L},\Delta} A$ as a mapping $\mathbf{L} \colon \wp(\mathcal{W}) \times \wp(\mathcal{W}) \to \wp(\mathcal{W})$. The first $\wp(\mathcal{W})$ takes care of Γ, the second $\wp(\mathcal{W})$ of Δ,[9] and the third $\wp(\mathcal{W})$ of the consequence set assigned by $\mathbf{L}$ to $\langle \Gamma, \Delta \rangle$. This construction is fine but it is not a logic $\mathbf{L}'$ for which $\mathbf{L}' \colon \wp(\mathcal{W}) \to \wp(\mathcal{W})$. So, if we take this road, we have to broaden the standard notion of a logic. (ii) We consider $\Gamma \vdash_{\mathbf{L},\Delta} A$ as a mapping $\mathbf{L} \times \wp(\mathcal{W}) \colon \wp(\mathcal{W}) \to \wp(\mathcal{W})$, in which the first $\wp(\mathcal{W})$ takes care of Δ, the second $\wp(\mathcal{W})$ of Γ, and the third $\wp(\mathcal{W})$ of the consequence set assigned by $\mathbf{L} \times \wp(\mathcal{W})$ to Γ. In this case, $\mathbf{L} \times \wp(\mathcal{W})$ is clearly a logic, but not a formal one because the consequence set defined by $\mathbf{L} \times \wp(\mathcal{W})$ does not depend on formal considerations alone but also on the contents of Δ. Summarizing: if we push Δ to the premise set, we need a couple of premise sets instead of a single one; if we push Δ to the logic, the logic is not a formal one.

Some readers might think that the dilemma obtained at the end of the previous paragraph is a result of bad will or, heaven beware, of dogmatism. They could not be more mistaken. All logicians, conservatives and progressives alike, agree that the transition from "John is a bachelor and John is bald" to "John is a bachelor" is formal, whereas the transition from "John is a bachelor" to "John is unmarried" is informal because it does not depend on the meaning of a logical symbol, such as "and", but on the meaning of the (non-logical) predicates "bachelor" and "unmarried". However, where $\Gamma = \{$John is a bachelor$\}$, $\Delta = \{$Bachelors are unmarried$\}$, and A is "John is unmarried", we obviously have $\Gamma \vdash_{\mathbf{CL},\Delta} A$. So this should not be a formal inference if formal is to mean anything.[10]

That adaptive logics are formal, viz. have the Punctual Uniformity property, depends essentially on the fact that the set of abnormalities, Ω, is characterized by a (possibly restricted) logical form and is not just an arbitrary (recursive) set of formulas.

5.6 Some Further Properties

Let us start with a theorem that is central for proving many others.

Theorem 5.6.1 $Cn_{\mathbf{LLL}}(Cn_{\mathbf{AL}}(\Gamma)) = Cn_{\mathbf{AL}}(\Gamma)$. *(Redundance of* **LLL** *with respect to* **AL**.*)*

Proof. By the reflexivity of **LLL**, $Cn_{\mathbf{AL}}(\Gamma) \subseteq Cn_{\mathbf{LLL}}(Cn_{\mathbf{AL}}(\Gamma))$.

To prove the converse, suppose that $A \in Cn_{\mathbf{LLL}}(Cn_{\mathbf{AL}}(\Gamma))$. So there are $B_1, \ldots, B_n \in Cn_{\mathbf{AL}}(\Gamma)$ such that $B_1, \ldots, B_n \vdash_{\mathbf{LLL}} A$.

Consider an **AL**-proof from Γ in which $B_1, \ldots, B_n$ have been finally derived. Let each B_i have been finally derived on the conditions $\Delta^i_1, \ldots, \Delta^i_{m_i}$ ($m_i \geq 1$). Let Σ_1 be the set of lines at which some B_i is so derived. As $B_1, \ldots, B_n \vdash_{\mathbf{LLL}} A$,

[9] In some cases one will need $\wp(\mathcal{W}_+)$ here and in (ii) below in the text. This is rather immaterial for the point I am making in the text.

[10] The reader might feel that it should not be too difficult to turn $\Gamma \vdash_{\mathbf{L},\Delta} A$ into a formal inference relation. It is indeed possible to *characterize* it by a formal logic, viz. an adaptive one, under a translation. We shall come to that in Section 9.10.

the proof can be extended with a set Σ_2 of lines at each of which A is derived on one of the conditions $\Delta^1_{j_1} \cup \ldots \cup \Delta^n_{j_n}$ in which, for each $\Delta^i_{j_i}$, $1 \leq j_i \leq m_i$.

Case 1: Reliability. As the formulas B_i of all lines in Σ_1 have been finally derived at those lines, the condition of each such line, say Δ^i_j, is such that $\Delta^i_j \cap U(\Gamma) = \emptyset$ (by Theorem 5.3.1). It follows that all lines in Σ_2 have a condition $\Delta^1_{j_1} \cup \ldots \cup \Delta^n_{j_n}$ for which $(\Delta^1_{j_1} \cup \ldots \cup \Delta^n_{j_n}) \cap U(\Gamma) = \emptyset$. So A is finally derivable from Γ in view of Theorem 5.3.1.[11]

Case 2: Minimal Abnormality. For every $\varphi \in \Phi(\Gamma)$ and for each B_i ($1 \leq i \leq n$), there is a line in Σ_1 of which the condition Δ^i_j is such that $\Delta^i_j \cap \varphi = \emptyset$ (by Theorem 5.3.3). Let Θ_φ be the union of these conditions and note that $\Theta_\varphi \cap \varphi = \emptyset$. On one of the lines in Σ_2, A is derived on the condition Θ_φ. As this holds for every $\varphi \in \Phi(\Gamma)$, A is finally derivable from Γ in view of Theorem 5.3.3. ■

This is a nice example of a metatheoretic proof that relies only on proof-theoretical considerations. It would be very elegant if all syntactic properties were so proved, but I shall not do so because this would make the book considerably longer. So I shall often rely on the soundness and completeness of the involved logics with respect to their semantics.

What was shown may be phrased as follows: every adaptive consequence set is closed under the lower limit of the adaptive logic. That Theorem 5.6.1 holds is essential, for example, for the provability of the fixed point property: Corollary 5.6.3.

Theorem 5.6.2 *$Dab(\Delta) \in Cn_{\mathbf{AL}}(\Gamma)$ iff $Dab(\Delta) \in Cn_{\mathbf{LLL}}(\Gamma)$. (**AL** is Dab-conservative with respect to **LLL**/Immunity.)*

Proof. If $Dab(\Delta) \in Cn_{\mathbf{LLL}}(\Gamma)$, then $Dab(\Delta)$ is derivable on the condition $\emptyset$ from Γ in an **AL**-proof from Γ and hence $Dab(\Delta) \in Cn_{\mathbf{AL}}(\Gamma)$. If $Dab(\Delta) \in Cn_{\mathbf{AL}}(\Gamma)$, there are two cases.

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Case 1: $Dab(\Delta)$ is derivable on the condition $\emptyset$ in an **AL**-proof from Γ. Then $Dab(\Delta) \in Cn_{\mathbf{LLL}}(\Gamma)$ in view of Lemma 4.4.1.

Case 2: $Dab(\Delta)$ is derivable in an **AL**-proof from Γ but only on non-empty conditions.

Case 2.1: the strategy is Reliability. Let Θ be a minimal such condition. In view of Lemma 4.4.1, (i) $Dab(\Delta \cup \Theta)$ is derivable on the condition $\emptyset$ in the **AL**-proof from Γ and (ii) $Dab(\Delta' \cup \Theta)$ is a minimal Dab-consequence of Γ for some $\Delta' \subset \Delta$. So $\Theta \subseteq U(\Gamma)$ and every line at which $Dab(\Delta)$ is derived on a condition $\Theta' \supseteq \Theta$ is marked.

Case 2.1: the strategy is Minimal Abnormality. Suppose that $Dab(\Delta)$ is finally derived on a condition Θ_0 at line i of an $\mathbf{AL}^m$-proof from Γ and that Θ_1, Θ_2, ... are the minimal conditions on which $Dab(\Delta)$ is derivable in the proof. So there are $\Delta_i \subseteq \Delta$ such that $Dab(\Delta_1 \cup \Theta_1)$, $Dab(\Delta_2 \cup \Theta_2)$, ... are minimal Dab-consequences of Γ. It is easily seen that some minimal choice set of these contains a member of every Θ_i, which contradicts the supposition. ■

This property is important in many respects. For example, it is essential for proving that adaptive consequence sets are fixed points—see Theorem 5.6.3

[11]It is sufficient that one line has such a condition for A to be finally derivable, but actually all lines in Σ_2 have such a condition.

below. The theorem can be read as stating: if adaptive consequences of the premises are added to the premises, then the derivable *Dab*-formulas are the same as the ones derivable from the premises. This may even be strengthened to the subsequent theorem: if any set of adaptive consequences of the premises are added to the premises, then the derivable *Dab*-formulas are the same as the ones derivable from the premises.

Theorem 5.6.3 *If* $\Gamma' \subseteq Cn_{\mathbf{AL}}(\Gamma)$*, then* $Dab(\Delta) \in Cn_{\mathbf{LLL}}(\Gamma \cup \Gamma')$ *iff* $Dab(\Delta) \in Cn_{\mathbf{LLL}}(\Gamma)$.

Proof. Suppose that $\Gamma' \subseteq Cn_{\mathbf{AL}}(\Gamma)$. We have to prove an equivalence.

$\Rightarrow$ By the supposition and the reflexivity of **LLL**, $\Gamma \cup \Gamma' \subseteq Cn_{\mathbf{AL}}(\Gamma)$, whence, by the monotonicity of **LLL**, $Cn_{\mathbf{LLL}}(\Gamma \cup \Gamma') \subseteq Cn_{\mathbf{LLL}}(Cn_{\mathbf{AL}}(\Gamma))$. So, if $Dab(\Delta) \in Cn_{\mathbf{LLL}}(\Gamma \cup \Gamma')$, then $Dab(\Delta) \in Cn_{\mathbf{LLL}}(Cn_{\mathbf{AL}}(\Gamma))$ and hence $Dab(\Delta) \in Cn_{\mathbf{AL}}(\Gamma)$ in view of Theorem 5.6.1 and hence $Dab(\Delta) \in Cn_{\mathbf{LLL}}(\Gamma)$ in view of Theorem 5.6.2.

$\Leftarrow$ Follows even without the supposition because **LLL** is monotonic. ■

Corollary 5.6.1 *If* $\Gamma' \subseteq Cn_{\mathbf{AL}}(\Gamma)$*, then* $U(\Gamma \cup \Gamma') = U(\Gamma)$ *and* $\Phi(\Gamma \cup \Gamma') = \Phi(\Gamma)$.

We are now ready to prove that all adaptive logics have three central Tarski-like properties: Reflexivity, Cumulative Transitivity, and Cumulative Monotonicity.

Theorem 5.6.4 $\Gamma \subseteq Cn_{\mathbf{AL}}(\Gamma)$. *(Reflexivity.)*

Proof. As **LLL** is reflexive, $\Gamma \subseteq Cn_{\mathbf{LLL}}(\Gamma)$. So $\Gamma \subseteq Cn_{\mathbf{AL}}(\Gamma)$ in view of Corollary 5.3.3. ■

Theorem 5.6.5 *If* $\Gamma' \subseteq Cn_{\mathbf{AL}}(\Gamma)$*, then* $Cn_{\mathbf{AL}}(\Gamma \cup \Gamma') \subseteq Cn_{\mathbf{AL}}(\Gamma)$. *(Cumulative Transitivity./Cautious Cut)*

Proof. Suppose that $\Gamma' \subseteq Cn_{\mathbf{AL}}(\Gamma)$ and that $A \in Cn_{\mathbf{AL}}(\Gamma \cup \Gamma')$. So $U(\Gamma \cup \Gamma') = U(\Gamma)$ and $\Phi(\Gamma \cup \Gamma') = \Phi(\Gamma)$ in view of Corollary 5.6.1.

Case 1: Reliability. By Theorem 5.3.1, there is a Δ such that $\Gamma \cup \Gamma' \vdash_{\mathbf{LLL}} A \check{\vee} Dab(\Delta)$ and $\Delta \cap U(\Gamma \cup \Gamma') = \emptyset$. So, as **LLL** is compact and $U(\Gamma \cup \Gamma') = U(\Gamma)$, there is a Δ and there are $C_1, \ldots, C_m \in \Gamma'$ such that $\Gamma \cup \{C_1, \ldots, C_m\} \vdash_{\mathbf{LLL}} A \check{\vee} Dab(\Delta)$ and $\Delta \cap U(\Gamma) = \emptyset$. Each of these C_i is itself finally derivable from Γ. So, by Theorem 5.3.1, there is, for each C_i, a Δ^i such that $\Gamma \vdash_{\mathbf{LLL}} C_i \check{\vee} Dab(\Delta^i)$ and $\Delta^i \cap U(\Gamma) = \emptyset$. It follows by **CL**-properties that $\Gamma \vdash_{\mathbf{LLL}} A \check{\vee} Dab(\Delta \cup \Delta_1 \cup \ldots \cup \Delta_n)$ and $(\Delta \cup \Delta_1 \cup \ldots \cup \Delta_n) \cap U(\Gamma) = \emptyset$, whence A is finally derivable from Γ in view of Theorem 5.3.1.

Case 2: Minimal Abnormality. This proof relates to the proof of case 1 in the same way as the proof of case 2 of Theorem 5.6.1 relates to the proof of case 1 of that theorem. We establish, for every $\varphi \in \Phi(\Gamma)$ instead of for $U(\Gamma)$, that there is a condition $\Delta \cup \Delta_1 \cup \ldots \cup \Delta_n$ for which $\Gamma \vdash_{\mathbf{LLL}} A \check{\vee} Dab(\Delta \cup \Delta_1 \cup \ldots \cup \Delta_n)$ and $(\Delta \cup \Delta_1 \cup \ldots \cup \Delta_n) \cap \varphi = \emptyset$—relying for the latter on $\Phi(\Gamma \cup \Gamma') = \Phi(\Gamma)$. So A is finally derivable from Γ in view of Theorem 5.3.3. ■

Theorem 5.6.6 *If* $\Gamma' \subseteq Cn_{\mathbf{AL}}(\Gamma)$*, then* $Cn_{\mathbf{AL}}(\Gamma) \subseteq Cn_{\mathbf{AL}}(\Gamma \cup \Gamma')$. *(Cumulative Monotonicity./Cautious Monotonicity)*

Proof. Suppose that $\Gamma' \subseteq Cn_{\mathbf{AL}}(\Gamma)$ and that $A \in Cn_{\mathbf{AL}}(\Gamma)$. So $U(\Gamma \cup \Gamma') = U(\Gamma)$ and $\Phi(\Gamma \cup \Gamma') = \Phi(\Gamma)$ by Corollary 5.6.1.

Case 1: Reliability. Suppose that $A \in Cn_{\mathbf{AL}}(\Gamma)$. In view of Theorem 5.3.1, there are $B_1, \ldots, B_n \in \Gamma$ such that $B_1, \ldots, B_n \vdash_{\mathbf{LLL}} A \check{\vee} Dab(\Delta)$ and $\Delta \cap U(Cn_{\mathbf{AL}}(\Gamma)) = \emptyset$. As **LLL** is monotonic and $U(\Gamma \cup \Gamma') = U(\Gamma)$, there are $B_1, \ldots, B_n \in \Gamma \cup \Gamma'$ such that $B_1, \ldots, B_n \vdash_{\mathbf{LLL}} A \check{\vee} Dab(\Delta)$ and $\Delta \cap U(Cn_{\mathbf{AL}}(\Gamma \cup \Gamma')) = \emptyset$. So A is finally derivable from Γ in view of Theorem 5.3.1.

Case 2: Minimal Abnormality. Similarly, reasoning for every $\varphi \in \Phi(\Gamma)$ and relying on $\Phi(\Gamma \cup \Gamma') = \Phi(\Gamma)$ and on Theorem 5.3.3. ■

The last two theorems together give us a corollary.

Corollary 5.6.2 *If* $\Gamma' \subseteq Cn_{\mathbf{AL}}(\Gamma)$*, then* $Cn_{\mathbf{AL}}(\Gamma \cup \Gamma') = Cn_{\mathbf{AL}}(\Gamma)$*. (Cumulative Indifference.)*

Moreover, as $Cn_{\mathbf{AL}}(\Gamma) \subseteq Cn_{\mathbf{AL}}(\Gamma)$ and $\Gamma \cup Cn_{\mathbf{AL}}(\Gamma) = Cn_{\mathbf{AL}}(\Gamma)$, a further corollary follows.

Corollary 5.6.3 $Cn_{\mathbf{AL}}(\Gamma)$ *is a fixed point (*$Cn_{\mathbf{AL}}(Cn_{\mathbf{AL}}(\Gamma)) = Cn_{\mathbf{AL}}(\Gamma)$*).*

The Fixed Point property guarantees that it does not make any difference whether one applies a logic **L** to a premise set Γ or to the **L**-consequence set of Γ. The Fixed Point property obviously holds in the case of a Tarski logic, but does not hold for all defeasible logics. Here is a simple example; the *Weak* consequences of $\{p \wedge q, \neg p\}$ comprise q as well as $\neg p$, but not $q \wedge \neg p$, whereas the *Weak* consequences of the *Weak* consequences of $\{p \wedge q, \neg p\}$ comprise $q \wedge \neg p$. So *Weak* leads to a consequence set that is not stable with respect to the *Weak* consequence relation.

For adaptive logics, the main danger would be that the defeasible consequences of the consequence set would result in the derivability of a different set of abnormalities. Well, it does not. The adaptive consequence set is closed under the lower limit logic and the addition of adaptive consequences to the premise set changes nothing to the derivable disjunctions of abnormalities (Theorem 5.6.3).

The importance of the Fixed Point property is among other things related, on the one hand, to the question whether a logic is suitable to serve as the underlying logic of an axiomatic 'theory' and, on the other hand, to the question whether a logic is suitable to be applied to a person's convictions. I shall first consider the more technical aspect, which is axiomatization, and then move on to organizing a person's convictions. In both cases, the Fixed Point property in itself will turn out to be really too weak. What we really need is the Cumulative Indifference property. But fortunately, all adaptive logics have this property (Corollary 5.6.2).

The traditional view on the axiomatization of a theory is as follows. A theory is a set T of formulas. An axiomatization of T is provided by locating a logic **L** and a decidable set of formulas $\Gamma \subseteq T$ such that (i) T is closed under **L**: $Cn_{\mathbf{L}}(T) = T$, (ii) all members of T are **L**-derivable from Γ: $T = Cn_{\mathbf{L}}(\Gamma)$, and (iii) there is a positive test for $\Gamma \vdash_{\mathbf{L}} A$. This view is not a matter of principle, but grew out of the theories that were around in the first half of the twentieth century and of what one had learned about them. For example, first order

Peano Arithmetic requires a recursive, rather than a finite, set of axioms and was found not to be effectively decidable.

Suppose that T is not recursively enumerable. Then there is no way to axiomatize it on the traditional view and T is not a theory. Still, it may be possible to consider T as a weaker kind of theory—let us call it an adaptive theory—and to find an adaptive logic **AL** and a recursive Γ such that $T = Cn_{\mathbf{AL}}(\Gamma)$. The couple $\langle \Gamma, \mathbf{AL} \rangle$ is not an axiomatization in the traditional sense because there is no positive test for being a member of T. So what is it?

First of all, $Cn_{\mathbf{AL}}(\Gamma)$ is a correct *definition* of T, just like second order theories are correct definitions.

That there is no positive test should not be blamed on the adaptive logic. T was not recursively enumerable in the first place; if it were, it might be axiomatized by means of a Tarski logic. Whether this axiomatization would be very enlightening is a different matter. The logic underlying the axiomatization might be as ugly as the set of S-rules used in the proof of Theorem 1.5.9 in Section 2.7.

The aforementioned properties (i) and (ii) hold uncurtailedly for **AL** and Γ. Note that the Fixed Point property is essential for (i). Property (iii) obviously cannot hold. For some A it will be possible to establish that $\Gamma \vdash_{\mathbf{AL}} A$, for other A it may be impossible to find out whether $\Gamma \vdash_{\mathbf{AL}} A$, even if this is the case. This is as expected. It results from the fact that there is no positive test for **AL**-derivability. However, once one established that $\Gamma \vdash_{\mathbf{AL}} A$, all doubts are removed. Incidentally, some people seem to think that establishing $\Gamma \vdash_{\mathbf{AL}} A$ is a semantic matter, relying essentially on a reasoning about the truth of A in the selected models of Γ. This is clearly mistaken. Any conclusive reasoning about models may be rephrased as a reasoning about dynamic proofs.

At this point the Cumulative Indifference property comes in. The Fixed Point property merely guarantees that the **AL**-consequences of $Cn_{\mathbf{AL}}(\Gamma)$ are identical to the **AL**-consequences of Γ. Any profit from this property is only available once all **AL**-consequences of Γ are located and we know beforehand that nothing new will result from applying **AL** to $Cn_{\mathbf{AL}}(\Gamma)$. The Cumulative Indifference property is much stronger. As soon as it is established that $\Gamma \vdash_{\mathbf{AL}} A$, one may rely on A, in other words reason from $\Gamma \cup \{A\}$. The same holds for any set Γ' of which the members were shown to be **AL**-consequences of Γ. One may rely on Γ', one may reason from $\Gamma \cup \Gamma'$ to find out what follows from Γ. This means, among other things, that one may forget about the conditions, however complex and diverse, on which the members of Γ' are derived from Γ; one may henceforth introduce those members as premises, so on the empty condition.

Before leaving the matter, two points are worth being mentioned. First, there is a sense in which adaptive theories are clearly simpler than second order theories. For one thing, the dynamic proofs of adaptive logics are governed by finitary rules. This makes it possible that final derivability is established at a finite proof stage (Theorem 5.4.2). The second point is that I tried to be friendly to opponents in the previous paragraph, but that this forced me to an unrealistic presupposition. The presupposition was that the set T is available. So the problem was in which way T may be axiomatized. But the presupposition is obviously unrealistic. No human being is able to hold even a non-decidable set *as such* in her mind. I even wonder what it might mean that a human holds the members of an infinite set in her finite mind. All we can have in our mind is either a name of an infinite set or a finite description of the infinite set, such as a finite set of axioms or axiom schemata and a logic that generates the full set from the set of axioms.

The argument concerning a person's convictions is even stronger. Some convictions are accepted for an independent reason, for example that the person experienced something, that someone whom she considers reliable informed her so, and so on. Other convictions are derived from the former ones. Obviously, no one is able, in the turmoil of real life, to remember which convictions were accepted for an independent reason and which were derived. So if a logic would not be cumulatively indifferent, humans would only be able to apply it to well-delineated premise sets, but not to their convictions in general.

It seems desirable to offer some further insights on Corollary 5.3.3, which states

$$Cn_{\mathbf{LLL}}(\Gamma) \subseteq Cn_{\mathbf{AL}^r}(\Gamma) \subseteq Cn_{\mathbf{AL}^m}(\Gamma) \subseteq Cn_{\mathbf{ULL}}(\Gamma)\,.$$

It is desirable to state precisely in which cases the subset relations are proper and in which cases an identity obtains. Moreover, it is desirable to specify this for both the standard language $\mathcal{L}$ and the extension $\mathcal{L}_+$, which comprises the classical logical symbols. Consider the case were $\Gamma = \mathcal{W}$. Clearly $Cn^{\mathcal{L}}_{\mathbf{LLL}}(\Gamma) = Cn^{\mathcal{L}}_{\mathbf{AL}^r}(\Gamma) = Cn^{\mathcal{L}}_{\mathbf{AL}^m}(\Gamma) = Cn^{\mathcal{L}}_{\mathbf{ULL}}(\Gamma) = \mathcal{W}$, but $Cn^{\mathcal{L}_+}_{\mathbf{LLL}}(\Gamma) = Cn^{\mathcal{L}_+}_{\mathbf{AL}^r}(\Gamma) = Cn^{\mathcal{L}_+}_{\mathbf{AL}^m}(\Gamma) \subset Cn^{\mathcal{L}_+}_{\mathbf{ULL}}(\Gamma) = \mathcal{W}_+$. If, in the following theorem, the language is not specified in an expression, as in the first half of item 1, the expression holds for $\mathcal{L}$ as well as for $\mathcal{L}_+$—the expression holds if *every* occurrence of *Cn* is replaced by $Cn^{\mathcal{L}}$ and also if every occurrence of *Cn* is replaced by $Cn^{\mathcal{L}_+}$.

Recall that a normal premise set (with respect to an adaptive logic **AL**) is defined by Definition 4.6.1: it has **ULL**-models. This means that no *Dab*-formula is **LLL**-derivable from a normal premise set and, *a fortiori*, that the **LLL**-closure of the premise set is not $\mathcal{L}_+$-trivial.

Theorem 5.6.7 *Each of the following holds:*

1. *If* Γ *is normal,* $\mathcal{M}^{\mathbf{ULL}}_\Gamma = \mathcal{M}^m_\Gamma = \mathcal{M}^r_\Gamma$ *and hence* $Cn_{\mathbf{AL}^r}(\Gamma) = Cn_{\mathbf{AL}^m}(\Gamma) = Cn_{\mathbf{ULL}}(\Gamma)$.
   If Γ *is abnormal and* $\mathcal{M}^{\mathbf{LLL}}_\Gamma \neq \emptyset$*, then* $\mathcal{M}^{\mathbf{ULL}}_\Gamma = \emptyset \subset \mathcal{M}^m_\Gamma$ *and hence* $Cn^{\mathcal{L}_+}_{\mathbf{AL}^m}(\Gamma) \subset Cn^{\mathcal{L}_+}_{\mathbf{ULL}}(\Gamma) = \mathcal{W}_+$.

2. *If* Γ *is normal,* $\mathcal{M}^{\mathbf{ULL}}_\Gamma = \mathcal{M}^m_\Gamma = \mathcal{M}^r_\Gamma$ *and hence* $Cn^{\mathcal{L}}_{\mathbf{AL}^r}(\Gamma) = Cn^{\mathcal{L}}_{\mathbf{AL}^m}(\Gamma) = Cn^{\mathcal{L}}_{\mathbf{ULL}}(\Gamma)$.
   If $Cn^{\mathcal{L}}_{\mathbf{LLL}}(\Gamma) = \mathcal{W}$ *(so* Γ *is* $\mathcal{L}$*-trivial), then* $Cn^{\mathcal{L}}_{\mathbf{AL}^m}(\Gamma) = Cn^{\mathcal{L}}_{\mathbf{ULL}}(\Gamma) = \mathcal{W}$.
   If Γ *is abnormal and* $Cn^{\mathcal{L}}_{\mathbf{LLL}}(\Gamma) \neq \mathcal{W}$*, then* $\mathcal{M}^{\mathbf{ULL}}_\Gamma = \emptyset \subset \mathcal{M}^m_\Gamma$ *and hence* $Cn^{\mathcal{L}}_{\mathbf{AL}^m}(\Gamma) \subset Cn^{\mathcal{L}}_{\mathbf{ULL}}(\Gamma) = \mathcal{W}$.

3. $Cn_{\mathbf{AL}^r}(\Gamma) \subset Cn_{\mathbf{AL}^m}(\Gamma)$ *iff there is an* A *such that (i) for all* $\varphi \in \Phi(\Gamma)$*, there is a* $\Delta \subseteq \Omega$ *for which* $\Gamma \vdash_{\mathbf{LLL}} A \check{\vee} Dab(\Delta)$ *and* $\Delta \cap \varphi = \emptyset$ *and (ii) for all* $\Delta \subseteq \Omega$*, if* $\Gamma \vdash_{\mathbf{LLL}} A \check{\vee} Dab(\Delta)$*, then* $\Delta \cap U(\Gamma) \neq \emptyset$*. Otherwise* $Cn_{\mathbf{AL}^r}(\Gamma) = Cn_{\mathbf{AL}^m}(\Gamma)$.

4. $Cn_{\mathbf{LLL}}(\Gamma) \subset Cn_{\mathbf{AL}^r}(\Gamma)$ *iff there is an* A *and a* $\Delta \subseteq \Omega$ *such that* $\Gamma \nvdash_{\mathbf{LLL}} A$*,* $\Gamma \vdash_{\mathbf{LLL}} A \check{\vee} Dab(\Delta)$*, and* $\Delta \cap U(\Gamma) = \emptyset$.
   $Cn^{\mathcal{L}_+}_{\mathbf{LLL}}(\Gamma) \subset Cn^{\mathcal{L}_+}_{\mathbf{AL}^r}(\Gamma)$ *iff there is an* $A \in \Omega - U(\Gamma)$.

5. $\mathcal{M}^r_\Gamma \subset \mathcal{M}^{\mathbf{LLL}}_\Gamma$ *iff* $\Gamma \cup \{A\}$ *is* **LLL***-satisfiable for some* $A \in \Omega - U(\Gamma)$.

6. *If $A \in \Omega - U(\Gamma)$, then $\check{\neg} A \in Cn^{\mathcal{L}_+}_{\mathbf{AL}^r}(\Gamma)$.*

7. $\mathcal{M}^r_\Gamma = \mathcal{M}^{\mathbf{LLL}}_{\Gamma \cup \{\check{\neg} A | A \in \Omega - U(\Gamma)\}}$.

8. *If $Cn_{\mathbf{LLL}}(\Gamma) \subset Cn_{\mathbf{AL}^r}(\Gamma)$, then $\mathcal{M}^r_\Gamma \subset \mathcal{M}^{\mathbf{LLL}}_\Gamma$.*
If $\mathcal{M}^r_\Gamma \subset \mathcal{M}^{\mathbf{LLL}}_\Gamma$, then $Cn^{\mathcal{L}_+}_{\mathbf{LLL}}(\Gamma) \subset Cn^{\mathcal{L}_+}_{\mathbf{AL}^r}(\Gamma)$.

9. *$\mathcal{M}^m_\Gamma \subset \mathcal{M}^{\mathbf{LLL}}_\Gamma$ iff there is a (possibly infinite) $\Delta \subseteq \Omega$ such that $\Gamma \cup \Delta$ is* **LLL***-satisfiable and there is no $\varphi \in \Phi(\Gamma)$ for which $\Delta \subseteq \varphi$.*

10. *If $\Phi(\Gamma) = \{\emptyset\}$, then $\mathcal{M}^m_\Gamma = \mathcal{M}^{\mathbf{LLL}}_{\Gamma \cup \Omega^{\check{\neg}}}$. If $\Phi(\Gamma) = \{\varphi_1, \ldots, \varphi_n\}$ $(n > 0)$, then $\mathcal{M}^m_\Gamma = \mathcal{M}^{\mathbf{LLL}}_{\Gamma \cup \{\check{\neg} A_1 \check{\vee} \ldots \check{\vee} \check{\neg} A_n | A_1 \in \Omega - \varphi_1, \ldots, A_n \in \Omega - \varphi_n\}}$.*

11. *If there are $A_1, \ldots, A_n \in \Omega$ $(n \geq 1)$ such that $\Gamma \cup \{A_1, \ldots, A_n\}$ is* **LLL***-satisfiable and, for every $\varphi \in \Phi(\Gamma)$, $\{A_1, \ldots, A_n\} \not\subseteq \varphi$, then $Cn^{\mathcal{L}_+}_{\mathbf{LLL}}(\Gamma) \subset Cn^{\mathcal{L}_+}_{\mathbf{AL}^m}(\Gamma)$.*

12. *$Cn_{\mathbf{AL}^m}(\Gamma)$ and $Cn_{\mathbf{AL}^r}(\Gamma)$ are not $\mathcal{L}_+$-trivial iff $\mathcal{M}^{\mathbf{LLL}}_\Gamma \neq \emptyset$.*

Proof. Ad 1. If Γ is normal, $U(\Gamma) = \emptyset$ and, by Corollary 5.2.1, $\Phi(\Gamma) = \{\emptyset\}$. So both strategies select exactly the **ULL**-models from the **LLL**-models of Γ. The proof-theoretic result follows by the soundness and completeness of all three logics with respect to their semantics. (More directly: as $U(\Gamma) = \emptyset$ and $\Phi(\Gamma) = \{\emptyset\}$, no line of a proof will ever be marked, etc.)

If Γ is abnormal, then $\mathcal{M}^{\mathbf{ULL}}_\Gamma = \emptyset$ in view of Lemma 4.6.2, whence $Cn^{\mathcal{L}_+}_{\mathbf{ULL}}(\Gamma)$ is $\mathcal{L}_+$-trivial by Theorem 4.6.1. If Γ has **LLL**-models, then it has $\mathbf{AL}^m$-models by Corollary 5.2.2, whence $Cn^{\mathcal{L}_+}_{\mathbf{AL}^m}(\Gamma)$ is not $\mathcal{L}_+$-trivial.

Ad 2. By the same reasoning as for item 1, except that we have to take into account the border case that $\Gamma = \mathcal{W}$. If Γ has **LLL**-models, verifying $\mathcal{W}$, then either Γ has **ULL**-models, also verifying $\mathcal{W}$, or it has none. In both cases $Cn^{\mathcal{L}}_{\mathbf{ULL}}(\Gamma) = \mathcal{W}$. If Γ has no **LLL**-models, then it has no **ULL**-models either.

Ad 3. Immediate in view of Theorems 5.3.1 and 5.3.3.

Ad 4. The first claim follows by Theorem 5.3.1. The second claim is a special case of this because $\Gamma \vdash_{\mathbf{LLL}} \check{\neg} A \check{\vee} A$ holds for all Γ.

Ad 5. $\Rightarrow$ Suppose that $\mathcal{M}^r_\Gamma \subset \mathcal{M}^{\mathbf{LLL}}_\Gamma$. So, by Definition 4.5.1, an **LLL**-model $M \in \mathcal{M}^{\mathbf{LLL}}_\Gamma - \mathcal{M}^r_\Gamma$ verifies an $A \in \Omega - U(\Gamma)$. $\Leftarrow$ Suppose that an **LLL**-model $M \in \mathcal{M}^{\mathbf{LLL}}_\Gamma - \mathcal{M}^r_\Gamma$ verifies an $A \in \Omega - U(\Gamma)$. So M is not selected in view of Definition 4.5.1.

Ad 6. If $A \in \Omega - U(\Gamma)$, all reliable models of Γ falsify A in view of Definition 4.5.1. So Corollary 5.3.1 warrants that $\check{\neg} A \in Cn_{\mathbf{AL}^r}(\Gamma)$.

Ad 7. $\Rightarrow$ Consider an $A \in \Omega - U(\Gamma)$. As $\Gamma \vDash_{\mathbf{LLL}} \check{\neg} A \check{\vee} A$, $\Gamma \vDash \check{\neg} A \check{\vee} Dab(\{A\})$, whence $\Gamma \vDash_{\mathbf{AL}} \check{\neg} A$ by Theorem 5.3.2. $\Leftarrow$ If $M \in \mathcal{M}^{\mathbf{LLL}}_{\Gamma \cup \{\check{\neg} A | A \in \Omega - U(\Gamma)\}}$, then $Ab(M) \subseteq U(\Gamma)$. So $M \in \mathcal{M}^r_\Gamma$ by Definition 4.5.1.

Ad 8. *First statement.* Suppose $A \in Cn_{\mathbf{AL}^r}(\Gamma) - Cn_{\mathbf{LLL}}(\Gamma)$. By the soundness and completeness of **LLL**, some model $M \in \mathcal{M}^{\mathbf{LLL}}_\Gamma$ falsifies A. So $M \notin \mathcal{M}^r_\Gamma$ by the soundness and completeness of $\mathbf{AL}^r$. *Second statement.* Suppose $\mathcal{M}^r_\Gamma \subset \mathcal{M}^{\mathbf{LLL}}_\Gamma$. So there is a model $M \in \mathcal{M}^{\mathbf{LLL}}_\Gamma - \mathcal{M}^r_\Gamma$. By Definition 4.5.1, there is an $A \in \Omega - U(\Gamma)$ such that $M \Vdash A$. So $\check{\neg} A \in Cn^{\mathcal{L}_+}_{\mathbf{AL}^r}(\Gamma)$ in view of Item 6.

Ad 9. $\Rightarrow$ If $\mathcal{M}^m_\Gamma \subset \mathcal{M}^{\mathbf{LLL}}_\Gamma$, there is a $M \in \mathcal{M}^{\mathbf{LLL}}_\Gamma - \mathcal{M}^m_\Gamma$. In view of Lemma 5.2.1, $Ab(M) - \varphi \neq \emptyset$ for all $\varphi \in \Phi(\Gamma)$. Let $\Delta = \bigcup \{A \mid A \in Ab(M) - \varphi;\ \varphi \in$

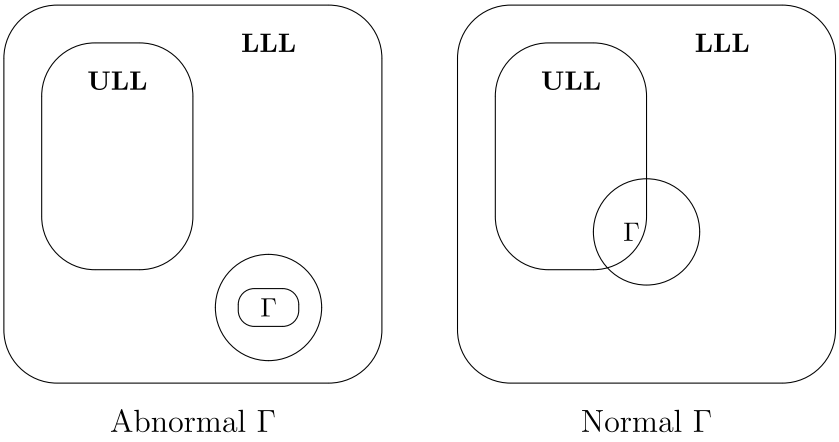

Figure 5.3: Comparison of Models

$\Phi(\Gamma)\}$. So $M \Vdash \Gamma \cup \Delta$ there is no $\varphi \in \Phi(\Gamma)$ for which $\Delta \subseteq \varphi$. $\Leftarrow$ If $\Delta \subseteq \Omega$ and $\Gamma \cup \Delta$ is **LLL**-satisfiable, there is a $M \in \mathcal{M}^{\mathbf{LLL}}_{\Gamma}$ such that $M \vdash \Gamma \cup \Delta$. As $\Delta - \varphi \neq \emptyset$ for all $\varphi \in \Phi(\Gamma)$, $M \notin \mathcal{M}^{m}_{\Gamma}$ in view of Lemma 5.2.1.

Ad 10. *Claim 1* is obvious as the **LLL**-models of $\Gamma \cup \Omega^{\check{\neg}}$ are the normal models (**ULL**-models) of Γ.

Claim 2. Suppose first that the antecedent is true. $\Rightarrow$ Consider a $M \in \mathcal{M}^{m}_{\Gamma}$. By Lemma 5.2.1, $Ab(M) \in \Phi(\Gamma)$. For all M' and for all $A \in \Omega - Ab(M')$, $M' \Vdash \check{\neg} A$. So, by Addition, $M \in \mathcal{M}^{\mathbf{LLL}}_{\Gamma \cup \{\check{\neg} A_1 \check{\vee} \ldots \check{\vee} \check{\neg} A_n \mid A_1 \in \Omega - \varphi_1, \ldots, A_n \in \Omega - \varphi_n\}}$. $\Leftarrow$ Consider a $M \notin \mathcal{M}^{m}_{\Gamma}$. In view of Definition 4.5.3 and Lemma 5.2.1, there is a $\psi \in \Phi(\Gamma)$ and an $B \in \Omega - \psi$ such that $M \Vdash \psi \cup \{B\}$. For all $\varphi_i \in \{\varphi_1, \ldots, \varphi_n\}$, let C_i be B if $\varphi_i = \psi$ and let C_i be an member of $\psi - \varphi_i$ if $\varphi_i \neq \psi$; indeed, $\psi - \varphi_i \neq \emptyset$ as $\psi, \varphi_i \in \Phi(\Gamma)$ and $\psi \neq \varphi_i$. Note that $M \nVdash \check{\neg} C_1 \check{\vee} \ldots \check{\vee} \check{\neg} C_n$ and that $\check{\neg} C_1 \check{\vee} \ldots \check{\vee} \check{\neg} C_n \in \{\check{\neg} A_1 \check{\vee} \ldots \check{\vee} \check{\neg} A_n \mid A_1 \in \Omega - \varphi_1, \ldots, A_n \in \Omega - \varphi_n\}$. So $M \notin \mathcal{M}^{\mathbf{LLL}}_{\Gamma \cup \{\check{\neg} A_1 \check{\vee} \ldots \check{\vee} \check{\neg} A_n \mid A_1 \in \Omega - \varphi_1, \ldots, A_n \in \Omega - \varphi_n\}}$.

Ad 11. Suppose the antecedent is true. Every $M \in \mathcal{M}^{m}_{\Gamma}$ falsifies some A_i and hence verifies $\check{\neg} A_1 \check{\vee} \ldots \check{\vee} \check{\neg} A_n$. So $\Gamma \vdash_{\mathbf{AL}^m} \check{\neg} A_1 \check{\vee} \ldots \check{\vee} \check{\neg} A_n$ in view of Theorem 5.3.4. However, as $\Gamma \cup \{A_1, \ldots, A_n\}$ is **LLL**-satisfiable, some $\Gamma \nvdash_{\mathbf{LLL}} \check{\neg} A_1 \check{\vee} \ldots \check{\vee} \check{\neg} A_n$.

Ad 12. Immediate from Corollary 5.2.2 and the fact that no **LLL**-model is $\mathcal{L}_{+}$-trivial. ■

Each item of this theorem deserves one or more comments, but I shall start with a general comment on the relation between the different kind of models. This is represented in Figure 5.3. The big 'rectangles' represent all **LLL**-models, the enclosed smaller rectangles the **ULL**-models. If Γ is normal, as on the right hand side, Γ has **ULL**-models as well as other **LLL**-models. In this case, the adaptive logic selects exactly the **ULL**-models of Γ, whence $Cn_{\mathbf{AL}^m}(\Gamma) = Cn_{\mathbf{ULL}}(\Gamma)$. If Γ is abnormal, it has no **ULL**-models. Nevertheless, the adaptive logic in general selects a subset of the **LLL**-models of Γ, which is represented by the smaller 'rectangle' inside the circle representing the **LLL**-models of Γ. As a result, the adaptive consequences in general extend the lower limit consequences.

I now comment on the separate items of Theorem 5.6.7. *Ad* 1. Note that $\mathcal{M}_\Gamma^{\mathbf{LLL}} = \emptyset$ iff $Cn_{\mathbf{LLL}}^{\mathcal{L}_+}(\Gamma) = \mathcal{W}_+$ and compare this to the comment on item 2. That the adaptive consequence sets of normal premise sets are identical to their upper limit consequence set is an important property. If, for example, the lower limit logic is paraconsistent and the upper limit logic is **CL**, then the adaptive consequence set is identical to the **CL**-consequence set. If the premise set is abnormal but has **LLL**-models, the adaptive consequence set is non-trivial, unlike the **CL**-consequence set. ev. voetn.: "Why bother?"

Note that the antecedents of the two statements are insufficient to further specify the relation between **LLL** and $\mathbf{AL}^r$. The reason for this is that I did not require, for example, that $\Omega \neq \emptyset$. So it is possible that $\mathcal{M}_\Gamma^r = \mathcal{M}_\Gamma^{\mathbf{LLL}}$ and hence that $Cn_{\mathbf{LLL}}(\Gamma) = Cn_{\mathbf{AL}^r}(\Gamma)$. The same holds for Item 2.

Ad 2. As is explained in Section 4.3, there may be $\mathcal{L}$-trivial **LLL**-models, viz. models verifying $\mathcal{W}$ and hence verifying every premise set $\Gamma \subseteq \mathcal{W}$. This obtains, for example, for **CLuN** and, in general, for paraconsistent logics in which classical negation is not definable. The proof of the item clarifies the situation.

Ad 3 and 4. These items specify the circumstances in which Minimal Abnormality leads to a stronger consequence set than Reliability and in which Reliability leads to a stronger consequence set than the lower limit logic. These circumstances depend on the lower limit logic, the set of abnormalities, the premise set, as well as the conclusion. That these items are phrased in terms of classical logical symbols does not prevent the statements to hold even for $\mathcal{L}$.

Items 5–12 list some further properties that are relevant for the preceding four items. The reason for listing the items is mainly that they provide further insight. *Ad* 5. This provides a much simpler criterion than Item 3 for deciding that some **LLL**-models are not selected by the Reliability strategy. The power of the statement becomes apparent if it is combined with the second half of Item 8. Remember that $\Omega \subseteq \mathcal{W}$ and that, in standard applications, $\Gamma \subseteq \mathcal{W}$ and $A \in \mathcal{W}$. So, for corrective adaptive logics, $\Gamma \cup \{A\}$ is always **LLL**-satisfiable. In other words, if **AL** is corrective, then $\mathcal{M}_\Gamma^r \subset \mathcal{M}_\Gamma^{\mathbf{LLL}}$ iff $U(\Gamma) \neq \Omega$.

Ad 6. This is a useful statement in preparation of the next Item, but it also is clarifying in itself: even on the Reliability strategy, every abnormality that is not a disjunct of a minimal *Dab*-consequence of Γ may finally be considered as false.

Ad 7 and 10. The Reliable models of Γ are the **LLL**-models of Γ that verify no abnormality except for those in $U(\Gamma)$. So is possible to characterize the reliable models of Γ as the **LLL**-models of a certain set of formulas.

This is not in general possible for the minimal abnormal models of Γ. Indeed, for some premise sets Γ, the minimal abnormal models of Γ falsify one out of an infinite set of abnormalities and this information cannot be expressed by a formula of $\mathcal{L}$. Let me first present an example. This is obvious from an example we met before: $\Gamma_3 = \{(p_i \wedge \neg p_i) \vee (p_j \wedge \neg p_j) \mid i \neq j; i, j \in \mathbb{N}\} \cup \{q \vee (p_i \wedge \neg p_i) \mid i \in \mathbb{N}\}$ from page 137. Every $\varphi \in \Phi(\Gamma_3)$ verifies all but one $p_i \wedge \neg p_i$; so it falsifies exactly one $p_i \wedge \neg p_i$. This can information only be expressed by an infinite disjunction: $\check{\bigvee}\{\check{\neg}(p_i \wedge \neg p_i) \mid i \in \mathbb{N}\}$. But this is not a formula of $\mathcal{L}$ because all formulas of $\mathcal{L}$ are finite. There is a good reason for this. Even the decidable infinite formulas—those $\check{\bigvee}(\Delta)$ for which Δ is a decidable set—are uncountable. So if such formulas are around, one would have to give up the requirement that proofs are chains of denumerably many stages.[12]

While the minimal abnormal models of Γ cannot in general be characterized as the **LLL**-models of Γ that verify a certain set of formulas,[13] they can be so characterized whenever $\Phi(\Gamma)$ is finite—Frederik Van De Putte first drew my attention to this.

Ad 8. This item completes Item 5 (with some help from Item 6). For example, if $U(\Gamma)$, the set of formulas that are unreliable with respect to Γ, does not comprise all abnormalities, then all abnormalities not contained in $U(\Gamma)$ may be considered as false on the Reliability strategy. Even if one considers only consequences that belong to $\mathcal{W}$, this may have dramatic consequences. Consider $\mathbf{CLuN}^r$. If $A \wedge \neg A \notin U(\Gamma)$ then all of the following are members of $Cn^{\mathcal{L}}_{\mathbf{CLuN}^r}(\Gamma)$: $(A \wedge \neg A) \supset B$, $((A \vee B) \wedge \neg A) \supset B$, $((B \supset A) \wedge \neg A) \supset \neg B$, etc.

Ad 9. Note that some reliable models of Γ may not be minimal abnormal models of Γ. Moreover, as $\Phi(\Gamma)$ and all $\varphi \in \Phi(\Gamma)$ may be infinite, it is possible that only an infinite Δ fulfils the condition. The premise set Γ_4 from page 138 provides a ready example.

Ad 11. Note that this item is stronger than item 10 in that the antecedent may even be true in case $\Phi(\Gamma)$ is infinite. For example $\{q \wedge \neg q\}$ will do for Γ_3—see the comments to items 7 and 10.

Item 11 cannot be proved if $A_1, \ldots, A_n \in \Omega$ is replaced by (an infinite) $\Delta \in \Omega$. Consider a logic $\mathbf{CLuN}^m_*$, which is just like the *propositional* fragment of $\mathbf{CLuN}^m$ except that its set of abnormalities is $\Omega* = \{A \wedge \neg A \mid A \in \mathcal{S}\}$ and let and $\Gamma_7 = \{A \vee B \mid A, B \in \Omega*; A \neq B\}$. Note that $\Phi(\Gamma_7)$ comprises all sets that contain all but one member of $\Omega*$. So $\mathcal{M}^m_{\Gamma_7} \subset \mathcal{M}^{\mathbf{CLuN}}_{\Gamma_7}$ because $\mathcal{M}^{\mathbf{CLuN}}_{\Omega*} \subset \mathcal{M}^{\mathbf{CLuN}}_{\Gamma_7} - \mathcal{M}^m_{\Gamma_7}$. In words, the **CLuN**-models that verify all members of $\Omega*$ are **CLuN**-models of Γ_7 but are not minimal abnormal (with respect to $\Omega*$). However, $Cn^{\mathcal{L}_+}_{\mathbf{CLuN}^m_*}(\Gamma_7) = Cn^{\mathcal{L}_+}_{\mathbf{CLuN}}(\Gamma_7)$ because no formula of $\mathcal{L}_S$ if verified by all members of $\mathcal{M}^m_{\Gamma_7}$ and falsified by the **CLuN**-models of $\Omega*$. Put differently, only the infinite formula $\check{\bigvee}\{\check{\neg} A \mid A \in \Omega*\}$ would separate the members of $\mathcal{M}^m_{\Gamma_7}$ from $\mathcal{M}^{\mathbf{CLuN}}_{\Omega*}$. This finishes the comments on Theorem 5.6.7.

The relation between adaptive logics and the Deduction Theorem is somewhat awkward. We shall see (Theorem 5.10.2) that the Deduction Theorem does not hold in general for Reliability, but it holds in special cases. Nevertheless, the Deduction Theorem holds for Minimal Abnormality. In preparation of Theorem 5.6.8, remark that the Deduction Theorem holds for **LLL**, viz. that $\Gamma \vdash_{\mathbf{LLL}} A \check{\supset} B$ if $\Gamma \cup \{A\} \vdash_{\mathbf{LLL}} B$—remember that $\check{\supset}$ is classical implication and that **LLL** contains **CL** and is compact.

Theorem 5.6.8 *If* $\Gamma \cup \{A\} \vdash_{\mathbf{AL}^m} B$ *then* $\Gamma \vdash_{\mathbf{AL}^m} A \check{\supset} B$. *(Deduction Theorem for* $\mathbf{AL}^m$*.)*

Proof. Suppose that the antecedent is true. By Theorem 5.3.3, there is, for every $\varphi \in \Phi(\Gamma \cup \{A\})$, a $\Delta \subset \Omega$ for which $\Gamma \cup \{A\} \vdash_{\mathbf{LLL}} B \check{\vee} Dab(\Delta)$ and $\Delta \cap \varphi = \emptyset$. It follows (by **CL**-properties) that $\Gamma \vdash_{\mathbf{LLL}} (A \check{\supset} B) \check{\vee} Dab(\Delta)$ for all these Δ.

[12] Not everyone will agree with the following claim, but I'll state it nevertheless. The real problem is not with the infinity of the formula, because the formula has a finite name, viz. $\check{\bigvee}\{\check{\neg}(p_i \wedge \neg p_i) \mid i \in \mathbb{N}\}$. The real problem is with the fact that some proofs containing such formulas are chains of more than a countable number of stages.

[13] That every minimal abnormal model of Γ verifies a *member* of the set $\{\{\check{\neg} A \mid A \in \Omega - \varphi\} \mid \varphi \in \Phi(\Gamma)\}$ is a different matter.

As **LLL** is monotonic, $Cn_{\mathbf{LLL}}(\Gamma) \subseteq Cn_{\mathbf{LLL}}(\Gamma \cup \{A\})$. Let Σ comprise the Δ for which $Dab(\Delta) \in Cn_{\mathbf{LLL}}(\Gamma)$ and let Σ' comprise the Δ for which $Dab(\Delta) \in Cn_{\mathbf{LLL}}(\Gamma \cup \{A\}) - Cn_{\mathbf{LLL}}(\Gamma)$. In view of Fact 5.1.5—see also page 152—$\Phi(\Gamma)$ is the set of minimal choice sets of Σ and $\Phi(\Gamma \cup \{A\})$ is the set of minimal choice sets of $\Sigma \cup \Sigma'$. Consider a $\psi \in \Phi(\Gamma)$.

Case 1: There is a $\varphi \in \Phi(\Gamma \cup \{A\})$ for which $\varphi \supseteq \psi$. As there is a $\Delta \subset \Omega$ for which $\Gamma \vdash_{\mathbf{LLL}} (A \check{\supset} B) \check{\vee} Dab(\Delta)$ and $\Delta \cap \varphi = \emptyset$, $\Delta \cap \psi = \emptyset$.

Case 2: There is no $\varphi \in \Phi(\Gamma \cup \{A\})$ for which $\varphi \supseteq \psi$. By Fact 5.1.8, there is a $\Theta \in \Sigma'$ such that $\Theta \cap \psi = \emptyset$ and, for every $B \in \Theta$, there is a $\psi' \in \Phi(\Gamma)$ for which $B \in \Theta \cap \psi'$ and $\psi \supseteq \psi' - \{B\}$. But then, as $\Gamma \cup \{A\} \vdash_{\mathbf{LLL}} Dab(\Theta)$ and the Deduction Theorem holds for **LLL**, $\Gamma \vdash_{\mathbf{LLL}} \check{\neg} A \check{\vee} Dab(\Theta)$. It follows that there is a $\Theta \subset \Omega$ for which $\Gamma \vdash_{\mathbf{LLL}} (A \check{\supset} B) \check{\vee} Dab(\Theta)$ and $\Theta \cap \psi = \emptyset$.

So in both cases, there is a Δ such that $\Gamma \vdash_{\mathbf{LLL}} (A \check{\supset} B) \check{\vee} Dab(\Delta)$ and $\Delta \cap \psi = \emptyset$. It follows that $\Gamma \vdash_{\mathbf{AL}^m} A \check{\supset} B$. ■

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Before ending this section, let us consider two further important properties of adaptive logics. The first concerns an important adequacy condition of dynamic proofs, the second is a closure property.

Adaptive proofs are somewhat unusual objects. Their most awkward property is that formulas considered as derived at some point may be considered as not derived at a later point and *vice versa*. That is unavoidable because it is typical for defeasible reasoning forms, but it is nevertheless odd. So it is important to show that adaptive proofs also have a lot of nice properties. One of them is Proof Invariance.

Suppose that Mary and John separately start a proof from the same premise set Γ and by means of the same adaptive logic **AL**. Suppose moreover that each of them establishes certain **AL**-consequences of Γ and that they inform each other of their results. It is obviously unproblematic that they would have reached different consequences. Maybe they were interested in different consequences in the first place. It would, however, be very problematic that Mary would be unable to establish in her proof the results reached by John, or *vice versa*. If this were the case, one of two problems would occur. A first possibility is that the logic **AL** and the premise set Γ jointly determine different consequence sets. In this case, **AL** would simply not be a logic in view of the very definition of a logic—see Section 1.1. A second possibility is that the different members of $Cn_{\mathbf{L}}(\Gamma)$ can only be established by means of different proofs. The way in which a proof is *started* would be determining for the consequences that may be established within the proof. Put differently, establishing certain **AL**-consequences of Γ would make it impossible to establish certain different consequences within the same reasoning process. The fact that the reasoning is defeasible should not be confused with any of these two possibilities. If a previously drawn conclusion is defeated, this means that the information on which its derivation relied was insufficient, that further information about the premises forces one to withdraw the conclusion (even if further information may be sufficient to establish the conclusion again). However, if both A and B can be established from the same premise set, it is simply unacceptable that the reasoning establishing A would prevent one to establish B. So it is unacceptable that B cannot be derived within the very proof in which A is derived. Incidentally, this was the reason why the first strategy discussed in this book, viz. in Section 2.3.1, was called a failing strategy.

Theorem 5.6.9 *If* $\Gamma \vdash_{\mathbf{AL}} A$*, then every* **AL**-*proof from* Γ *can be extended in such a way that* A *is finally derived in it. (Proof Invariance)*

Proof. Let $\mathsf{p}_1 = \langle l_1, l_2, \ldots \rangle$ be the (stage of the) proof in which A is finally derived from Γ at line l_k and let $\mathsf{p}_2 = \langle l'_1, l'_2, \ldots \rangle$ be a (stage of an) arbitrary proof from Γ. (If p_1 is finite, there is a last element in the sequence; similarly for p_2.)

In view of Definitions 4.4.2 and 4.4.1, the following is obvious. Whether B is derived at a stage s in a proof from Γ depends on the lines that occur in the proof, not on the order in which these lines occur. So the sequence $\mathsf{p}_3 = \langle l_1, l'_1, l_2, l'_2, \ldots \rangle$ (if there are more l_i than l'_j, the sequence will contain only l_i from some point on, etc.) is an extension of p_1 as well as of p_2. So, as A is finally derived in p_1, it follows by Definitions 4.4.4 and 4.4.5 that p_3 as well as every extension of p_3 in which line l_k is marked has a further extension in which line l_k is unmarked. ■

The last two properties I wish to establish in this section may be seen as concerning the equivalence of premise sets. In this sense it would be also at home in the next section, but is easily provable at this point.

Theorem 5.6.10 *For all* Γ*,* $Cn_{\mathbf{AL}}(Cn_{\mathbf{LLL}}(\Gamma)) = Cn_{\mathbf{AL}}(\Gamma)$*. (***LLL** *is conservative with respect to* **AL***.)*

Proof. By Corollary 5.3.3, $Cn_{\mathbf{LLL}}(\Gamma) \subseteq Cn_{\mathbf{AL}}(\Gamma)$. So, by Corollary 5.6.2, $Cn_{\mathbf{AL}}(\Gamma \cup Cn_{\mathbf{LLL}}(\Gamma)) = Cn_{\mathbf{AL}}(\Gamma)$. As **LLL** has static proofs, $\Gamma \subseteq Cn_{\mathbf{LLL}}(\Gamma)$ by Theorem 1.5.4. So $\Gamma \cup Cn_{\mathbf{LLL}}(\Gamma) = Cn_{\mathbf{LLL}}(\Gamma)$, whence the theorem follows. ■

In words: it does not make any difference whether an adaptive logic is applied to a premise set or to the **LLL**-closure of this premise set. This is as expected, but it still had to be proved.

bewijs kan wellicht verkort in fct van corollaries:

Theorem 5.6.11 *For all* Γ*,* $Cn_{\mathbf{ULL}}(Cn_{\mathbf{AL}}(\Gamma)) = Cn_{\mathbf{ULL}}(\Gamma)$*. (***AL** *is conservative with respect to* **ULL***.)*

Proof. In view of Theorem 5.6.4 and Corollary 5.3.3, $\Gamma \subseteq Cn_{\mathbf{AL}}(\Gamma) \subseteq Cn_{\mathbf{ULL}}(\Gamma)$. So, as **ULL** is monotonic, $Cn_{\mathbf{ULL}}(\Gamma) \subseteq Cn_{\mathbf{ULL}}(Cn_{\mathbf{AL}}(\Gamma)) \subseteq Cn_{\mathbf{ULL}}(Cn_{\mathbf{ULL}}(\Gamma))$. In view of Theorem 4.6.3 **ULL** has static proofs, whence, in view of Corollary 1.5.3, it is a Tarski logic. As Tarski logics have the Fixed Point property by Lemma 1.5.1, $Cn_{\mathbf{ULL}}(Cn_{\mathbf{ULL}}(\Gamma)) = Cn_{\mathbf{ULL}}(\Gamma)$. It follows that $Cn_{\mathbf{ULL}}(Cn_{\mathbf{AL}}(\Gamma)) = Cn_{\mathbf{ULL}}(\Gamma)$. ■

Actually this theorem is mentioned for completeness sake, but is not very fascinating. If Γ is normal, $Cn_{\mathbf{AL}}(\Gamma) = Cn_{\mathbf{ULL}}(\Gamma)$; if Γ is abnormal, $Cn_{\mathbf{ULL}}(\Gamma)$ is trivial (see Items 1 and 2 of Theorem 5.6.7).

5.7 Equivalent Premise Sets

In this section[14] I present criteria for the equivalence of two premise sets with respect to an adaptive logic. The results also highlight one of the advantages

[14] This section and the next rely on joint work with Peter Verdée and Christian Straßer—see [BSV09].

of adaptive logics, viz. their transparency in comparison to other approaches to defeasible reasoning forms.

Theories may have different *formulations*; the *same* theory may be presented in different ways. Remember that a theory T is a couple $\langle \Gamma, \mathbf{L} \rangle$, in which Γ is a set of statements (the non-logical axioms of T) and $\mathbf{L}$ is a logic. The claims made by T are the members of $Cn_{\mathbf{L}}(\Gamma) = \{A \mid \Gamma \vdash_{\mathbf{L}} A\}$. That $T = \langle \Gamma, \mathbf{L} \rangle$ and $T' = \langle \Gamma', \mathbf{L} \rangle$ are different formulations of the same theory obviously means that $Cn_{\mathbf{L}}(\Gamma) = Cn_{\mathbf{L}}(\Gamma')$. Similarly, people may come to the conclusion that they fully agree on a subject. If they are serious about the matter, they mean that all one person believes on the subject is derivable from the statements made (or agreed to) by the other. We may safely take it that the agreeing parties share the underlying logic $\mathbf{L}$, at least in the context of their present communication. So their agreement may be formally expressed by a statement of the form $Cn_{\mathbf{L}}(\Gamma) = Cn_{\mathbf{L}}(\Gamma')$.

Definition 5.7.1 Γ *and* Γ' *are* $\mathbf{L}$-equivalent premise sets *iff* $Cn_{\mathbf{L}}(\Gamma) = Cn_{\mathbf{L}}(\Gamma')$.

As a theory may be formulated in many ways, criteria for identifying equivalent theories are important. Offering a *direct* proof of $Cn_{\mathbf{L}}(\Gamma) = Cn_{\mathbf{L}}(\Gamma')$ is obviously out of the question: it is impossible for humans to enumerate the members of $Cn_{\mathbf{L}}(\Gamma)$ and to demonstrate for each of them that it also belongs to $Cn_{\mathbf{L}}(\Gamma')$. So humans need to rely on shortcuts for establishing that Γ and Γ' are $\mathbf{L}$-equivalent premise sets. This is the reason why our inferential habits tell us that certain transformations of the premises should not affect their consequence set. Let us now consider the three most common criteria that govern such transformations.

First, I have to define a phrase that was used intuitively until now. A logic $\mathbf{L_1}$ is *weaker than* a logic $\mathbf{L_2}$ (and $\mathbf{L_2}$ is *stronger than* $\mathbf{L_1}$) iff $Cn_{\mathbf{L_1}}(\Gamma) \subset Cn_{\mathbf{L_2}}(\Gamma)$ for some Γ and $Cn_{\mathbf{L_1}}(\Gamma) \subseteq Cn_{\mathbf{L_2}}(\Gamma)$ for all Γ. The two logics are identical to each other iff $Cn_{\mathbf{L_1}}(\Gamma) = Cn_{\mathbf{L_2}}(\Gamma)$.

If $\mathbf{L}$ is a Tarski logic, three simple criteria for the $\mathbf{L}$-equivalence of premise sets are available:

C1 If $\Gamma' \subseteq Cn_{\mathbf{L}}(\Gamma)$ and $\Gamma \subseteq Cn_{\mathbf{L}}(\Gamma')$, then Γ and Γ' are $\mathbf{L}$-equivalent.

C2 If $\mathbf{L}'$ is a Tarski logic weaker than $\mathbf{L}$, and Γ and Γ' are $\mathbf{L}'$-equivalent, then Γ and Γ' are $\mathbf{L}$-equivalent.

C3 If every $Cn_{\mathbf{L}}(\Delta)$ is closed under a Tarski logic $\mathbf{L}'$ (viz. $Cn_{\mathbf{L}'}(Cn_{\mathbf{L}}(\Delta)) = Cn_{\mathbf{L}}(\Delta)$ for all Δ), and Γ and Γ' are $\mathbf{L}'$-equivalent, then Γ and Γ' are $\mathbf{L}$-equivalent.

According to criterion C1, that all members of Γ are $\mathbf{L}$-derivable from Γ' and *vice versa* is sufficient for the equivalence of Γ and Γ'. In terms of theories: T and T' are equivalent if they share the underlying logic $\mathbf{L}$ and the axioms of each theory are theorems of the other. Similarly for the mutual agreement of two persons on some subject. C1 obviously holds for all transitive logics $\mathbf{L}$.

Criterion C2 states that if two premise sets are equivalent with respect to a Tarski logic weaker than $\mathbf{L}$, then they are equivalent with respect to $\mathbf{L}$. It is easily seen that C2 holds for all Tarski logics $\mathbf{L}$. Suppose indeed that the antecedent of C2 is true. As $Cn_{\mathbf{L}'}(\Gamma) \subseteq Cn_{\mathbf{L}}(\Gamma)$, $Cn_{\mathbf{L}'}(\Gamma) \cup \Gamma \subseteq Cn_{\mathbf{L}}(\Gamma)$ by the reflexivity of $\mathbf{L}$ and hence $Cn_{\mathbf{L}}(Cn_{\mathbf{L}'}(\Gamma) \cup \Gamma) \subseteq Cn_{\mathbf{L}}(\Gamma)$ by the transitivity of **L**. So, by the monotonicity of **L**, $Cn_{\mathbf{L}}(Cn_{\mathbf{L}'}(\Gamma) \cup \Gamma) = Cn_{\mathbf{L}}(\Gamma)$. Finally, as $Cn_{\mathbf{L}'}(\Gamma) \cup \Gamma = Cn_{\mathbf{L}'}(\Gamma)$ by the reflexivity of **L**′, $Cn_{\mathbf{L}}(Cn_{\mathbf{L}'}(\Gamma)) = Cn_{\mathbf{L}}(\Gamma)$. By the same reasoning $Cn_{\mathbf{L}}(Cn_{\mathbf{L}'}(\Gamma')) = Cn_{\mathbf{L}}(\Gamma')$. As $Cn_{\mathbf{L}'}(\Gamma) = Cn_{\mathbf{L}'}(\Gamma')$, $Cn_{\mathbf{L}}(\Gamma) = Cn_{\mathbf{L}}(\Gamma')$.

Criterion C3 is related to the fact that we expect operations under which **L**-consequence sets are closed to define a logic that is weaker than **L** or identical to it, which triggers C2. If, for all Δ, $A \wedge B \in Cn_{\mathbf{L}}(\Delta)$ just in case $A \in Cn_{\mathbf{L}}(\Delta)$ and $B \in Cn_{\mathbf{L}}(\Delta)$, then we expect $\Gamma \cup \{p \wedge q\}$ and $\Gamma \cup \{p, q\}$ to be **L**-equivalent premise sets.

The following lemma, due to Christian Straßer, establishes that C2 and C3 are coextensive whenever **L** is reflexive and $Cn_{\mathbf{L}}(\Gamma)$ is a fixed point. All Tarski logics are obviously a fixed point.

Lemma 5.7.1 *If* **L** *is reflexive,* $Cn_{\mathbf{L}}(\Gamma)$ *is a fixed point, and* **L**′ *is reflexive and monotonic, then* $Cn_{\mathbf{L}'}(Cn_{\mathbf{L}}(\Gamma)) = Cn_{\mathbf{L}}(\Gamma)$ *for all* Γ *iff* $Cn_{\mathbf{L}'}(\Gamma) \subseteq Cn_{\mathbf{L}}(\Gamma)$ *for all* Γ. *(Closure Lemma)*

Proof. Suppose that the antecedent is true. So, for all Γ, $\Gamma \subseteq Cn_{\mathbf{L}}(\Gamma)$ by the reflexivity of **L** and hence, for all Γ, $Cn_{\mathbf{L}'}(\Gamma) \subseteq Cn_{\mathbf{L}'}(Cn_{\mathbf{L}}(\Gamma))$ by the monotonicity of **L**′. We have to prove an equivalence.

$\Rightarrow$ Suppose that, for all Γ, $Cn_{\mathbf{L}'}(Cn_{\mathbf{L}}(\Gamma)) = Cn_{\mathbf{L}}(\Gamma)$. $\Gamma \subseteq Cn_{\mathbf{L}}(\Gamma)$ by the reflexivity of **L**. So $Cn_{\mathbf{L}'}(\Gamma) \subseteq Cn_{\mathbf{L}'}(Cn_{\mathbf{L}}(\Gamma))$ by the monotonicity of **L**′. From this and the supposition follows that, for all Γ, $Cn_{\mathbf{L}'}(\Gamma) \subseteq Cn_{\mathbf{L}}(\Gamma)$.

$\Leftarrow$ Suppose that, for all Γ, $Cn_{\mathbf{L}'}(\Gamma) \subseteq Cn_{\mathbf{L}}(\Gamma)$ and hence $Cn_{\mathbf{L}'}(Cn_{\mathbf{L}}(\Gamma)) \subseteq Cn_{\mathbf{L}}(Cn_{\mathbf{L}}(\Gamma))$. As $Cn_{\mathbf{L}}(\Gamma)$ is a fixed point, it follows that $Cn_{\mathbf{L}'}(Cn_{\mathbf{L}}(\Gamma)) \subseteq Cn_{\mathbf{L}}(\Gamma)$ for all Γ. So, by the reflexivity of **L**′, $Cn_{\mathbf{L}'}(Cn_{\mathbf{L}}(\Gamma)) = Cn_{\mathbf{L}}(\Gamma)$ for all Γ. ■

That **L**′ is a Tarski logic is essential for both C2 and C3; they do not hold for an arbitrary logic **L**′. To see this, let **L**′ be defined by $Cn_{\mathbf{L}'}(\Gamma) = \{A \in \Gamma \mid$ for all $B \in \mathcal{W}_s$, $B \notin Cn_{\mathbf{CL}}(\{A\})$ or $B \in \Gamma\}$. In words, the **L**′-consequence set of Γ are those members of Γ of which all **CL**-consequences are members of Γ. Obviously, it holds for all Δ that $Cn_{\mathbf{L}'}(\Delta) \subseteq Cn_{\mathbf{CL}}(\Delta)$ and also that $Cn_{\mathbf{L}'}(Cn_{\mathbf{CL}}(\Delta)) = Cn_{\mathbf{CL}}(\Delta)$. However, there are infinitely many Γ for which no $A \in \Gamma$ is such that $Cn_{\mathbf{CL}}(A) \subseteq \Gamma$. For all of them $Cn_{\mathbf{L}'}(\Gamma) = Cn_{\mathbf{L}'}(\emptyset)$ but $Cn_{\mathbf{CL}}(\Gamma) \neq Cn_{\mathbf{CL}}(\emptyset)$.

Obviously, C1 may be combined with C2 or C3. Thus if **L**′ is a Tarski logic weaker than **L**, $\Gamma' \subseteq Cn_{\mathbf{L}'}(\Gamma)$ and $\Gamma \subseteq Cn_{\mathbf{L}'}(\Gamma')$, then Γ and Γ' are **L**-equivalent.

C1–C3 do *not* hold in general for *defeasible* logics. Consider first the Strong (also called Inevitable) and Weak consequence relations from [RM70]—see also [BDP97]. Given a possibly inconsistent set of premises Γ, $\Delta \subseteq \Gamma$ is a maximal consistent subset of Γ iff, for all $A \in \Gamma - \Delta$, $\Delta \cup \{A\}$ is inconsistent. $\Gamma \vdash_{Strong} A$ iff A is a **CL**-consequence of every maximal consistent subset of Γ and $\Gamma \vdash_{Weak} A$ iff A is a **CL**-consequence of some maximal consistent subset of Γ.

C1 does not hold for the Weak consequence relation. Here is an example: $\{p, q, \neg p\} \subseteq Cn_{Weak}(\{p \wedge q, \neg p\})$ and $\{p \wedge q, \neg p\} \subseteq Cn_{Weak}(\{p, q, \neg p\})$, but $\neg p \wedge q \in Cn_{Weak}(\{p, q, \neg p\}) - Cn_{Weak}(\{p \wedge q, \neg p\})$.

C3 fails for the Strong consequence relation. Let **LC** be the Tarski logic that consists, apart from the Premise rule, of the rules Adjunction and Simplification. All Strong consequence sets are closed under **LC**, viz. $Cn_{Strong}(\Gamma) =$

$Cn_{\mathbf{LC}}(Cn_{Strong}(\Gamma))$ for all Γ. However, $Cn_{\mathbf{LC}}(\{p, q, \neg p\}) = Cn_{\mathbf{LC}}(\{p \wedge q, \neg p\})$ but $Cn_{Strong}(\{p, q, \neg p\}) \neq Cn_{Strong}(\{p \wedge q, \neg p\})$, for example $p, q, \neg p \vdash_{Strong} q$ whereas $p \wedge q, \neg p \nvdash_{Strong} q$.

For an example of a logic for which C2 does not hold, I shall remain close to the Rescher-Manor consequence relations, adding a (weak) Schotch-Jennings flavour—see for example [SJ89]. A partition of Γ is a set of sets $\{\Gamma_1, \ldots, \Gamma_n\}$ $(n \geq 1)$ such that $\Gamma = \Gamma_1 \cup \ldots \cup \Gamma_n$ and $\Gamma_i \cap \Gamma_j = \emptyset$ for all different $i, j \in \{1, \ldots, n\}$. A partition $\{\Gamma_1, \ldots, \Gamma_n\}$ of Γ is *consistent* iff every Γ_i $(1 \leq i \leq n)$ is consistent. Obviously, Γ has a consistent partition iff all $A \in \Gamma$ are consistent. The *regular* partitions of Γ are the consistent ones or, if there are no consistent ones, all partitions of Γ. Define: $A \in Cn_{\mathbf{R}}(\Gamma)$ iff there is a regular partition $\{\Gamma_1, \ldots, \Gamma_n\}$ of Γ and an i $(1 \leq i \leq n)$ such that $A \in Cn_{\mathbf{CL}}(\Gamma_i)$. Define $Cn_{\mathbf{Q}}(\Gamma) = Cn_{\mathbf{CL}^+}(Cn_{\mathbf{R}}(\Gamma))$, in which $\mathbf{CL}^+$ is full positive **CL**.[15] If $\{\Gamma\}$ is a regular partition of Γ, $Cn_{\mathbf{Q}}(\Gamma) = Cn_{\mathbf{CL}}(\Gamma)$; if some $A \in \Gamma$ is inconsistent, $Cn_{\mathbf{Q}}(\Gamma)$ is trivial; if Γ is inconsistent but all $A \in \Gamma$ are consistent, $Cn_{\mathbf{Q}}(\Gamma)$ is inconsistent but non-trivial, border cases aside. Note that $\mathbf{CL}^+$ is a Tarski logic and that it is weaker than **Q**, viz. $Cn_{\mathbf{CL}^+}(\Gamma) \subseteq Cn_{\mathbf{Q}}(\Gamma)$ for all Γ. C2 does not hold for the defeasible logic **Q**. Indeed, $Cn_{\mathbf{CL}^+}(\{p, \neg p\}) = Cn_{\mathbf{CL}^+}(\{p \wedge \neg p\})$, but $Cn_{\mathbf{Q}}(\{p \wedge \neg p\})$ is trivial whereas $Cn_{\mathbf{Q}}(\{p, \neg p\})$ is not.

These examples are rather 'generous' because the situation is actually worse for certain systems describing defeasible reasoning forms. For example, for the many kinds of default logics the criteria C1–3 should be reformulated in order to make a chance to be applicable. The set of defaults has to enter the picture and 'facts' and defaults are to some extent exchangeable. The situation is similar for many other logics characterizing defeasible reasoning forms, even for the very transparent pivotal-assumption consequences defined in [Mak05].

That C1–C3 hold for all adaptive logics is easily provable in view of the fact that they have the properties Reflexivity, Cumulative Transitivity, Cautious Monotonicity, and Fixed Point.

Theorem 5.7.1 *C1 holds for all adaptive logics.*

Proof. Suppose that $\Gamma' \subseteq Cn_{\mathbf{AL}}(\Gamma)$ and $\Gamma \subseteq Cn_{\mathbf{AL}}(\Gamma')$. By Corollary 5.6.2, $Cn_{\mathbf{AL}}(\Gamma) = Cn_{\mathbf{AL}}(\Gamma \cup \Gamma')$ and $Cn_{\mathbf{AL}}(\Gamma') = Cn_{\mathbf{AL}}(\Gamma \cup \Gamma')$. So $Cn_{\mathbf{AL}}(\Gamma) = Cn_{\mathbf{AL}}(\Gamma')$. ■

Note that C1 has an implicative form. The converse of the implication, however, follows immediately from the Reflexivity of **AL** (Theorem 5.6.4). This gives us the following corollary.

Corollary 5.7.1 Γ *and* Γ' *are* **AL**-*equivalent* ($Cn_{\mathbf{AL}}(\Gamma) = Cn_{\mathbf{AL}}(\Gamma')$) *iff* $\Gamma' \subseteq Cn_{\mathbf{AL}}(\Gamma)$ *and* $\Gamma \subseteq Cn_{\mathbf{AL}}(\Gamma')$. *(Formulation Independence.)*

Theorem 5.7.2 *C2 and C3 hold for all adaptive logics.*

Proof. C2 and C3 are coextensive for all adaptive logics because of Lemma 5.7.1 together with Theorems 5.6.4 and 5.6.3. So it suffices to prove that C2 holds for all adaptive logics.

[15] $\mathbf{CL}^+$ is **CL** with both axioms for negation removed.

Suppose that the antecedent of C2, $Cn_{\mathbf{L}'}(\Gamma) \subseteq Cn_{\mathbf{AL}}(\Gamma)$, holds true for all Γ. As **AL** is reflexive (Theorem 5.6.4), it follows that

$$\Gamma \cup Cn_{\mathbf{L}'}(\Gamma) \subseteq Cn_{\mathbf{AL}}(\Gamma)\,.$$

From this, by Corollary 5.6.2,

$$Cn_{\mathbf{AL}}(\Gamma) = Cn_{\mathbf{AL}}(\Gamma \cup Cn_{\mathbf{L}'}(\Gamma))\,,$$

whence, as $\mathbf{L}'$ is reflexive,

$$Cn_{\mathbf{AL}}(\Gamma) = Cn_{\mathbf{AL}}(Cn_{\mathbf{L}'}(\Gamma))\,.$$

By the same reasoning

$$Cn_{\mathbf{AL}}(\Gamma') = Cn_{\mathbf{AL}}(Cn_{\mathbf{L}'}(\Gamma'))\,.$$

So, as $Cn_{\mathbf{L}'}(\Gamma) = Cn_{\mathbf{L}'}(\Gamma')$ by the supposition,

$$Cn_{\mathbf{AL}}(\Gamma) = Cn_{\mathbf{AL}}(\Gamma')\,.$$

■

Note that, for every adaptive logic **AL**, **LLL** is a Tarski logic weaker than or identical to **AL**. So if two premise sets are **LLL**-equivalent, they are also **AL**-equivalent in view of C2. For some premise sets, however, one needs to rely directly on C1. An example is that $Cn_{\mathbf{CLuN}^m}(\{p\}) = Cn_{\mathbf{CLuN}^m}(\{p \vee (q \wedge \neg q)\})$. While $Cn_{\mathbf{CLuN}}(\{p\}) \neq Cn_{\mathbf{CLuN}}(\{p \vee (q \wedge \neg q)\})$, it is easy enough to show that $\{p\} \vdash_{\mathbf{CLuN}^m} p \vee (q \wedge \neg q)$ and that $\{p \vee (q \wedge \neg q)\} \vdash_{\mathbf{CLuN}^m} p$.

That C1–C3 hold for all adaptive logics is a particularly interesting and somewhat unexpected property. Cumulative Indifference—see Corollary 5.6.2—provides itself a criterion related to C1, but the criterion has rather limited applications.

What precedes should not be confused with the behaviour of extensions of premise sets. In this respect adaptive logics do not behave like Tarski logics. At first sight, they seem to behave just as strangely as other formal approaches to defeasible reasoning. So let us have a closer look.

Fact 5.7.1 *If* **L** *is a Tarski logic, then* $Cn_{\mathbf{L}}(\Gamma_1) = Cn_{\mathbf{L}}(\Gamma_2)$ *warrants that* $Cn_{\mathbf{L}}(\Gamma_1 \cup \Delta) = Cn_{\mathbf{L}}(\Gamma_2 \cup \Delta)$.

Fact 5.7.2 $Cn_{\mathbf{AL}}(\Gamma_1) = Cn_{\mathbf{AL}}(\Gamma_2)$ *does not warrant that* $Cn_{\mathbf{AL}}(\Gamma_1 \cup \Delta) = Cn_{\mathbf{AL}}(\Gamma_2 \cup \Delta)$.

The proof of the first fact is left as an easy exercise for the reader. The second fact is established by the following example:

$$Cn_{\mathbf{CLuN}^m}(\{p\}) = Cn_{\mathbf{CLuN}^m}(\{p \vee (q \wedge \neg q)\}) \text{ but}$$
$$Cn_{\mathbf{CLuN}^m}(\{p, q \wedge \neg q\}) \neq Cn_{\mathbf{CLuN}^m}(\{p \vee (q \wedge \neg q), q \wedge \neg q\}).$$

Note that the example may be adjusted to any adaptive logic in which classical disjunction is present or definable. The example clearly indicates the most straightforward reason why the fact holds. The formula $q \wedge \neg q$ is an abnormality and hence is supposed to be false ‘unless and until proven otherwise’. So the original premise sets are equivalent because $p \vee (q \wedge \neg q)$ comes to p on the supposition. If, however, $q \wedge \neg q$ is added to the premise sets, $\{p, q \wedge \neg q\}$ still gives us p because $\mathbf{CLuN}^m$ is reflexive, but p is not derivable from $\{p \vee (q \wedge \neg q), q \wedge \neg q\}$ because the extended premise set requires $q \wedge \neg q$ to be true. To the negative fact corresponds a positive result which is very similar to it.

Theorem 5.7.3 *If* **L** *is a Tarski logic weaker than or identical to* **AL** *and* $Cn_{\mathbf{L}}(\Gamma_1) = Cn_{\mathbf{L}}(\Gamma_2)$*, then* $Cn_{\mathbf{AL}}(\Gamma_1 \cup \Delta) = Cn_{\mathbf{AL}}(\Gamma_2 \cup \Delta)$ *for all* Δ.

Proof. Suppose that the antecedent is true. So, in view of Fact 5.7.1, $Cn_{\mathbf{L}}(\Gamma_1 \cup \Delta) = Cn_{\mathbf{L}}(\Gamma_2 \cup \Delta)$. But then $Cn_{\mathbf{AL}}(\Gamma_1 \cup \Delta) = Cn_{\mathbf{AL}}(\Gamma_2 \cup \Delta)$ by C2 (Theorem 5.7.2). ■

For adaptive logics there is a weaker alternative for Fact 5.7.1. For this, we need another definition.

Definition 5.7.2 *A set of formulas* Θ *is an* **AL**-monotonic extension *of a set of formulas* Γ *iff* $\Gamma \subset \Theta$ *and* $Cn_{\mathbf{AL}}(\Gamma) \subseteq Cn_{\mathbf{AL}}(\Theta)$.

Theorem 5.7.4 *If* $\Gamma_1 \cup \Delta$ *is an* **AL***-monotonic extension of* Γ_1 *and* $\Gamma_2 \cup \Delta$ *is an* **AL***-monotonic extension of* Γ_2*, then* $Cn_{\mathbf{AL}}(\Gamma_1) = Cn_{\mathbf{AL}}(\Gamma_2)$ *warrants that* $Cn_{\mathbf{AL}}(\Gamma_1 \cup \Delta) = Cn_{\mathbf{AL}}(\Gamma_2 \cup \Delta)$

Proof. Suppose that the antecedent is true and that $Cn_{\mathbf{AL}}(\Gamma_1) = Cn_{\mathbf{AL}}(\Gamma_2)$. By Definition 5.7.2, the supposition implies that $Cn_{\mathbf{AL}}(\Gamma_1) \subseteq Cn_{\mathbf{AL}}(\Gamma_1 \cup \Delta)$, and the Reflexivity of **AL** (Theorem 5.6.4) gives us $\Delta \subseteq Cn_{\mathbf{AL}}(\Gamma_1 \cup \Delta)$. So

$$Cn_{\mathbf{AL}}(\Gamma_1) \cup \Delta \subseteq Cn_{\mathbf{AL}}(\Gamma_1 \cup \Delta)\,,$$

whence, by Corollary 5.6.2,

$$Cn_{\mathbf{AL}}(\Gamma_1 \cup Cn_{\mathbf{AL}}(\Gamma_1) \cup \Delta) = Cn_{\mathbf{AL}}(\Gamma_1 \cup \Delta)\,.$$

As $\Gamma_1 \subseteq Cn_{\mathbf{AL}}(\Gamma_1)$ by Reflexivity of **AL** (Theorem 5.6.4), it follows that

$$Cn_{\mathbf{AL}}(\Gamma_1 \cup \Delta) = Cn_{\mathbf{AL}}(Cn_{\mathbf{AL}}(\Gamma_1) \cup \Delta)\,.$$

By the same reasoning,

$$Cn_{\mathbf{AL}}(\Gamma_2 \cup \Delta) = Cn_{\mathbf{AL}}(Cn_{\mathbf{AL}}(\Gamma_2) \cup \Delta)\,.$$

The second half of the supposition implies

$$Cn_{\mathbf{AL}}(Cn_{\mathbf{AL}}(\Gamma_1) \cup \Delta) = Cn_{\mathbf{AL}}(Cn_{\mathbf{AL}}(\Gamma_2) \cup \Delta)\,.$$

So $Cn_{\mathbf{AL}}(\Gamma_1 \cup \Delta) = Cn_{\mathbf{AL}}(\Gamma_2 \cup \Delta)$. ■

There are criteria for deciding whether an extension is **AL**-monotonic. The criteria depend on the strategy of **AL**. The criteria introduced below may not be the sharpest possible ones, but they are obviously correct. Let Γ be the original premise set and Γ' the extended premise set.

For the Reliability strategy, the criterium reads: If $\Gamma \subseteq \Gamma'$ and $U(\Gamma') \subseteq U(\Gamma)$ then Γ' is an **AL**-monotonic extension of Γ. In words: if Γ' includes Γ and every abnormality that is unreliable with respect to Γ' is also unreliable with respect to Γ, then Γ' is an **AL**-monotonic extension of Γ. In terms of the proof theory, this means that every unmarked line in a proof from Γ remains unmarked if the premise set is extended to Γ'. This warrants that the final consequences of Γ are also final consequences of Γ'. Obviously, some lines that are marked in a proof from Γ may be unmarked in a proof from Γ'. The effect of this is that the latter premise set has more, but not less, consequences than the former.

For the Minimal Abnormality strategy, the criterium reads: If $\Gamma \subseteq \Gamma'$ and for every $\varphi' \in \Phi(\Gamma')$, there is a $\varphi \supseteq \Delta'$ for which $\varphi \in \Phi(\Gamma)$, then Γ' is an **AL**-monotonic extension of Γ. The criterion is most easily understood from a semantic point of view. The antecedent warrants that every **AL**-model of Γ' is a **AL**-model of Γ and hence verifies every formula verified by all **AL**-models of Γ.

It is instructive to illustrate the difference between the criteria in terms of $\mathbf{CLuN}^r$ and $\mathbf{CLuN}^m$. Let $\Gamma = \{(p \wedge \neg p) \vee (q \wedge \neg q), (p \wedge \neg p) \vee (r \wedge \neg r), s \vee (p \wedge \neg p), s \vee (q \wedge \neg q)\}$ and let $\Gamma' = \Gamma \cup \{q \wedge \neg q\}$. As $U(\Gamma) = U(\Gamma') = \{p \wedge \neg p, q \wedge \neg q, r \wedge \neg r\}$, Γ' is a $\mathbf{CLuN}^r$-monotonic extension of Γ. However, $\Phi(\Gamma) = \{\{p \wedge \neg p\}, \{q \wedge \neg q, r \wedge \neg r\}\}$ whereas $\Phi(\Gamma') = \{\{q \wedge \neg q, p \wedge \neg p\}, \{q \wedge \neg q, r \wedge \neg r\}\}$. So Γ' is not a $\mathbf{CLuN}^m$-monotonic extension of Γ and actually $\Gamma \vdash_{\mathbf{CLuN}^m} s$ whereas $\Gamma' \nvdash_{\mathbf{CLuN}^m} s$.

5.8 Maximality of the Lower Limit Logic

As **LLL** is a Tarski logic weaker than **AL**, Theorem 5.7.3 entails the following.

Corollary 5.8.1 *If* $Cn_{\mathbf{LLL}}(\Gamma_1) = Cn_{\mathbf{LLL}}(\Gamma_2)$, *then* $Cn_{\mathbf{AL}}(\Gamma_1 \cup \Delta) = Cn_{\mathbf{AL}}(\Gamma_2 \cup \Delta)$ *for all* Δ.

Moreover, the lower limit logic may be relied upon for applications of C2 or, in view of Theorem 5.6.1, C3. However, the lower limit logic **LLL** of an adaptive logic **AL** is not only a Tarski logic that is weaker than **AL**. Every monotonic logic **L** that is weaker than **AL** is weaker than **LLL** or identical to **LLL**. The proof of the following theorem relies on the compactness of **LLL**, but does not require **L** to be compact.

Theorem 5.8.1 *For all* monotonic *logics* **L** *weaker than or identical to* $\mathbf{AL}^m$ *and for all* Γ, $Cn_{\mathbf{L}}(\Gamma) \subseteq Cn_{\mathbf{LLL}}(\Gamma)$.

Proof. Suppose that **L** is a monotonic logic weaker than $\mathbf{AL}^m$ or identical to it, and that there is a Γ and a B for which the following three hold.

$$\Gamma \nvdash_{\mathbf{LLL}} B \tag{5.1}$$

$$\Gamma \vdash_{\mathbf{L}} B \tag{5.2}$$

$$\Gamma \vdash_{\mathbf{AL}^m} B \tag{5.3}$$

Let $\Gamma' = \{Dab(\Delta) \mid \Gamma \vdash_{\mathbf{LLL}} B \check{\vee} Dab(\Delta)\}$. In view of the definition of Γ', (5.1) entails (5.4); (5.5) follows from (5.2) by the monotonicity of **L**, and (5.6) follows from (5.5) by the supposition.

$$\Gamma \cup \Gamma' \nvdash_{\mathbf{LLL}} B \tag{5.4}$$

$$\Gamma \cup \Gamma' \vdash_{\mathbf{L}} B \tag{5.5}$$

$$\Gamma \cup \Gamma' \vdash_{\mathbf{AL}^m} B \tag{5.6}$$

In view of Theorem 5.3.3, it follows from (5.6) and (5.4) that, for every $\varphi \in \Phi(\Gamma \cup \Gamma')$, there is a $\Delta \subset \Omega$ such that

$$\Gamma \cup \Gamma' \vdash_{\mathbf{LLL}} B \check{\vee} Dab(\Delta),\ \Delta \neq \emptyset \text{ and } \Delta \cap \varphi = \emptyset\,. \tag{5.7}$$

In view of the compactness and monotonicity of **LLL** there are $Dab(\Delta_1), \ldots, Dab(\Delta_n) \in \Gamma'$ such that

$$\Gamma \cup \{Dab(\Delta_1), \ldots, Dab(\Delta_n)\} \vdash_{\mathbf{LLL}} B \check{\vee} Dab(\Delta)\,. \tag{5.8}$$

As $\Gamma \vdash_{\mathbf{LLL}} B \check{\vee} Dab(\Delta_i)$ for every $i \in \{1, \ldots, n\}$,

$$\Gamma \vdash_{\mathbf{LLL}} B \check{\vee} (Dab(\Delta_1) \check{\wedge} \ldots \check{\wedge} Dab(\Delta_n)). \tag{5.9}$$

From (5.8) follows

$$\Gamma \cup \{Dab(\Delta_1) \check{\wedge} \ldots \check{\wedge} Dab(\Delta_n)\} \vdash_{\mathbf{LLL}} B \check{\vee} Dab(\Delta), \tag{5.10}$$

whence, by the Deduction Theorem,

$$\Gamma \vdash_{\mathbf{LLL}} (Dab(\Delta_1) \check{\wedge} \ldots \check{\wedge} Dab(\Delta_n)) \check{\supset} (B \check{\vee} Dab(\Delta))\,. \tag{5.11}$$

From (5.9) and (5.11) follows

$$\Gamma \vdash_{\mathbf{LLL}} B \check{\vee} Dab(\Delta)\,, \tag{5.12}$$

whence $Dab(\Delta) \in \Gamma'$. But then every $\varphi \in \Phi(\Gamma \cup \Gamma')$ contains at least one member of Δ, which contradicts (5.7). ■

The crucial step in the proof is the transition from (5.5) to (5.6). If the monotonic **L** is indeed weaker than $\mathbf{AL}^m$ or identical to it, then a consequence of the monotonicity of **L**, viz. (5.5) may be carried over to $\mathbf{AL}^m$, and this leads to a contradiction.

By Corollary 5.3.3, this result also holds when the third element of **AL** is Reliability. Hence we obtain the following corollary.

Corollary 5.8.2 *Every monotonic logic* **L** *that is weaker than or identical to* **AL** *is weaker than* **LLL** *or identical to* **LLL**.

Lemma 5.7.1 gives us a further corollary.

Corollary 5.8.3 *If* $Cn_{\mathbf{AL}}(\Gamma)$ *is closed under a monotonic logic* **L**, *then* **L** *is weaker than* **LLL** *or identical to* **LLL**.

This corollary has immediate consequences for the previous section. The lower limit logic **LLL** allows for very sharp applications of C2 and C3. Moreover, the lower limit logic is the strongest Tarski logic **L** for which holds: if two premise sets are **L**-equivalent and both are extended with the same set for formulas, then these extensions are **AL**-equivalent. But the corollary has an even greater import. For one thing, it defines the largest subset of $Cn_{\mathbf{AL}}(\Gamma)$ that may be provided from Γ by a monotonic logic. All this highlights the pivotal role of the lower limit logic.

Corollary 5.8.3 has a further important consequence. By Theorem 1.5.6, logics that have static proofs are monotonic. This guarantees that genuine adaptive logics do not have static proofs, as the next corollary states. In other words, no logic that has static proofs agrees with the function $\mathbf{AL}\colon \wp(\mathcal{W}) \to \wp(\mathcal{W})$. This establishes the specific character of adaptive logics.

Corollary 5.8.4 *If* $Cn_{\mathbf{LLL}}(\Gamma) \subset Cn_{\mathbf{AL}}(\Gamma)$, *then* **AL** *does not have static proofs.*

5.9 Border Cases

Everything proved in this chapter and the preceding one was proved for all adaptive logics in standard format. So the standard format, defined in Section 4.2, turns out to be a very powerful tool. Nevertheless, the standard format allows for certain border cases that are not adaptive in a serious sense because they have static proofs. In this section, I review the most important border cases.

Let me begin with a harmless border case. If Γ comprises all $\mathcal{L}$-formulas verified by a **LLL**-model M, then all models of Γ verify the same formulas. These formulas comprise the members of Γ, the classical negation of all non-members of Γ, and every formula **CL**-derivable from these.

Fact 5.9.1 *If there is a* **LLL**-*model* M *such that* $\Gamma = \{A \mid A \in \mathcal{W}; M \Vdash A\}$, *then (i)* $Cn^{\mathcal{L}_+}_{\mathbf{LLL}}(\Gamma) = Cn^{\mathcal{L}_+}_{\mathbf{AL}^r}(\Gamma) = Cn^{\mathcal{L}_+}_{\mathbf{AL}^m}(\Gamma)$ *and (ii) if* M *is a* **ULL**-*model,* $Cn^{\mathcal{L}_+}_{\mathbf{AL}^m}(\Gamma) = Cn^{\mathcal{L}_+}_{\mathbf{ULL}}(\Gamma)$, *otherwise* $Cn^{\mathcal{L}_+}_{\mathbf{AL}^m}(\Gamma) \subset Cn^{\mathcal{L}_+}_{\mathbf{ULL}}(\Gamma) = \mathcal{W}_+$.

Obviously $Cn^{\mathcal{L}_+}_{\mathbf{ULL}}(\Gamma)$ and $Cn^{\mathcal{L}}_{\mathbf{ULL}}(\Gamma)$ are trivial if M is not a **ULL**-model. The fact holds *a fortiori* if $\Gamma = \{A \mid A \in \mathcal{W}_+; M \Vdash A\}$. This border case is harmless because it does not affect the logic, but only the consequence sets of a specific premise set. So this is not the kind of border case that interests us in the present section. Yet, one should remember that premise sets may cause border cases—see also Section 6.1.1. But let us turn the attention to border cases that affect the adaptive logic.

Nothing in Section 4.2 excludes that **LLL** is the trivial logic **Tr**—see Section 1.5. Writing any formula of $\mathcal{W}$ constitutes a **Tr**-proof of the formula. So **Tr** has static proofs.

Fact 5.9.2 *If* **LLL** *is* **Tr**, *then* **AL** *has static proofs and* $Cn_{\mathbf{LLL}}(\Gamma) = Cn_{\mathbf{AL}}(\Gamma) = Cn_{\mathbf{ULL}}(\Gamma)$.

Remember (from Section 4.3) that the classical logical symbols are superimposed rather than intertwined. So, where **Tr** is defined with respect to $\mathcal{L}$, $Cn^{\mathcal{L}}_{\mathbf{Tr}}(\Gamma) = \mathcal{W}$ but $Cn^{\mathcal{L}_+}_{\mathbf{Tr}}(\Gamma) \neq \mathcal{W}_+$. This is most easily seen in semantic terms. $\Gamma \vDash_{\mathbf{Tr}} A$ comes out true, but $\Gamma \vDash_{\mathbf{Tr}} \check{\neg} A$ comes out false because it now means that no **Tr**-model of Γ verifies A whereas every **Tr**-model of Γ verifies A.

The opposite border case is the empty logic **Em** from Section 1.5, according to which $Cn_{\mathbf{Em}}(\Gamma) = \emptyset$ for all Γ. One might expect a similar situation as for **Tr**. However, **Em** does not have static proofs because, being non-reflexive, it necessarily misses the premise rule.

Let us now turn to border cases of the set of abnormalities. If $\Omega = \emptyset$, all models are selected by both Reliability and Minimal Abnormality because $U(\Gamma) = \emptyset$ and $\Phi(\Gamma) = \{\emptyset\}$ for all Γ. So all lower limit models are upper limit models and every line of an **AL**-proof has an empty condition, whence no line is ever marked.

Fact 5.9.3 *If* $\Omega = \emptyset$, *then* **AL** *has static proofs,* $\mathcal{M}^{\mathbf{LLL}}_{\Gamma} = \mathcal{M}^{r}_{\Gamma} = \mathcal{M}^{m}_{\Gamma} = \mathcal{M}^{\mathbf{ULL}}_{\Gamma}$, *and* $Cn_{\mathbf{LLL}}(\Gamma) = Cn_{\mathbf{AL}}(\Gamma) = Cn_{\mathbf{ULL}}(\Gamma)$.

If $\Omega = \mathcal{W}$, then $A \in U(\Gamma)$ whenever $\Gamma \vdash_{\mathbf{LLL}} A$ or $\Gamma \vdash_{\mathbf{LLL}} A \check{\vee} B$. Suppose that $\Gamma \nvdash_{\mathbf{LLL}} A$ and $\Gamma \vdash_{\mathbf{AL}^r} A$. It follows that there is a minimal $\Delta \subseteq \Omega$ for which $\Gamma \vdash_{\mathbf{LLL}} A \check{\vee} Dab(\Delta)$ and $\Delta \cap U(\Gamma) = \emptyset$. But then $\Gamma \vdash_{\mathbf{LLL}} Dab(\{A\} \cup \Delta)$ in which $Dab(\{A\} \cup \Delta)$ is a minimal *Dab*-consequence of Γ. So $\Delta \subseteq U(\Gamma)$, which contradicts $\Delta \cap U(\Gamma) = \emptyset$. It follows that $Cn_{\mathbf{AL}^r}(\Gamma) = Cn_{\mathbf{LLL}}(\Gamma)$.

The reasoning for Minimal Abnormality is similar but slightly more complicated. Suppose that $\Gamma \nvdash_{\mathbf{LLL}} A$ and $\Gamma \vdash_{\mathbf{AL}^m} A$. So there is a *minimal* Δ_1 and a $\varphi_1 \in \Phi(\Gamma)$ for which $\Gamma \vdash_{\mathbf{LLL}} A \check{\vee} Dab(\Delta_1)$ and $\Delta_1 \cap \varphi_1 = \emptyset$. It follows that $Dab(\{A\} \cup \Delta_1)$ is a minimal *Dab*-consequence of Γ and, in view of Fact 5.1.1, that $A \in \varphi_1$. By Fact 5.1.9, there is a $\varphi_2 \in \Phi(\Gamma)$ such that $A \notin \varphi_2$ and hence $\Delta_1 \cap \varphi_2 \neq \emptyset$. So it follows from the supposition that there is a minimal Δ_2 for which $\Gamma \vdash_{\mathbf{LLL}} A \check{\vee} Dab(\Delta_2)$ and $\Delta_2 \cap \varphi_2 = \emptyset$. But then $Dab(\{A\} \cup \Delta_2)$ is a minimal *Dab*-consequence of Γ, whence, by Fact 5.1.9, either A or a member of Δ_2 is a member of φ_2. But this is impossible as $A \notin \varphi_2$ and $\Delta_2 \cap \varphi_2 = \emptyset$. It follows that $Cn_{\mathbf{AL}^m}(\Gamma) = Cn_{\mathbf{LLL}}(\Gamma)$.

Finally, consider the upper limit logic. If $\Omega = \mathcal{W}$, then **ULL** is the trivial logic **Tr** in view of Definition 4.2.1. Note, however, that the lower limit logic may itself be **Tr**. In the following fact and later on, I write $\mathcal{M}^{\mathbf{AL}}_{\Gamma}$ to denote the **LLL**-models of Γ selected by the adaptive logic **AL** (independent of whether the strategy of **AL** is Reliability or minimal Abnormality).

Fact 5.9.4 *If* $\Omega = \mathcal{W}$*, then* **AL** *has static proofs,* $\mathcal{M}^{\mathbf{LLL}}_{\Gamma} = \mathcal{M}^{r}_{\Gamma} = \mathcal{M}^{m}_{\Gamma}$*, and* $Cn_{\mathbf{LLL}}(\Gamma) = Cn_{\mathbf{AL}}(\Gamma)$*. If moreover* **LLL** *is* **Tr** *or* $Cn^{\mathcal{L}_+}_{\mathbf{LLL}}(\Gamma) = \mathcal{W}_+$*, then* $\mathcal{M}^{\mathbf{AL}}_{\Gamma} = \mathcal{M}^{\mathbf{ULL}}_{\Gamma}$ *and* $Cn_{\mathbf{AL}}(\Gamma) = Cn_{\mathbf{ULL}}(\Gamma)$*; otherwise* $\mathcal{M}^{\mathbf{AL}}_{\Gamma} \supset \mathcal{M}^{\mathbf{ULL}}_{\Gamma}$ *and* $Cn_{\mathbf{AL}}(\Gamma) \subset Cn_{\mathbf{ULL}}(\Gamma)$*.*

Incidentally, nothing interesting changes if $\Omega = \mathcal{W}_+$ instead of $\Omega = \mathcal{W}$. The complication is that most *Dab*-formulas actually represent several *Dab*-formulas. For example if $A_1, \ldots, A_n \in \Omega$, then $A_1 \check{\vee} A_2, A_3 \check{\vee} A_4, \ldots, A_1 \check{\vee} \ldots \check{\vee} A_n \in \Omega$. So if $Dab(\{A_1, \ldots, A_n\})$ is a *Dab*-consequence of Γ, then all those formulas should be members of $U(\Gamma)$ and the different ways in which the continuous disjunction may be split up will play their role in $\Phi(\Gamma)$.

Let us turn to the border case in which every member of Ω is a **LLL**-theorem. In this case, every line that has a non-empty condition will be marked in an extension of the proof as well as in all extensions of the extension. So the only formulas that are finally derivable from Γ are those that are derivable on the empty condition, viz. are **LLL**-derivable from Γ. Obviously the upper limit logic is again **Tr** in view of Definition 4.2.1. Note that also the lower limit logic may be **Tr** or, which is a more general case, that $Cn^{\mathcal{L}_+}_{\mathbf{LLL}}(\Gamma)$ may be trivial.

Fact 5.9.5 *If* $\Omega \subseteq Cn^{\mathcal{L}}_{\mathbf{LLL}}(\emptyset)$*, then* **AL** *has static proofs and* $\mathcal{M}^{\mathbf{LLL}}_{\Gamma} = \mathcal{M}^{r}_{\Gamma} = \mathcal{M}^{m}_{\Gamma}$*, and* $Cn_{\mathbf{LLL}}(\Gamma) = Cn_{\mathbf{AL}}(\Gamma)$*. If moreover* $Cn^{\mathcal{L}_+}_{\mathbf{LLL}}(\Gamma) = \mathcal{W}_+$*, then* $\mathcal{M}^{\mathbf{AL}}_{\Gamma} = \mathcal{M}^{\mathbf{ULL}}_{\Gamma}$ *and* $Cn_{\mathbf{AL}}(\Gamma) = Cn_{\mathbf{ULL}}(\Gamma)$*; otherwise* $\mathcal{M}^{\mathbf{AL}}_{\Gamma} \supset \mathcal{M}^{\mathbf{ULL}}_{\Gamma}$ *and* $Cn_{\mathbf{AL}}(\Gamma) \subset Cn_{\mathbf{ULL}}(\Gamma)$*.*

Suppose that every member of Ω is a **LLL**-falsehood. This means that, if $A \in \Omega$, then $Cn_{\mathbf{LLL}}(\{A\})$ is trivial. So, if Γ is abnormal, $Cn_{\mathbf{LLL}}(\Gamma)$ is trivial. More importantly, it follows that the upper limit logic is identical to the lower limit logic and hence, by Corollary 5.3.3, that the adaptive logic is identical to the lower limit logic. Moreover, no **LLL**-model verifies any abnormality. That the adaptive logic is identical to its lower limit may also be seen directly. Whether the strategy is Reliability or Minimal Abnormality, if $\Gamma \nvdash_{\mathbf{LLL}} A$ and

$\Gamma \vdash_{\mathbf{AL}} A$, there must be one or more $\Delta \subseteq \Omega$ such that $\Gamma \vdash_{\mathbf{LLL}} A \check{\vee} Dab(\Delta)$ and the Δ fulfil certain conditions. If all members of Δ are **LLL**-falsehoods, however, $\Gamma \vdash_{\mathbf{LLL}} A$ follows from $\Gamma \vdash_{\mathbf{LLL}} A \check{\vee} Dab(\Delta)$ by **CL**-properties. It is instructive to rephrase this with respect to unmarked lines of a proof in view of Lemma 4.4.1.

Fact 5.9.6 *If* $\Omega \subseteq \{A \mid A \vdash_{\mathbf{LLL}} B \text{ for all } B\}$*, then* $\mathcal{M}^{\mathbf{LLL}}_{\Gamma} = \mathcal{M}^{r}_{\Gamma} = \mathcal{M}^{m}_{\Gamma} = \mathcal{M}^{\mathbf{ULL}}_{\Gamma}$*, and* $Cn_{\mathbf{LLL}}(\Gamma) = Cn_{\mathbf{AL}}(\Gamma) = Cn_{\mathbf{ULL}}(\Gamma)$*.*

If only *some* members of Ω are **LLL**-theorems, the adaptive logic will not be identical to its lower limit. The upper limit logic, however, will still be **Tr** and in this sense we are dealing with a border case. Let the set of abnormalities be defined by $\Omega = \{\mathsf{A} \mid \ldots\}$ in which A is a logical form and ... a condition. Imposing a further condition may normalize even the upper limit logic.

Fact 5.9.7 *If* **AL** *has* $\Omega = \{\mathsf{A} \mid \ldots\}$ *as its set of abnormalities and* $\Omega \cap Cn^{\mathcal{L}}_{\mathbf{LLL}}(\emptyset) \neq \emptyset$*, then* **ULL** *is* **Tr**. *By redefining* $\Omega = \{\mathsf{A} \mid \ldots; \nvdash_{\mathbf{LLL}} \mathsf{A}\}$*,* **AL** *remains unchanged and* **ULL** *is only* **Tr** *if it is for another reason.*

If Ω is redefined as in Fact 5.9.7, we shall say that the **LLL**-theorems are filtered out from Ω. In view of the insights provided by Fact 5.9.6, it is wise to filter out both the **LLL**-theorems and the **LLL**-falsehoods from Ω.[16] In a sense, this makes the situation more perspicuous. For the adaptive logic, however, this filtering out does not make any difference and so is by no means necessary. Let me state this in a precise way before explaining it.

Fact 5.9.8 *If* **AL1** *has* $\Omega_1 = \{\mathsf{A} \mid \ldots\}$ *as its set of abnormalities and* **AL2** *is the result of replacing in* **AL1** *the set of abnormalities* Ω_1 *by* $\Omega_2 = \{\mathsf{A} \mid \ldots; \nvdash_{\mathbf{LLL}} \mathsf{A}; \text{ for some } B, \mathsf{A} \nvdash_{\mathbf{LLL}} B\}$*, then* **AL1** *and* **AL2** *assign the same consequence set to every premise set.*

The **LLL**-theorems in Ω are verified by all models and, for every premise set Γ, they are members of $U(\Gamma)$ as well as of every $\varphi \in \Phi(\Gamma)$. The **LLL**-logical falsehoods in Ω are falsified by all models and they are only members of $U(\Gamma)$ or of a $\varphi \in \Phi(\Gamma)$ if Γ is **LLL**-trivial anyway. The last statement may require some explanation. Suppose that a premise set Γ has a *Dab*-consequence $Dab(\Delta \cup \Delta')$ and that the members of Δ' are **LLL**-falsehoods, which means that all formulas are **LLL**-derivable from them. So obviously $Dab(\Delta)$ is also a *Dab*-consequence of Γ. So if a **LLL**-falsehood is a member of $U(\Gamma)$ or of a $\varphi \in \Phi(\Gamma)$, then it is **LLL**-derivable from Γ. In this case, $Cn_{\mathbf{LLL}}(\Gamma)$ is trivial anyway, and so is identical to $Cn_{\mathbf{AL}}(\Gamma)$ as well as to $Cn_{\mathbf{ULL}}(\Gamma)$.

Note that Fact 5.9.7 also applies in the situation described by Fact 5.9.5. As the Ω considered there contains only **LLL**-theorems, weeding out the **LLL**-theorems from Ω results in $\Omega = \emptyset$, which is described by Fact 5.9.3.

Some of the aforementioned facts may be phrased differently. Thus every premise set is normal iff $\Omega = \emptyset$. So Fact 5.9.3 may be rephrased with the antecedent "every $\Gamma \subseteq \mathcal{W}$ is normal". That every premise set is abnormal means that a *Dab*-formula is derivable from it. This, however, need not have any dramatic consequences. Sometimes it does, for example if every set is

[16] Note that the case in which some members of Ω are **LLL**-falsehoods but others are **LLL**-contingent, is not in general a border case.

abnormal because $\Omega = \mathcal{W}$—see Fact 5.9.4. But sometimes it leads to adaptive logics that behave very well. All that can be proved in general is the following fact.

Fact 5.9.9 *If every* $\Gamma \subseteq \mathcal{W}$ *is abnormal, then* **ULL** *is* **Tr**. *If every* $\Gamma \subseteq \mathcal{W}$ *is abnormal and* $Cn^{\mathcal{L}_+}_{\mathbf{LLL}}(\Gamma) \neq \mathcal{W}_+$*, then* $Cn^{\mathcal{L}_+}_{\mathbf{AL}}(\Gamma) \subset Cn^{\mathcal{L}_+}_{\mathbf{ULL}}(\Gamma)$*. If every* $\Gamma \subseteq \mathcal{W}$ *is abnormal and* $Cn^{\mathcal{L}}_{\mathbf{LLL}}(\Gamma) \neq \mathcal{W}$*, then* $Cn^{\mathcal{L}}_{\mathbf{AL}}(\Gamma) \subset Cn^{\mathcal{L}}_{\mathbf{ULL}}(\Gamma)$*.*

Obviously, that $Cn^{\mathcal{L}_+}_{\mathbf{LLL}}(\Gamma) \neq \mathcal{W}_+$ does not warrant that $Cn^{\mathcal{L}}_{\mathbf{LLL}}(\Gamma) \neq \mathcal{W}$. That every $\Gamma \subseteq \mathcal{W}$ is abnormal, also need not prevent that $Cn_{\mathbf{LLL}}(\Gamma) \subset Cn_{\mathbf{AL}}(\Gamma)$ for some Γ, as is illustrated by $\mathbf{LI}^r$ and $\mathbf{LI}^m$. So we ended this section as we started it, with a harmless border case.

5.10 Some Negative Theorems

In view of the border cases, it is impossible to show, for example, that all adaptive logics are non-monotonic. They are not because some are identical to their lower limit logic. That an adaptive logic is stronger than its lower limit logic is required for the adaptive logic to have sensible adaptive properties *and* seems sufficient to prove the interesting negative properties. One may wonder whether there is a general criterion, for example in terms of border cases, for deciding that an adaptive logic is stronger than its lower limit, viz. $Cn_{\mathbf{LLL}}(\Gamma) \subset Cn_{\mathbf{AL}}(\Gamma)$ for some Γ. Such criteria are stated as Items 4 and 11 of Theorem 5.6.7. Moreover, Items 3 and 4 of Theorem 5.6.7 provide indirect criteria for Minimal Abnormality in view of Corollary 5.3.3.

It is not difficult to prove that adaptive logics that do not coincide with their lower limit logic have certain negative properties. Thus non-monotonicity comes as a corollary to Corollary 5.8.2 and non-transitivity is also provable.

Corollary 5.10.1 *If* $Cn_{\mathbf{LLL}}(\Gamma) \subset Cn_{\mathbf{AL}}(\Gamma)$ *for some* Γ*, then* **AL** *is non-monotonic.*

Theorem 5.10.1 *If* $Cn_{\mathbf{LLL}}(\Gamma) \subset Cn_{\mathbf{AL}}(\Gamma)$ *for some* Γ*, then* **AL** *is non-transitive.*

Proof. Suppose that the antecedent is true and let $A \in Cn_{\mathbf{AL}}(\Gamma) - Cn_{\mathbf{LLL}}(\Gamma)$. I shall show that there is a specific Γ' for which $A \notin Cn_{\mathbf{AL}}(\Gamma \cup \Gamma')$. As $\Gamma \subseteq Cn_{\mathbf{AL}}(\Gamma \cup \Gamma')$ by the Reflexivity of **AL** (Theorem 5.6.4) and $A \in Cn_{\mathbf{AL}}(\Gamma)$ by the supposition, it follows that **AL** is not transitive.

Let $\Gamma' = \{Dab(\Delta_i) \mid \Gamma \vdash_{\mathbf{LLL}} A \check{\vee} Dab(\Delta_i)\}$ and suppose that $\Gamma \cup \Gamma' \vdash_{\mathbf{AL}} A$. I shall show that this supposition leads to a contradiction. We have to consider two cases.

Case 1: the strategy of **AL** is Reliability. By Theorem 5.3.1 there is Θ such that $\Gamma \cup \Gamma' \vdash_{\mathbf{LLL}} A \check{\vee} Dab(\Theta)$ and $\Theta \cap U(\Gamma \cup \Gamma') = \emptyset$. As **LLL** is compact, there are $\Delta_1, \ldots, \Delta_n$ such that $\Gamma \cup \{Dab(\Delta_1) \check{\wedge} \ldots \check{\wedge} Dab(\Delta_n)\} \vdash_{\mathbf{LLL}} A \check{\vee} Dab(\Theta)$. So, by the Deduction Theorem for **CL**,

$$\Gamma \vdash_{\mathbf{LLL}} (Dab(\Delta_1) \check{\wedge} \ldots \check{\wedge} Dab(\Delta_n)) \check{\supset} (A \check{\vee} Dab(\Theta)) . \tag{5.13}$$

By the definition of Γ',

$$\Gamma \vdash_{\mathbf{LLL}} A \check{\vee} (Dab(\Delta_1) \check{\wedge} \ldots \check{\wedge} Dab(\Delta_n)) . \tag{5.14}$$

From (5.13) and (5.14) follows, by **CL**-properties, that $\Gamma \vdash_{\mathbf{LLL}} A \check{\vee} Dab(\Theta)$, whence $Dab(\Theta) \in \Gamma'$. But this contradicts $\Theta \cap U(\Gamma) = \emptyset$.

Case 2: the strategy of **AL** is Minimal Abnormality. By Theorem 5.3.3 there is, for every $\varphi \in \Phi(\Gamma)$, a Θ such that $\Gamma \cup \Gamma' \vdash_{\mathbf{LLL}} A \check{\vee} Dab(\Theta)$ and $\Theta \cap \varphi = \emptyset$. The rest of the proof proceeds exactly as for Case 1, except that we now rely on Fact 5.1.1 to show that $\Theta \cap \varphi = \emptyset$ is contradicted. ■

Let us now turn to Compactness. Here is an example that illustrates the non-compactness of $\mathbf{CLuN}^r$. Let $\Gamma = \{(p \vee (q \wedge \neg q)) \wedge ((q \wedge \neg q) \vee (r_1 \wedge \neg r_1))\} \cup \{(r_i \wedge \neg r_i) \wedge ((q \wedge \neg q) \vee (r_{i+1} \wedge \neg r_{i+1})) \mid i \in \mathbb{N}\}$. Every **CLuN**-model of Γ verifies $r_i \wedge \neg r_i$ for all $i \in \mathbb{N}$. Moreover, every model verifying p as well as all $r_i \wedge \neg r_i$ $(i \in \mathbb{N})$ and falsifying all other abnormalities, is a **CLuN**-model of Γ. It follows that $U(\Gamma) = \{r_i \wedge \neg r_i \mid i \in \mathbb{N}\}$, whence $\Gamma \vDash_{\mathbf{CLuN}^r} p$. However, for all finite $\Gamma' \subseteq \Gamma$, $(q \wedge \neg q) \in U(\Gamma')$, whence $\Gamma' \nvDash_{\mathbf{CLuN}^r} p$. So $\mathbf{CLuN}^r$ is not compact. The same example shows that $\mathbf{CLuN}^m$ is not compact either: the only minimal abnormal model M of Γ is the one for which $Ab(M) = \{r_i \wedge \neg r_i \mid i \in \mathbb{N}\}$. The example may be adjusted for all adaptive logics provided $Cn_{\mathbf{LLL}}(\Gamma) \subset Cn_{\mathbf{AL}}(\Gamma)$ and classical disjunction and conjunction can be defined in $\mathcal{L}$.

If a logic **L** is compact and monotonic, it is also *relatively compact*: if $\Gamma \cup \Gamma' \vdash_{\mathbf{L}} A$, then there are $B_1, \ldots, B_n \in \Gamma'$ such that $\Gamma \cup \{B_1, \ldots, B_n\} \vdash_{\mathbf{L}} A$. Logics that are not compact, may still be relatively compact. Adaptive logics are neither. A simple example shows this for $\mathbf{CLuN}^r$ and $\mathbf{CLuN}^m$. It is obtained by choosing $\{p \vee (q \wedge \neg q)\} \cup \{(q \wedge \neg q) \vee (r_i \wedge \neg r_i) \mid i \in \mathbb{N}\}$ for Γ, $\{r_i \wedge \neg r_i \mid i \in \mathbb{N}\}$ for Γ', and p for A. Adaptive logics also miss many related properties. Here is one example of such a property: if $\Gamma \vdash_{\mathbf{AL}} A$ and $\Gamma \cup \Delta \cup \Theta \vdash_{\mathbf{AL}} A$, then there is a finite $\Delta' \subseteq \Delta$ and a finite $\Theta' \subseteq \Theta$ such that $\Delta' \cup \Theta' \neq \emptyset$ and $\Gamma \cup \Delta' \cup \Theta' \vdash_{\mathbf{AL}} A$. A counterexample for $\mathbf{CLuN}^r$ and $\mathbf{CLuN}^m$ is obtained by choosing $\{p \vee (q \wedge \neg q)\}$ for Γ, $\{(q \wedge \neg q) \vee (r_0 \wedge \neg r_0)\}$ for Δ, and $\{(r_i \wedge \neg r_i) \wedge ((q \wedge \neg q) \vee (r_{i+1} \wedge \neg r_{i+1})) \mid i \in \mathbb{N}\}$ for Θ, and p for A.

The reader may feel that the last counterexample, just like the counterexample to Compactness itself, brings us very close to a kind of relative compactness. If Θ is reformulated as $\{r_i \wedge \neg r_i, (q \wedge \neg q) \vee (r_{i+1} \wedge \neg r_{i+1}) \mid i \in \mathbb{N}\}$, then choosing $\Delta' = \Delta$ and, for example, $\Theta' = \{r_0 \wedge \neg r_0\}$ gives one $\Gamma \cup \Delta' \cup \Theta' \vdash_{\mathbf{AL}} A$. This observation is correct. So let us say that a logic **L** is *pseudo-compact* iff the following holds: if $\Gamma \vdash_{\mathbf{L}} A$, then there is a Γ' such that Γ and Γ' are **L**-equivalent and there is a finite $\Gamma'' \subseteq \Gamma'$ such that $\Gamma'' \vdash_{\mathbf{L}} A$. Adaptive logics are quasi-compact. This property is not very deep, however, as the utterly simple proof reveals. For every adaptive logic **AL**, if $\Gamma \vdash_{\mathbf{AL}} A$, then Γ and $\Gamma' \cup \{A\}$ are **AL**-equivalent, $\{A\} \subseteq \Gamma' \cup \{A\}$, and $A \vdash_{\mathbf{AL}} A$. So adaptive logics share this property with all logics that are reflexive, cautiously monotonic and cautiously transitive.

A slightly deeper compactness-like notion is shared by many adaptive logics. The idea is that a premise set Γ is first transformed, for example, to $\Gamma' = \{Cn_{\mathbf{LLL}}(\{A\}) \mid A \in \Gamma\}$ or to $\Gamma' = \{B \mid B$ is a duly quantified conjunct of the PCNF of $A \in \Gamma\}$; see page 246 for PCNF. Next, **AL** is said to be *quasi-compact* iff, whenever $\Gamma \vdash_{\mathbf{AL}} A$, then there are $B_1, \ldots, B_n \in \Gamma'$ such that $B_1, \ldots, B_n \vdash_{\mathbf{AL}} A$. Many adaptive logics, possibly all, can be shown to be quasi-compact in one of these senses. I do not elaborate on the matter because Compactness is not really a very sensible property for defeasible logics.[17] Its

[17]Still, there is an unsolved problem that may be important. Apparently, an as yet unidenti-

function is to guarantee, for logics that have static proofs, that whatever is derivable from a premise set is derivable from it within a finite proof. The properties that most closely correspond to compactness are the following. First, only finite entities are used to build proofs: all rules are finitary and so are the conditions of the lines. Next, if A is adaptively derivable from any premise set Γ, then A is finally derived in a finite proof from Γ. That this holds can be seen from Definition 4.4.4 and also from Section 5.4.

In Section 5.6, I showed that the Deduction Theorem holds for Minimal Abnormality and announced that it does not for Reliability.

Theorem 5.10.2 *The Deduction Theorem does not hold for some adaptive logics that have Reliability as their third element.*

Proof. A ready example is $\mathbf{CLuN}^r$: $(q\wedge\neg q)\vee(r\wedge\neg r), (r\wedge\neg r)\vee p, q\wedge\neg q \vdash_{\mathbf{CLuN}^r} p$ but $(q\wedge\neg q)\vee(r\wedge\neg r), (r\wedge\neg r)\vee p \nvdash_{\mathbf{CLuN}^r} (q\wedge\neg q) \check{\supset} p$. Note that $U(\{(q\wedge\neg q)\vee(r\wedge\neg r), (r\wedge\neg r)\vee p, q\wedge\neg q\}) = \{q\wedge\neg q\}$, whereas $U(\{(q\wedge\neg q)\vee(r\wedge\neg r), (r\wedge\neg r)\vee p\}) = \{r\wedge\neg r, q\wedge\neg q\}$. ■

Note that the Deduction Theorem holds for Reliability in specific cases. Here is one of them.

Fact 5.10.1 *If* $U(\Gamma\cup\{A\}) = U(\Gamma)$ *and* $\Gamma\cup\{A\}\vdash_{\mathbf{AL}^r} B$, *then* $\Gamma\vdash_{\mathbf{AL}^r} A\check{\supset} B$.

Indeed, as $\Gamma\cup\{A\}\vdash_{\mathbf{AL}^r} B$, there is a finite $\Delta\subset\Omega - U(\Gamma\cup\{A\})$ for which $\Gamma\cup\{A\}\vdash_{\mathbf{LLL}} B\check{\vee}Dab(\Delta)$. So $\Gamma\vdash_{\mathbf{LLL}} (A\check{\supset} B)\check{\vee}Dab(\Delta)$. As $U(\Gamma\cup\{A\}) = U(\Gamma)$, it follows that $\Gamma\vdash_{\mathbf{AL}^r} A\check{\supset} B$.

The reason why the Deduction Theorem fails for Reliability but holds for Minimal Abnormality may be clarified in terms of the semantics. Consider the example from the proof of Theorem 5.10.2. Both $(q\wedge\neg q)\vee(r\wedge\neg r), (r\wedge\neg r)\vee p, q\wedge\neg q\vdash_{\mathbf{CLuN}^r} p$ and $(q\wedge\neg q)\vee(r\wedge\neg r), (r\wedge\neg r)\vee p, q\wedge\neg q\vdash_{\mathbf{CLuN}^m} p$ hold true. There are **CLuN**-reliable models of $\{(q\wedge\neg q)\vee(r\wedge\neg r), (r\wedge\neg r)\vee p\}$ that verify both $q\wedge\neg q$ and $r\wedge\neg r$ and some of these falsify p. So $(q\wedge\neg q)\vee(r\wedge\neg r), (r\wedge\neg r)\vee p\nvdash_{\mathbf{CLuN}^r} (q\wedge\neg q)\check{\supset} p$. However, the minimally abnormal models of $\{(q\wedge\neg q)\vee(r\wedge\neg r), (r\wedge\neg r)\vee p\}$ verify either $q\wedge\neg q$ or $r\wedge\neg r$ but not both. So the minimally abnormal models that verify $q\wedge\neg q$ falsify $r\wedge\neg r$ and hence verify p. So they verify $(q\wedge\neg q)\check{\supset} p$. The minimal abnormal models that verify $r\wedge\neg r$ falsify $q\wedge\neg q$. So they too verify $(q\wedge\neg q)\check{\supset} p$. It follows that all minimally abnormal models of $\{q\wedge\neg q\check{\vee}(r\wedge\neg r), (r\wedge\neg r)\check{\vee} p\}$ verify $q\wedge\neg q\check{\supset} p$.

It seems appropriate to end this section with a warning. As we have seen, sensible adaptive logics extend their lower limit and all sensible adaptive logics have certain negative properties, for example non-monotonicity. We have also seen, in Section 1.2, that some defeasible reasoning forms are monotonic—the example was the Weak Consequence relation. This does not rule out, however, that such a reasoning form is characterized by an adaptive logic *under a translation*—see Section \ref{s:var:strat} for details. REF

fied property that is similar to Relative Compactness holds for Reliability, but not for Minimal Abnormality. This property may be responsible for the higher computational complexity of Minimal Abnormality, as explained in Section 10.1, and for the complication with the proof theory for combined logics discussed in Section 6.2.4.