Chapter 6

Strategies and Combinations

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In this chapter some more generic results will be presented. On the one hand, I shall introduce adaptive strategies other than Reliability and Minimal Abnormality. Some of them are border cases. As mentioned before, other strategies are mainly intended to characterize defeasible logics from the literature in terms of adaptive logics. Applications follow in Chapter 9. On the other hand, combined adaptive logics will be presented in a systematic way. Some fascinating consequences are defined by these, as we saw in Chapter 3.

6.1 Other Strategies

Two strategies were presented and studied in Chapter 4: Reliability and Minimal Abnormality. Among the other known strategies, I shall first describe the Simple strategy, which is a special case of the previous two. Other strategies are mainly useful for characterizing defeasible consequence relations from the literature in terms of an adaptive logic. These strategies will be seen at work in other chapters, but are systematically described here.

All adaptive logics considered in this section are defined by a triple: a lower limit logic **LLL**, a set of abnormalities Ω, and one of the new strategies.

6.1.1 Simple

Certain combinations of a lower limit logic **LLL**, a set of abnormalities, and possibly a type of premise set, warrant that $\Gamma \vdash_{\mathbf{LLL}} Dab(\Delta)$ iff there is an $A \in \Delta$ such that $\Gamma \vdash_{\mathbf{LLL}} A$. Where this is the case, the Reliability strategy and the Minimal Abnormality strategy come to the same, and are then called the *Simple strategy.*

That, under the described condition, Reliability and Minimal abnormality reduce to the same is easy to see. If $\Gamma \vdash_{\mathbf{LLL}} Dab(\Delta)$ just in case some $A \in \Delta$ is such that $\Gamma \vdash_{\mathbf{LLL}} A$, then Δ is a singleton whenever $Dab(\Delta)$ is a *minimal Dab*-consequence of Γ. Let $Dab(\Delta_1)$, $Dab(\Delta_2)$, ... be the minimal *Dab*-consequences of Γ with every $Dab(\Delta_i)$ ($i \in \{1, 2, \ldots\}$) a singleton. While $U(\Gamma) = \Delta_1 \cup \Delta_2 \cup \ldots$ as always, $\Phi(\Gamma)$ is a singleton; its only member is $\Delta_1 \cup \Delta_2 \cup \ldots$ because this is the only minimal choice set (as well as the only choice set) of $\{\Delta_1, \Delta_2, \ldots\}$. So, in a proof in which all minimal *Dab*-consequences of Γ are derived, a line is marked for Reliability iff it is marked for Minimal Abnormality, viz. just in case its condition overlaps with $\Delta_1 \cup \Delta_2 \cup \ldots$. So Reliability and Minimal Abnormality lead to the same set of *finally derivable* formulas. The formulas derivable at a stage will still be different, because at a stage a minimal *Dab*-formula may count several disjuncts, but this is immaterial.

Incidentally, the failing strategy described in Section 2.3.1 would coincide with the Simple strategy, and hence would work fine, in situations where $\Phi(\Gamma) = \{U(\Gamma)\}$.[1]

The Simple strategy is suitable whenever $\Phi(\Gamma) = \{U(\Gamma)\}$ because it then leads to the same result as both Reliability and Minimal Abnormality. So, where it is suitable, the Simple strategy leads to a logic in standard format. Moreover, whatever was proved in Chapter 5 for all logics that have Reliability as their strategy or for all logics that have Minimal Abnormality as their strategy also holds for those that have Simple as their strategy, provided the strategy is suitable. Where $\mathbf{AL}^s$ is an adaptive logic that has Simple as its strategy and the strategy is suitable, we have

$$Cn_{\mathbf{LLL}}(\Gamma) \subseteq Cn_{\mathbf{AL}^r}(\Gamma) = Cn_{\mathbf{AL}^s}(\Gamma) = Cn_{\mathbf{AL}^m}(\Gamma) \subseteq Cn_{\mathbf{ULL}}(\Gamma)\,.$$

The interest of the Simple strategy lies obviously at the object level. It has a drastically simpler marking definition than Reliability and Minimal Abnormality and a simpler model selection mechanism as well. As $\Phi(\Gamma) = \{U(\Gamma)\}$, Reliability and Minimal Abnormality select the same **LLL**-models of Γ, viz. the models M for which $Ab(M) = \{A \in \Omega \mid \Gamma \vdash_{\mathbf{LLL}} A\}$. Here are two handy definitions.

Definition 6.1.1 *A* **LLL**-*model* M *of* Γ *is* simply all right *iff* $Ab(M') = \{A \in \Omega \mid \Gamma \vDash_{\mathbf{LLL}} A\}$.

Definition 6.1.2 $\Gamma \vDash_{\mathbf{AL}^s} A$ *iff* A *is verified by all simply all right models of* Γ.

The marking definition theory is also drastically simplified—the generic rules remain obviously unchanged.

Definition 6.1.3 *Marking for the Simple strategy: Line i is marked at stage s iff, where Δ is its condition, stage s contains a line of which an $A \in \Delta$ is the formula and ∅ is the condition.*

It is straightforward to show that final derivability on the Simple strategy, where this strategy is suitable, corresponds to final derivability on the Reliability strategy as well as to final derivability on the Minimal Abnormality strategy. The proof of the corresponding semantics statements is equally straightforward. From these follow soundness and completeness.

It seems wise to compare the Simple strategy with Reliability and Minimal abnormality with respect to proofs at a stage. The rules are the same, but the marking definitions are different. As a result of this, they also lead to different

[1] It is worth mentioning the reason for this. If every minimal *Dab*-consequence of every premise set is a singleton, every *Dab*-formula that is derivable in a proof from a Γ on a non-empty condition at an unmarked line is also derivable in a proof from Γ on the empty condition.

proofs at a stage. Suppose that **AL** has Simple as its strategy and that this strategy is suitable. Although we know that, for every premise set Γ, every *minimal* *Dab*-consequence of Γ is a singleton, this does not mean that non-minimal *Dab*-formulas cannot occur in the proof. Consider a proof at stage 15 in which the following lines occur, no *Dab*-formula occurs in lines 1–10, $A_1, A_2 \in \Omega$, $B_1, B_2 \notin \Omega$, and disjunction classical.

⋮			
11	B_1	...	$\{A_1\}$
12	B_2	...	$\{A_2\}$
13	$B_1 \vee B_2$	11; RU	$\{A_1\}$
14	$B_1 \vee B_2$	12; RU	$\{A_2\}$
15	$A_1 \check{\vee} A_2$	...	$\emptyset$

On the Reliability strategy, lines 11–14 are marked; on the Minimal Abnormality strategy, lines 11 and 12 are marked whereas lines 13 and 14 are not; on the Simple strategy, no line is marked because no singleton *Dab*-formula was derived. Which strategy got it right at this point depends on whether A_1, A_2 or both are minimal *Dab*-consequences of the premise set. With respect to *final derivability*, however, the three strategies coincide. Moreover, as we know that all minimal *Dab*-consequences of the premise set are singletons (because the Simple strategy is suitable), each of the three strategies instructs one to try to derive A_1 and to try to derive A_2. More on this is said in Chapter 10.

There are few adaptive logics that have Simple as their strategy. One of them is $\mathbf{AN}^s$, which is described at the end of Section 7.3. There are, however, many adaptive logics that may be given Simple as their strategy in view of a restriction on the considered premise sets. Actually, we met an example of this on page 97. There are, however, more interesting examples, which we shall come across in Chapter 9. As we shall see there, many defeasible consequence relations can be characterized by an adaptive logic under a translation. This means that, for the characterization, we are only interested in very specific (translated) premise sets. Precisely this may cause the situation that is required for the application of the Simple strategy.

A nice example is provided by the logic of compatibility—see Section 9.2. Let $\Gamma \vdash_{\mathbf{COMPAT}} A$ express that A is compatible with Γ. It is shown that $\Gamma \vdash_{\mathbf{COMPAT}} A$ is characterized by $\Gamma^{\Box} \vdash_{\mathbf{COM}} \Diamond A$ where $\Gamma^{\Box} = \{\Box A \mid A \in \Gamma\}$ as elsewhere and the adaptive logic **COM** is defined by (i) a specific predicative version of **S5**, (ii) $\Omega = \{\Box A \mid A \in \mathcal{W}_s;\ \nvdash_{\mathbf{S5}} \Box A\}$, and (iii) the Simple strategy. As defined here, the adaptive logic **COM** would lead to disaster if it were combined to an arbitrary premise set $\Gamma \in \mathcal{W}_m$. To be more precise, many modal premise sets have *Dab*-consequences that comprise more than one disjunct. So an adaptive logic that has Simple as its strategy would assign the trivial consequence set to many premise sets. However, in order to define **COMPAT** we are only interested in premise sets $\Gamma^{\Box}$. The only *Dab*-consequences (relative to the specified Ω) of such premise sets comprise a single disjunct. So, for these premise sets, Reliability , Minimal Abnormality, and Simple define the same consequence sets.

Some people may be puzzled by the fact that the Simple strategy might result in triviality. The simplest example is obtained if **AL** is an adaptive logic that has **CL** as its lower limit and Simple as its strategy. Let $A_1, A_2 \in \Omega$ and

$B \notin \Omega$, and let $\{B \vee A_1, \neg B \vee A_2\}$ be the premise set.

1	$B \vee A_1$	Prem	$\emptyset$
2	$\neg B \vee A_2$	Prem	$\emptyset$
3	B	1; RC	$\{A_1\}$
4	$\neg B$	2; RC	$\{A_2\}$
5	$A_1 \check{\vee} A_2$	3, 4; RD	$\emptyset$
6	C	3, 4; RU	$\{A_1, A_2\}$

Obviously, no singleton *Dab*-consequence is derivable from the premises. So, in view of Definition 6.1.3, no line 1–6 is marked in any extension of this proof. But note that C is just any formula whatsoever.

6.1.2 Blindness

This strategy, which was already mentioned in Section 4.8, is the simplest of all.

Definition 6.1.4 *Marking for the Blindness strategy: no line is marked.*

In other words, although formulas may be derived on a condition and although the condition may prove to be problematic in view of derived *Dab*-formulas, the problem is not seen and no line is marked. So, where $\mathbf{AL}^b$ is an adaptive logic that has Blindness as its strategy, the $\mathbf{AL}^b$-consequences of a premise set coincide with its **ULL**-consequences. The semantics is equally simple: from the **LLL**-models of the premise set, the Blindness strategy selects the **ULL**-models. Take this literally: if Γ has no **ULL**-models, the set of selected models is $\emptyset$.

The name of the strategy is self-explanatory: one derives *Dab*-formulas from which it follows that previously drawn conclusions are mistaken, but one does not see this, or refuses to see it. The reader may easily apply this to the logics of inductive generalization from Chapter 3 or to the inconsistency-adaptive logics from Chapter 2. The Blindness strategy leads to the refusal to review one's conclusions in the face of evidence that the conclusions are mistaken. Blindness leads to or results from dogmatism, which is a form of stupidity.

If the premise set Γ has no **ULL**-models, there are no adaptive models and triviality results. By the completeness and soundness of **ULL** with respect to its semantics, a Γ that has no **ULL**-models has *Dab*-consequences. Suppose that $Dab(\Delta)$ is a *Dab*-consequence of Γ. From this follows, for any A, $A \check{\vee} Dab(\Delta)$, whence A is derivable on the condition Δ at a line of the adaptive proof from Γ and this line will not be marked—see Definition 6.1.4. So every formula A is finally derivable from an abnormal premise set.

Where $\mathbf{AL}^b$ is as before, we obviously have $Cn_{\mathbf{AL}^b}(\Gamma) = Cn_{\mathbf{ULL}}(\Gamma)$. So the adequacy of $\mathbf{AL}^b$ with respect to its semantics is obvious. The Blindness strategy leads to a border case adaptive logic, but not one that fits into the standard format. Indeed, Reassurance and Strong Reassurance obviously do not holds for it.

6.1.3 Normal Selections

This strategy is most easily introduced semantically. Remember that $\Gamma \vDash_{\mathbf{AL}^m} A$ iff A is verified by every Minimal Abnormal **LLL**-model of Γ. As follows from Lemma 5.2.1, M is a minimal abnormal model of Γ iff $Ab(M) = \varphi$ for some

$\varphi \in \Phi(\Gamma)$. So $\Gamma \vDash_{\mathbf{AL}^m} A$ iff $M \Vdash A$ for every **LLL**-model M such that $M \Vdash \Gamma$ and $Ab(M) \in \Phi(\Gamma)$.

We know, again from Lemma 5.2.1, that, for every $\varphi \in \Phi(\Gamma)$, there is a minimal abnormal **LLL**-model M such that $M \Vdash \Gamma$ and $Ab(M) = \varphi$. However, it is obviously not excluded there are several **LLL**-models M and M' of Γ such that that $Ab(M) = Ab(M') = \varphi$ for a $\varphi \in \Phi(\Gamma)$. of course, some formulas may be verified by the considered M and not by the considered M', and vice versa. So this leads naturally to the question which formulas are verified by all **LLL**-models of Γ that share the same abnormal part—where M is such that $M \Vdash \Gamma$ and $Ab(M) \in \Phi(\Gamma)$, all M' such that $M' \Vdash \Gamma$ and $Ab(M') = Ab(M)$.

If this question is taken at face value, the answer will obviously be a logic that is not Proof Invariant—see Theorem 5.6.9. Indeed, consider a premise set Γ and consider all **LLL**-models M of Γ for which $Ab(M) = \varphi$ for a $\varphi \in \Phi(\Gamma)$. It is obviously possible that all these M verify A, whereas A is falsified for another **LLL**-model M' of Γ for which $Ab(M') \in \Phi(\Gamma)$.

Proof Invariance may be obviously restored by defining: $\Gamma \vdash A$ iff there is a $\varphi \in \Phi(\Gamma)$ such that $M \Vdash A$ whenever $M \Vdash \Gamma$ and $Ab(M) = \varphi$. There is a prize to be paid: we loose other desirable properties. First, the obtained consequence set may not be closed under the lower limit logic. Indeed, it is very well possible that there are $\varphi, varphi' \in \Phi(\Gamma)$ such A is verified by all **LLL**-models M of Γ for which $Ab(M) = \varphi$ and B is verified by all **LLL**-models M of Γ for which $Ab(M) = \varphi'$, whereas there is no $\varphi'' \in \Phi(\Gamma)$ such $A \wedge B$ is verified by all **LLL**-models M of Γ for which $Ab(M) = \varphi''$.[2] The second property we loose in general is Reassurance. Allow me to recycle an example: let **AL** be an adaptive logic that has **CL** as its lower limit and for which $\Gamma \vdash A$ is defined as in the first sentence of this paragraph. Let $A_1, A_2 \in \Omega$ and $B \notin \Omega$, and let $\{B \vee A_1, \neg B \vee A_2\}$ be the premise set. As is shown by the last proof of Section 6.1.1, the resulting consequence set is trivial. As the trivial set has no **CL**-models, Reassurance fails. Dialetheists will take this to be an argument for their position. Indeed, is the lower limit logic has models for all premise sets, including the trivial one, Reassurance is not lost. And indeed, there are such logics. There even is an abundance of them: **CLuN**, **CLuNs**, **LP**, and so on. And yet something goes basically wrong here: by loosing Proof Invariance, we loose the Fixed Point property as well as others.

Researchers in artificial intelligence have nevertheless chosen for the approach outlined in the previous paragraph. The straightforward reconstruction of their work in terms of adaptive logics requires that one invokes the Normal Selections strategy—but see below.

The semantics for an adaptive logic that has Normal Selections as its strategy, $\mathbf{AL}^n$, is defined as follows.

Definition 6.1.5 *A set* $\mathcal{M}$ *of* **LLL***-models of* Γ *is a* normal selection *iff, for some* $\varphi \in \Phi(\Gamma)$*,* $\mathcal{M} = \{M \mid M \Vdash \Gamma; Ab(M) = \varphi\}$*.*

Definition 6.1.6 $\Gamma \vDash_{\mathbf{AL}^n} A$ *iff* A *is verified by every model of a normal selection of* **LLL***-models of* Γ*.*

Consider the premise set $\Gamma_1 = \{\neg p, \neg q, p \vee r, q \vee s, p \vee q\}$ from page 51. We have seen in Section 2.3.3 that $r \vee s$ is a final $\mathbf{CLuN}^m$-consequence of it, while

[2]For the sake of example, I presuppose that Adjunction is verified by the lower limit logic. It it is not, replace $A \wedge B$ by a C for which $A, B \vdash_{\mathbf{LLL}} C$.

neither r nor s is. However, both r and s are final $\mathbf{CLuN}^n$-consequences of this premise set. To see this, remember that $\Phi(\Gamma_1) = \{\{p \wedge \neg p\}, \{q \wedge \neg q\}\}$. All minimal abnormal models that verify $q \wedge \neg q$ falsify p; so $\Gamma_1 \vDash_{\mathbf{AL}^n} r$. Moreover, all minimal abnormal models that verify $p \wedge \neg p$ falsify q; so $\Gamma_1 \vDash_{\mathbf{AL}^n} s$.

Let us turn to the proof theory. The inference rules of $\mathbf{AL}^n$ are as those of $\mathbf{AL}^r$ and $\mathbf{AL}^m$. The marking definition goes as follows.

Definition 6.1.7 *Marking for Normal Selections: Line i is marked at stage s iff, where Δ is the condition of line i, $Dab(\Delta)$ has been derived on the condition $\emptyset$ at stage s.*

While it is obvious that, where the Simple strategy is suitable, $\Gamma \vdash_{\mathbf{AL}^s} A$ iff $\Gamma \vDash_{\mathbf{AL}^s} A$ and, in general, $\Gamma \vdash_{\mathbf{AL}^b} A$ iff $\Gamma \vDash_{\mathbf{AL}^b} A$, it might not be obvious that $\mathbf{AL}^n$ is sound and complete with respect to its semantics. Yet, to prove so is extremely simple.

Consider a line l of an $\mathbf{AL}^n$-proof from Γ at which A is derived on the condition Δ. Suppose first that l is marked. So $Dab(\Delta)$ is a **LLL**-consequence of Γ. In view of Fact 5.1.1, it holds for every $\varphi \in \Phi(\Gamma)$ that $\varphi \cap \Delta \neq \emptyset$.[3] So the derivability of A on the condition Δ does not warrant that A is verified by all members of every normal selection $\mathcal{M}$. Suppose next that line l is unmarked extension of the proof. So $Dab(\Delta)$ is not a **LLL**-consequence of Γ. It follows that a **LLL**-model M of Γ falsifies $Dab(\Delta)$. By Strong Reassurance for Minimal Abnormality, Theorem 5.2.1, there is a minimal abnormal model M' of Γ that falsifies $Dab(\Delta)$. So there is a $\varphi \in \Phi(\Gamma)$ such that $Ab(M') = \varphi$ and $\varphi \cap \Delta = \emptyset$. But then every member of the normal selection $\mathcal{M} = \{M \mid M \Vdash \Gamma; Ab(M) = \varphi\}$ falsifies $Dab(\Delta)$, whence all members of $\mathcal{M}$ verify A. So we have established the following theorem.

Theorem 6.1.1 $\Gamma \vdash_{\mathbf{AL}^n} A$ *iff* $\Gamma \vDash_{\mathbf{AL}^n} A$. *(Adequacy for Normal Selections.)*

We have seen that the Normal Selections strategy does not give us adaptive logics in standard format. Nevertheless, every adaptive logic that has Normal Selections as its strategy is characterized by an adaptive logic in standard format. This will be shown in Section 9.4.

6.1.4 The Flip-flop Strategy

Sometimes we need an adaptive logic that is a flip-flop; we saw an example in Section 3.6. So it would be handy to have a general means to obtain a flip-flop logic from a given lower limit logic and a given set of abnormalities. There are indeed several such means. Below I describe two of them, the first follows an
ref idea of Hans Lycke, the other an idea of Frederik Van De Putte. In Section 9.5, I present a way to reduce the Flip-flop strategy to the Simple strategy, and hence to an adaptive logic in standard format.

Both ideas are obtained by a modification of the Reliability strategy. The rules of inference are the same as for Reliability and Minimal abnormality. For the first idea, define $F(\Gamma)$ as the set comprising the disjuncts of the Dab-consequences of Γ, and define, with respect to a proof from Γ, $F_s(\Gamma)$ as the set

[3] Whether $Dab(\Delta)$ is a minimal Dab-consequence of Γ is immaterial. If it is not, some members of Δ may not occur in any $\varphi \in \Phi(\Gamma)$, but every such φ will still contain a member of Δ.

comprising the disjuncts of the *Dab*-formulas that occur in the proof at stage s.[4] A line of a stage s of a proof from Γ is defined as marked iff, where Δ is its condition, $\Delta \cap F(\Gamma) \neq \emptyset$. Obviously, every abnormality is a disjunct of a *Dab*-consequence of Γ iff Γ has *Dab*-consequences. So $F(\Gamma) = \emptyset$ or $F(\Gamma) = \Omega$. Proofs at a stage are equally transparent. Suppose that Γ has *Dab*-consequences, for example $Dab(\Delta)$, and that A is derived on the condition Θ at line l of a proof from Γ. By extending the proof with $Dab(\Delta \cup \Theta)$ line l is marked. In other words, if Γ has *Dab*-consequences, no formula is finally derived at a line that has a non-empty condition. The semantics is also similar to that for Reliability: a **LLL**-model M of Γ is selected iff $Ab(M) \subseteq F(\Gamma)$.

The second idea proceeds in terms of $U(\Gamma)$ but requires a different marking definition and a different definition of the semantics. A line is marked at stage s of a proof from Γ iff $U(\Gamma) \neq \emptyset$ and the condition of the line is not empty. The selected models of Γ are the **ULL**-models of Γ if there are any; otherwise they are the**LLL**-models of Γ.

6.1.5 Counting

Zie tegenvb. Frederik / opsplitsen in 3 soorten / hoe dan ook de minimaal abnormale modellen nemen

Counting is typically a quantitative strategy. Nevertheless, it can be nicely integrated into the adaptive logics program. The idea behind Counting is that a **LLL**-model M of a premise set Γ is selected iff no other **LLL**-model M' of Γ verifies less abnormalities than M. The **LLL**-models of Γ that are selected by the Counting strategy will be the ones for which $Ab(M) \in \Phi^{\#}(\Gamma)$.

The strategy relates to the so-called majority rule. In some cases a board is considered to have taken a decision if a majority of its members has voted in favour of the decision. Similarly, if equally trustworthy witnesses contradict each other, one might take a statement to be plausible if the witnesses affirming it outnumber those denying it. To approach such situations in terms of the counting strategy may require that the statements by the different witnesses are kept apart, for example by conjoining all statements a witness made and by putting in from of the conjunction a $\Diamond$, expressing plausibility. In other cases, a multi-modal approach may be required.

In defining $\Phi^{\#}(\Gamma)$, some troubles relating to infinity are to be expected. This is why I shall define four Counting strategies: Counting$_1$, ..., Counting$_4$. Let the resulting logics be called $\mathbf{AL}^{c1}$, ..., $\mathbf{AL}^{c4}$. The definition of the consequence relation is the same for all these strategies.

Definition 6.1.8 $\Gamma \vDash_{\mathbf{AL}^{ci}} A$ *iff every Counting$_i$-model of* Γ *verifies* A.

The first variant is straightforward. Let $\#\Delta$ denote the cardinality of Δ.

Definition 6.1.9 *A model* $M \in \mathcal{M}_\Gamma^{\mathbf{LLL}}$ *is a Counting$_1$ model of* Γ *iff there is no* $M' \in \mathcal{M}_\Gamma^{\mathbf{LLL}}$ *for which* $\#Ab(M') < \#Ab(M)$.

If all lower limit models of Γ verify an *infinite* set of abnormalities, they are all Counting$_1$-models of Γ. Indeed, in this case $Ab(M)$ is denumerable for

[4]An alternative is to define $F(\Gamma) \in \{\emptyset, \Omega\}$, stating that $F(\Gamma) = \emptyset$ iff $U(\Gamma) = \emptyset$.

every M, and any two denumerable sets have the same cardinality. A criterion that is effective to compare infinite $Ab(M)$ was already invoked for minimal abnormality. Combining this with Counting$_1$ gives us the following, more refined, definition.

Definition 6.1.10 *A model* $M \in \mathcal{M}^{\mathbf{LLL}}_{\Gamma}$ *is a Counting*$_2$ *model of* Γ *iff there is no* $M' \in \mathcal{M}^{\mathbf{LLL}}_{\Gamma}$ *for which* $\#Ab(M') < \#Ab(M)$ *or* $Ab(M') \subseteq Ab(M)$.

This comes to: $Ab(M) \in \Phi(\Gamma)$ and there is no $M' \in \mathcal{M}^{\mathbf{LLL}}_{\Gamma}$ for which $\#Ab(M') < \#Ab(M)$. The definition warrants that $Cn_{\mathbf{AL}^m}(\Gamma) \subseteq Cn_{\mathbf{AL}^{c2}}(\Gamma)$.

A still more refined form of Counting is possible—one that selects in general a smaller set of **LLL**-models. Remember that $U(\Gamma) = \bigcup \Phi(\Gamma)$ (Corollary 5.2.1). So every abnormality verified by a minimal abnormal model of Γ is a member of $U(\Gamma)$. It is very well possible that a minimal abnormal model of Γ verifies all but finitely many members of $U(\Gamma)$. If M is such a model, that another **LLL**-model of Γ *falsifies* a larger number of members of $U(\Gamma)$, may be taken as a reason not to select M.

Definition 6.1.11 *A model* $M \in \mathcal{M}^{\mathbf{LLL}}_{\Gamma}$ *is a Counting*$_3$ *model of* Γ *iff there is no* $M' \in \mathcal{M}^{\mathbf{LLL}}_{\Gamma}$ *for which* $\#Ab(M') < \#Ab(M)$ *or* $Ab(M') \subseteq Ab(M)$ *or* $\#(U(\Gamma) - Ab(M')) > \#(U(\Gamma) - Ab(M))$.

Christian Straßer formulated a still sharper criterion, which has the advantage to incorporate the criteria used for the three former forms of Counting—it is instructive to check this. Suppose that the minimal *Dab*-consequences of Γ are $\{A \check{\vee} B, A \check{\vee} C, D_1 \check{\vee} D_2, D_3 \check{\vee} D_4, \ldots\}$, where $A, B, C, D_1, D_2, \ldots \in \Omega$ are all different formulas. Note that every minimal choice set φ of $\{\{A, B\}, \{A, C\}, \{D_1, D_2\}, \ldots\}$ is infinite, whereas $U(\Gamma) - \varphi$ is also infinite. So Counting$_1$ to Counting$_3$ select the same models as Minimal Abnormality. However, the following definition selects, for example, $\{A, D_1, D_3, D_5, \ldots\}$ but not $\{B, C, D_1, D_3, D_5, \ldots\}$.

Definition 6.1.12 *A model* $M \in \mathcal{M}^{\mathbf{LLL}}_{\Gamma}$ *is a Counting*$_4$ *model of* Γ *iff there is no* $M' \in \mathcal{M}^{\mathbf{LLL}}_{\Gamma}$ *for which* $\#(Ab(M') - Ab(M)) < \#(Ab(M) - Ab(M'))$.

In order to compare the Minimal Abnormality strategy and the four Counting strategies, it is instructive to realize that all abnormalities in $Ab(M) \cup Ab(M')$ belong to one of the three following (disjoint) sets.

$Ab(M') - Ab(M)$	$Ab(M) \cap Ab(M')$	$Ab(M) - Ab(M')$

The union of the left and middle set comprise the abnormalities in $Ab(M')$; the union of the middle and right set comprises the abnormalities in $Ab(M)$. Counting$_4$ only numerically compares the left set and the right set and hence deselects one of the models as soon as one of these sets is finite and smaller than the other.[5] Minimal Abnormality merely checks whether the left or right set is empty. Counting$_1$ requires that the left set and the middle set are finite or that the middle set and the right set are finite, and next numerically compares the left and the right set. Counting$_2$ combines the two previous criteria. Counting$_3$ uses an in-between criterion, but Counting$_4$ may deselect more, and only more,

[5]Note that a model M' may 'defeat' M without affecting any other models. Indeed, it is possible that $\#(Ab(M') - Ab(M)) < \#(Ab(M) - Ab(M'))$ whereas both $Ab(M') - Ab(M'')$ and $Ab(M'') - Ab(M')$ are infinite and hence have the same cardinality.

lower limit models of some premise sets. To see this one should realize that (i) if $\#(U(\Gamma) - Ab(M')) > \#(U(\Gamma) - Ab(M))$, then $\#(Ab(M') - Ab(M)) < \#(Ab(M) - Ab(M'))$ and (ii) there may be $M, M' \in \mathcal{M}_\Gamma^{\mathbf{LLL}}$ such that $Ab(M) \cap Ab(M')$, $U(\Gamma) - Ab(M')$, and $U(\Gamma) - Ab(M)$ are all infinite, but $Ab(M) - Ab(M')$ is not. The following diagram is helpful. It represents three disjoint sets the union of which is identical to $U(\Gamma) - (Ab(M) \cap Ab(M'))$.

$U(\Gamma) - (Ab(M) \cup Ab(M'))$	$Ab(M') - Ab(M)$	$Ab(M) - Ab(M')$

Note that $U(\Gamma) - Ab(M') = U(\Gamma) - (Ab(M) \cup Ab(M')) \cup Ab(M) - Ab(M')$ and $U(\Gamma) - Ab(M) = U(\Gamma) - (Ab(M) \cup Ab(M')) \cup Ab(M') - Ab(M)$.

The transition to proofs is easy. The rules are obviously the same as for adaptive logics in standard format.

To devise the marking definition, it is wise to make a short detour. Remember the relation between $\Phi(\Gamma)$ and the $Ab(M)$ of a minimal abnormal model M of Γ: if Γ has **LLL**-models, then $M \in \mathcal{M}_\Gamma^m$ iff $Ab(M) \in \Phi(\Gamma)$ (from Lemma 5.2.1). This, combined with Definitions 6.1.10–6.1.12, enables us at once to define sets $\Phi^{c2}(\Gamma)$–$\Phi^{c4}(\Gamma)$. For example, $\varphi \in \Phi^{c2}(\Gamma)$ iff $\varphi \in \Phi(\Gamma)$ and there is no $\varphi' \in \Phi(\Gamma)$ for which $\#\varphi' < \#\varphi$ or $\varphi' \subseteq \varphi$. If the function of the sets $\Phi^{ci}(\Gamma)$ is not obvious at once: they are such that $M \in \mathcal{M}_\Gamma^{ci}$ iff $Ab(M) \in \Phi^{ci}(\Gamma)$. What about $\Phi^{c1}(\Gamma)$? Even this is easy. If Γ has Counting$_1$ models, then they are obviously Minimally Abnormal models of Γ.[6] So we define $\varphi \in \Phi^{c1}(\Gamma)$ iff $\varphi \in \Phi(\Gamma)$ iff there is a $\varphi' \in \Phi(\Gamma)$ for which $\#\varphi < \#\varphi'$ and there is no $\varphi' \in \Phi(\Gamma)$ for which $\#\varphi' < \#\varphi$ or $\varphi' \subseteq \varphi$. So if all members of $\Phi(\Gamma)$ are infinite, $\Phi^{c1}(\Gamma) = \emptyset$.

Next, we define $\Phi_s^{ci}(\Gamma)$ ($i \in \{1, 2, 3, 4\}$) with respect to a proof at a stage s. The definition is just the same as that for $\Phi^{ci}(\Gamma)$, except that we refer to the minimal *Dab*-formulas that occur in stage s instead of to the minimal *Dab*-consequences of Γ. Given this, the marking definition for Counting$_i$ is a faithful analogue of that for Minimal Abnormality.

Definition 6.1.13 *Marking for the Counting$_i$ strategy ($i \in \{1, 2, 3, 4\}$): Line l is marked at stage s iff, where A is derived on the condition Δ at line l, (i) no $\varphi \in \Phi_s^{ci}(\Gamma)$ is such that $\varphi \cap \Delta = \emptyset$, or (ii) for some $\varphi \in \Phi_s^{ci}(\Gamma)$, there is no line at which A is derived on a condition Θ for which $\varphi \cap \Theta = \emptyset$.*

For every finite proof at a stage—and that is what one may write down—all four marking definitions come to the same. Remember, however, that final derivability is established from a finite stage by a reasoning in the metalanguage. The difference between the four marking definitions will show up in that reasoning in the metalanguage.

Let us now move to a simple propositional example, in which all four Counting strategies do better than Minimal Abnormality. Let $\Gamma_1 = \{p, q, r, \neg p \vee s, \neg q \vee t, \neg p \vee \neg q, \neg p \vee \neg r\}$ and let the logic be $\mathbf{CLuN}^{ci}$ (with Ω as for $\mathbf{CLuN}^m$).

1	p	Prem	$\emptyset$
2	q	Prem	$\emptyset$
3	r	Prem	$\emptyset$
4	$\neg p \vee s$	Prem	$\emptyset$
5	$\neg q \vee t$	Prem	$\emptyset$

[6] The weakness of the Counting$_1$ strategy is that it selects *all* lower limit models of Γ iff $Ab(M)$ is infinite for all Minimal Abnormal models of Γ.

6	$\neg p \vee \neg q$	Prem	$\emptyset$	
7	$\neg p \vee \neg r$	Prem	$\emptyset$	
8	s	1, 4; RC	$\{p \wedge \neg p\}$	✓
9	t	2, 5; RC	$\{q \wedge \neg q\}$	
10	$(p \wedge \neg p) \check{\vee} (q \wedge \neg q)$	1, 2, 6; RU	$\emptyset$	
11	$(p \wedge \neg p) \check{\vee} (r \wedge \neg r)$	1, 3, 7; RU	$\emptyset$	

At stage 10 of the proof lines 8 and 9 are marked because $\Phi^{ci}_{10}(\Gamma_1) = \{\{p \wedge \neg p\}, \{q \wedge \neg q\}\}$. At stage 11 line 9 is unmarked because $\Phi_{11}(\Gamma_1) = \{\{p \wedge \neg p\}, \{q \wedge \neg q, r \wedge \neg r\}\}$, whence $\Phi^{ci}_{11}(\Gamma_1) = \{\{p \wedge \neg p\}\}$. The proof at stage 11 is obviously stable with respect to all its lines.

Suppose that more *Dab*-formula generating premises are added, even infinitely many, but in such a way that 10 and 11 are the only (non-equivalent) *Dab*-formulas in which occur $p \wedge \neg p$, $q \wedge \neg q$, and $r \wedge \neg r$. One may even consider adding $\{(p_i \wedge \neg p_i) \vee (p_{i+1} \wedge \neg p_{i+1}) \mid i \in \{1, 3, 5, \ldots\}\}$. The result stays exactly the same as in the previous proof. If one knows that $p \wedge \neg p$, $q \wedge \neg q$, and $r \wedge \neg r$ do not occur in the further *Dab*-formulas, there is even no point in deriving them (or introducing the premises from which they follow). Every newly introduced *Dab*-formula will increase the number of members of $\varphi \in \Phi^{ci}_s(\Gamma)$, but they all will contain $p \wedge \neg p$ and none of them will contain either $q \wedge \neg q$ or $r \wedge \neg r$.

Soundness and completeness are pretty obvious. They proceed exactly as for Minimal Abnormality, except that $\Gamma \vdash_{\mathbf{AL}^{ci}} A$ and $\Gamma \vDash_{\mathbf{AL}^{ci}} A$ hold iff, for every $\varphi \in \Phi^{ci}(\Gamma)$, there is a finite $\Delta \subset \Omega - \varphi$ such that $\Gamma \vdash_{\mathbf{LLL}} A \check{\vee} Dab(\Delta)$.

In Section 9.6, I shall consider ways to incorporate Counting strategies into the standard format.

6.2 Combined Adaptive Logics

Several examples of combined adaptive logics occur in Chapter 3. It is time to consider them in a more rigourous way. As in Chapters 4 and 5, I shall describe a set of logics without referring to their specific properties. This time, however, I shall give more examples of specific logics. As simple adaptive logics were thoroughly studied and the properties of combined adaptive logics are easily derivable, I shall describe properties in the text rather than displaying them as theorems or lemmas and shall skip metatheoretic proofs or summarize them in the text.

To see the importance of combined adaptive logics, remember that adaptive logics are basically tools. Many actions require that tools are combined. The situation is not different for the mental actions involved in reasoning. In the methodology of the sciences, for example, agreement grew during the second half of the twentieth century that doing science is essentially a form of problem solving—this culminated in Larry Laudan's [Lau77]. A diversity of potential problems has to be taken into account and most of them require a diversity of reasoning tools. In view of all this, the natural habitat of adaptive logics is within so-called formal problem-solving processes. This matter has been given some thought, for example in [Bat99a, Bat03b, Bat06a, Bat07a], but the topic deserves further and systematic attention.

This book is not the right place to spell out formal problem-solving processes. All I can to do here is describe the properties and functioning of separate adaptive logics, simple ones as well as combined ones. However, the combined ones offer glimpses beyond. They show ways in which simple adaptive logics may be made to cooperate. The most important feature is that this cooperation is possible without fundamentally complicating the ensuing dynamic proofs. A proof at a stage is still the result of applying rules and letting the marking definition operate on it. This is a marvellous result, because, as was said before, the proofs provide the explication for actual human reasoning.

As we shall have to consider several adaptive logics that are combined in some way or other, it is useful to introduce a few *terminological conventions* for the rest of this chapter. Where the adaptive logics are called $\mathbf{AL}_1$, $\mathbf{AL}_2$, etc., their lower limit logics will be $\mathbf{LLL}_1$, $\mathbf{LLL}_2$, etc., and their sets of abnormalities Ω^1, Ω^2, etc. The expression $Dab^i(\Delta)$ will abbreviate the classical disjunction of the members of a finite $\Delta \subset \Omega^i$—this will be called a Dab^i-formula. The generic rules will be distinguished by a subscript, as in RU^i and RC^i.

Combining (and splitting) logics is known to be a touchy business. Those who doubt this should consult [CCG+08], a 600 page book by Walter Carnielli, Marcelo Coniglio, and associates. This book concerns only Tarski logics. Combining defeasible logics is obviously a more complex and harder task. Phrased in terms of adaptive logics, one has not only to take care of the formulas that are derivable by the combined logics, but also of the abnormalities that are so derivable. Many combinations are straightforward for Tarski logics, but lead to circularity when applied to defeasible logics. That I nevertheless engage in this tricky business is because I am interested in actual reasoning and actual reasoning combines different methods, explicated by different adaptive logics.

As will be seen in the sequel of this chapter, many combined adaptive logics do not display properties that are considered desirable for Tarski logics and separate adaptive logics. The combinations may, for example, miss the Fixed Point property. A closer consideration suggests that this should not be taken too heavily. Dealing with problem-solving, which at best may be explicated by formal problem-solving processes, humans stumble around in this world, trying to devise sensible theories. Often a criterion for final derivability is absent. Even if it is present, one may decide to rush forward, relying on present insights, to a theory that, if it proves viable, will drastically change our insights in the domain. This is what all great scientists did—as well as many others, who were unlucky to end up with the wrong theories. Let me try to clarify this. A scientist may stop his or her present reasoning, rely on the provisional insights gained, however non-final, provisional, and defeasible these may be, and restart reasoning from there. In view of this, Fixed Point and similar properties may be less important than they are for simple adaptive logics.

6.2.1 Unions of Sets of Abnormalities

If adaptive logics share their lower limit and their strategy, the most straightforward combination of them is obtained by defining the set of abnormalities of the combined logic as the union of the sets of abnormalities of the combining logics. Let Σ be a set of adaptive logics in standard format that share their lower limit logic and strategy and let $\mathbf{C}$ be defined by the same lower limit logic and strategy and by the set of abnormalities that is the union of the sets of abnormalities of the members of $\Sigma = \{\mathbf{AL}1, \mathbf{AL}2, \ldots\}$. So $\Omega = \Omega_1 \cup \Omega_2 \cup \ldots$ is the set of abnormalities of the combined logic. The result of this 'combination' is a simple adaptive logic in standard format: $\mathbf{C}$ has al the features and properties described in Chapters 4 and 5.

As expected, **C** will often be stronger than any member $\mathbf{AL}i \in \Sigma$. This is most easily seen in terms of the dynamic proofs. A **AL***i*-proof will contain unmarked lines at which a formula is derived on a condition comprising members of Ω^i. Transforming the **AL***i*-proof to a **C**-proof may enable one to push certain disjuncts of so derived formulas towards the condition, without the resulting line being marked.

I shall present one example of this kind of combination here, referring to Chapter 8 for further transparent and simple examples. There are, however, other types of combined adaptive logics. These do not lead to an adaptive logic in standard format. The most important ones (of the known ones) are described in subsequent sections.

Example: Combining Inductive Generalization and Abduction

A nice study of abduction is presented by Atocha Aliseda [Ali06]. Adaptive logics of abduction have been studied especially by Joke Meheus—see [MVVDP02, MB06, Meh07, Meh10]. These logics describe consequence relations that connect sets of premises, containing singular statements as well as generalizations, with (disjunctions of) singular statements that can be abduced from them. Other approaches to abduction are more closely related to [Ali06] and Hintikka's work, for example [Hin98, Hin99, HH05]. They often refer to the *process* of explanation. They describe procedures, which operate in terms of **CL** and lead from a given set of premises to an abduced statement. Those procedures and the ones from [BM01a, MP07] refer to tableau-methods or to the prospective dynamics—see Sections 10.3 and 10.4. A recipe for integrating the inconsistent case is presented in [Bat05c]. As is explained in [MB06], one should distinguish between logics and procedures that lead to potential explanations and those that lead to the weakest explanation, viz. the disjunction of all available potential explanations. Among the adaptive logics of the latter kind, the most interesting available proposal was made in [Meh10]. I shall present it here and combine its Reliability variant, $\mathbf{LAs}^r$,[7] with a simple logic of inductive generalization, viz. $\mathbf{IL}^r$.

The adaptive logic $\mathbf{LAs}^r$ is defined by (i) the lower limit logic **CL**, (ii) $\Omega = \{\forall\alpha(A(\alpha) \supset B(\alpha)) \wedge (B(\beta) \wedge \neg A(\beta) \mid \beta \in \mathcal{C}$; A and B do share a member of $\mathcal{P}\}$ and (iii) Reliability. In the sequel, I shall abbreviate $\forall\alpha(A(\alpha) \supset B(\alpha)) \wedge (B(\beta) \wedge \neg A(\beta))$ to $(A \supset B)^\alpha_\beta$; with some notational abuse, a formula like $\forall x(Px \supset Qx) \wedge (Qa \wedge \neg Pa)$ will be abbreviated by $(P \supset Q)^x_a$.

It is useful to make the reader somewhat acquainted with the properties of this logic. I shall more particularly illustrate that it delivers exactly what one expects. Consider the premise set $\Gamma_2 = \{\forall x(Px \supset Qx), \forall x(Rx \supset Sx), \forall x(Tx \supset Sx), Qa, Sb, Qc, \neg Pc\}$. Let us first consider a proof in which the premises are introduced and the first premise is invoked for two abductions.

1	$\forall x(Px \supset Qx)$	Prem	$\emptyset$
2	$\forall x(Rx \supset Sx)$	Prem	$\emptyset$
3	$\forall x(Tx \supset Sx)$	Prem	$\emptyset$
4	Qa	Prem	$\emptyset$

[7] The name refers to the fact that we are dealing with a logic of abduction which delivers (disjunctions of) singular explanations and has Reliability as its strategy.

5	Sb	Prem	$\emptyset$	
6	Qc	Prem	$\emptyset$	
7	$\neg Pc$	Prem	$\emptyset$	
8	Pa	1, 4; RC	$\{(P \supset Q)^x_a\}$	
9	Pc	1, 6; RC	$\{(P \supset Q)^x_c\}$	$\checkmark^{10}$
10	$(P \supset Q)^x_c$	1, 6, 7; RU	$\emptyset$	

Note that line 9 is justly marked: the abduced Pc contradicts the premise $\neg Pc$. However, that an object's Q-hood cannot be explained by its P-hood does not prevent that the Q-hood of other objects can be explained by their P-hood, as is illustrated by line 8, which will not be marked in any extension of the proof.

Next consider the behaviour of $\mathbf{LAs}^r$ with respect to Sb. I proceed slowly to arrive at line 15; line 16 is obtained by similar steps. Note that 14 is a **CL**-theorem.

11	Rb	2, 5; RC	$\{(R \supset S)^x_b\}$	$\checkmark^{15}$
12	Tb	3, 5; RC	$\{(T \supset S)^x_b\}$	$\checkmark^{16}$
13	$\forall x((Tx \wedge \neg Rx) \supset Sx)$	3; RU	$\emptyset$	
14	$\neg Rb \vee \neg(Tb \wedge \neg Rb)$	RU	$\emptyset$	
15	$(R \supset S)^x_b \vee ((T \wedge \neg R) \supset S)^x_b$	2, 13, 14; RU	$\emptyset$	
16	$(T \supset S)^x_b \vee ((R \wedge \neg T) \supset S)^x_b$	2, 3; RU	$\emptyset$	

Do not think, however, that it is impossible to find an explanation for Sb. The explanation is as it should be: the disjunction of the singular explanations.

17	$\forall x((Rx \vee Tx) \supset Sx)$	2, 3; RU	$\emptyset$	
18	$Rb \vee Tb$	17, 5; Rc	$\{((R \vee T) \supset S)^x_b\}$	

Line 18 will not be marked in any extension of the proof.

The same logical features that prevent Rb and Tb separately be be final consequences, also prevents explanations that are too 'thick' as appears from the following extension of the proof.

19	$\forall x((Px \wedge Ux) \supset Qx)$	1; RU	$\emptyset$	
20	$Pa \wedge Ua$	19, 4; RC	$\{((P \wedge U) \supset Q)^x_b\}$	$\checkmark^{21}$
21	$((P \wedge U) \supset Q)^x_b \vee ((P \wedge \neg U) \supset Q)^x_b$	1; RU	$\emptyset$	

The abduction of inconsistent hypotheses is prevented by a similar mechanism. Note for example that $Qa \vdash_{\mathbf{CL}} (S \wedge \neg S) \supset Q)^x_a$; so the abnormality is derivable whenever the explanandum is derivable.

The literature on explanation distinguishes between potential and actual explanations—see already [Hem65]. A potential explanation requires merely a specific relation between the explanation, a theory—here a set of generalizations–and an explanandum (a fact to be explained). A potential explanation A becomes an actual explanation if A is indeed known to be the case. The premise set Γ_2 is somewhat special in that it only enables one to derive potential explanations, in other words abductive hypotheses. It is not difficult to modify $\mathbf{LAs}^r$ in such a way that actual and potential explanations are distinguished, but I shall not do so here and retain the distinction only in comments on proofs. By all means, it would be a mistake to consider potential explanations as inferior. One often *accepts* "the best explanation", which is the $\mathbf{LAs}^r$-derivable explanation. In this sense abduction is a clear ampliative form of reasoning which transcends explanation in the traditional sense.

As $\mathbf{LAs}^r$ is now sufficiently clarified, let us combine it with $\mathbf{IL}^r$. Let us call the combined logic $\mathbf{ILA}^r$. Its lower limit is **CL**. Its strategy is Reliability. Its set of abnormalities is $\Omega = \{\exists(A_0 \vee \ldots \vee A_n) \wedge \exists\neg(A_0 \vee \ldots \vee A_n) \mid A_0, \ldots, A_n \in \mathcal{A}^{f1}; n \geq 0\} \cup \{(A \supset B)^{\alpha}_{\beta} \mid \alpha \in \mathcal{V}; \beta \in \mathcal{C};\ A$ and B do share a member of $\mathcal{P}\}$. As was repeatedly remarked in Chapter 3, realistic applications of inductive generalization require that background knowledge is taken into account—see Section 3.6. This holds even more so if inductive generalization is combined with abduction. However, handling background knowledge requires a combination by JA ? sequential superposition, which is discussed in subsequent sections.

Nevertheless, the combined logic $\mathbf{ILA}^r$ will enable me to present interesting consequences of the present combination. Moreover, sometimes a new domain is explored. One then tries to find a set of generalizations that, on the empirical side, enable one to predict and explain whatever can be predicted or explained in the domain. So it seems natural that one combines a logic of inductive generalization with a logic of abduction.

Consider the premise set $\Gamma_3 = \{(Pa \wedge \neg Qa) \wedge \neg Ra, (\neg Pb \wedge Qb) \wedge Rb, Pc \wedge Rc, Qd, \neg Pe \wedge \neg Qe\}$. I shall first derive the only relevant generalization[8] that are $\mathbf{IL}^r$-final consequences of Γ_2 as well as the new prediction that is derivable from it. As the reader is sufficiently familiar with $\mathbf{IL}^r$ by now, I proceed rather quickly.

1	$(Pa \wedge \neg Qa) \wedge \neg Ra$	Prem	$\emptyset$
2	$(\neg Pb \wedge Qb) \wedge Rb$	Prem	$\emptyset$
3	$Pc \wedge Rc$	Prem	$\emptyset$
4	Qd	Prem	$\emptyset$
5	$\neg Pe \wedge \neg Qe$	Prem	$\emptyset$
6	$\forall x(Qx \supset Rx)$	2; RC	$\{!(\neg Qx \vee Rx)\}$
7	Rd	4, 6; RU	$\{!(\neg Qx \vee Rx)\}$

We now turn to the singular statements that can be abduced. I shall proceed in small portions in order to be able to comment.

8	$\neg Ra$	1, 6; RC	$\{(\neg R \supset \neg Q)^x_a, !(\neg Qx \vee Rx)\}$
9	Qb	2, 6; RC	$\{(Q \supset R)^x_b, !(\neg Qx \vee Rx)\}$
10	Qd	7, 6; RU	$\{(Q \supset R)^x_d, !(\neg Qx \vee Rx)\}$

The three abduced formulas are actual explanations, which provide no new information. The following abductions deliver potential explanations and hence provide new information.

11	Qc	3, 6; RU	$\{(Q \supset R)^x_c, !(\neg Qx \vee Rx)\}$
12	$\neg Re$	5, 6; RU	$\{(\neg R \supset \neg Q)^x_e, !(\neg Qx \vee Rx)\}$

The cooperation between prediction and abduction resulted in a serious enrichment. It is not specified determined whether Pd holds, but all other objects named in the premise set are specified with respect to all predicates that occur in the premise set. Some people will judge that the combination of inductive generalization and abduction delivers too much. I have three comments in reply. The first is that we are considering an unrealistic situation in which background knowledge is absent. The second comment is more philosophical in nature.

[8]All generalizations and disjunctions of generalizations that are $\mathbf{IL}^r$-final consequences of Γ_2 happen to be derivable from $\forall x(Qx \supset Rx)$.

Applying ampliative reasoning forms is one thing. If the reasoning forms are justified, their outcome informs us about the best decisions we can take, *given our present insights*. A fully different matter is whether we try to verify (or falsify) the outcome of our ampliative reasoning. As I explained in Section 3.3 and elsewhere, the results of ampliative reasoning should urge one to empirically check certain statements. If the check turns out negative, the new premises will force one to revise the results of inductive generalization and hence also the results of abductive reasoning. In the presence of this attitude, the richness of the combination of inductive generalization and abduction is not an objection.

The third comment is slightly longer. The nice result obtained for this premise set does not generalize to all premise sets. If, for example, one adds Re to Γ_3, the generalization $\forall x(\neg Px \supset Rx)$ is finally $\mathbf{IL}^r$-derivable, viz. on the condition $\{!(Px \vee Rx)\}$. As a result Re is finally $\mathbf{IL}^r$-derivable on the same condition. However, precisely this reduces the $\mathbf{ILA}^r$-consequence set to the **CL**-consequence set. Indeed, Re conflicts with $\neg Re$, which still can be abduced at a stage—see line 12 of the proof. The conflict is readily solved if one realizes that the following minimal *Dab*-formula

$$(\neg R \supset \neg Q)^x_e \vee !(\neg Qx \vee Rx) \vee !(Px \vee Rx)$$

is **CL**-derivable from the extended premise set. This means that all lines of the proof that have a non-empty condition are marked. So even the generalizations are not finally derivable any more. This suggests that it seems more advisable to combine $\mathbf{IL}^r$ and $\mathbf{LAs}^r$ in a different way, viz. to apply the abductive logic on the $\mathbf{IL}^r$-closure of the data set. In this way, the occurrence of a 'problem' at the level of abduction will not affect the results of inductive generalization.

The last paragraph may sound like a negative result, but the logic $\mathbf{ILA}^r$ allowed me to illustrate that unions of sets of abnormalities may produce nice results in some cases, but obviously make it more likely that *Dab*-formulas are derivable from a premise set. As announced, more examples of such unions occur in Chapter 8.

6.2.2 Sequential Superpositions

zie ref naar deze afdeling in Hfst 1

The idea behind these logics is that they are obtained by a combination with the following structure

$$Cn_{\mathbf{C}}(\Gamma) = \ldots Cn_{\mathbf{AL3}}(Cn_{\mathbf{AL2}}(Cn_{\mathbf{AL1}}(\Gamma))) \ldots$$

in which the last dots abbreviate only right brackets. This seems like a combination of infinitely many logics, and in some cases the set of combined logics is indeed a recursive infinite set. In many specific cases, the premises or the intended conclusion will enable one to consider at most a finite number of combined logics. Incidentally, the subsequent comments also apply in case the combination is truncated at a finite point.

The recipe displayed above is lacking—this was first shown by Peter Verdée. It fails to specify the language to which the consequence sets belong. As spelled out in footnote 3 of page 114, this means that the statement is true for $\mathcal{L}$ as well as for $\mathcal{L}_+$ and that is not correct—$\mathcal{L}$ and $\mathcal{L}_+$ are used here exactly as in Chapters 4 and 5. The general case requires that consequences in $\mathcal{L}_+$ are included in the premise set of the superposed adaptive logic. Of course, we are still interested in members of $Cn_{\mathbf{C}}(\Gamma)$ that belong to $\mathcal{L}$, viz. are members of $\mathcal{W}$. This makes the definition a bit tiresome.

$$Cn^{\mathcal{L}}_{\mathbf{C}}(\Gamma) = \mathcal{W} \cap (\ldots Cn^{\mathcal{L}_+}_{\mathbf{AL}3}(Cn^{\mathcal{L}_+}_{\mathbf{AL}2}(Cn^{\mathcal{L}_+}_{\mathbf{AL}1}(\Gamma)))\ldots) \tag{6.1}$$

Whenever the combined logic is truncated to finitely many combining logics, the $\mathcal{L}_+$-consequences are included into the input-set (premise set) of the next combining logic, whereas the last combining logic delivers $\mathcal{L}$-consequences only.

In Chapter 4 we have seen that the lower limit logic of an adaptive logic has static proofs. In the construction $Cn^{\mathcal{L}_+}_{\mathbf{AL}2}(Cn^{\mathcal{L}_+}_{\mathbf{AL}1}(\Gamma))$, the role of **AL**1 should not be confused with the lower limit logic of **AL**2. The adaptive logic **AL**2 operates on the premise set $Cn^{\mathcal{L}_+}_{\mathbf{AL}1}(\Gamma)$, which is closed under **AL**1 and of course also under **L**1, the lower limit of **AL**1. The lower limit logic of **AL**2 is a logic that has static proofs. It plays its normal role with respect to **AL**2.

Let us again suppose that all combined logics share their lower limit as well as their strategy.

The reader will remember from Chapter 3 that some combined logics **AL**1, **AL**2, ... are obtained by starting from a simple adaptive logic **AL** and splitting up its set of abnormalities Ω into several subsets Ω_1, Ω_2, ..., which may be disjoint or not. One moves to the combined logic because it offers more or more interesting consequences than the simple one. For some logics and premise sets, however, there are consequences that are derivable by **AL** but not by any **AL**i. When logics have this property, (6.1) is better modified to the following.

$$Cn^{\mathcal{L}}_{\mathbf{C}}(\Gamma) = \mathcal{W} \cap (\ldots Cn^{\mathcal{L}_+}_{\mathbf{AL}3}(Cn^{\mathcal{L}_+}_{\mathbf{AL}2}(Cn^{\mathcal{L}_+}_{\mathbf{AL}1}(Cn^{\mathcal{L}_+}_{\mathbf{AL}}(\Gamma))))\ldots)$$

If the combining logics **AL**1, **AL**2, ... are not the result of splitting up the set of abnormalities of a simple adaptive logic **AL**, one may define **AL** by (i) the lower limit logic shared by the **AL**i, (ii) $\Omega_1 \cup \Omega_2 \cup \ldots$, and (iii) the strategy
verberg
shared by the **AL**i—see also Section 6.2.1.

While (6.1) is nice as a definition, it is essential that the dynamic proof theory of **C** does not follow this line. If the proof theory would require that all members of $Cn_{\mathbf{AL}1}(\Gamma)$ are derived before one starts applying **AL**2, then it would be impossible to obtain even a single specific **AL**2-consequence. We shall see that the proof theory of **C** circumvents this complication in an astoundingly simple way. However, let us first have a look at the semantics.

What about the semantics? The structure of (6.1) suggests the following. Any Γ has a set of **LLL**-models—in the worst case the set is empty. Call this set $\mathcal{M}^0_\Gamma$. First, **AL**1 selects the members of $\mathcal{M}^0_\Gamma$ that are minimally abnormal with respect to Ω^1. Let $\mathcal{M}^1_\Gamma$ be the set comprising these models and note that $Cn_{\mathbf{AL}1}(\Gamma)$ is the set of all formulas verified by every member of $\mathcal{M}^1_\Gamma$. From $\mathcal{M}^1_\Gamma$, **AL**2 selects the members that are minimally abnormal with respect to Ω^2. The set comprising these models is called $\mathcal{M}^2_\Gamma$ and $Cn_{\mathbf{AL}2}(Cn_{\mathbf{AL}1}(\Gamma))$ is the set of all formulas verified by every member of $\mathcal{M}^2_\Gamma$. And so on. The **C**-models are obtained with the last applicable composing logic, if there is one, or else after denumerably many steps. So every Γ has a set of **C**-models, which is a subset of the **LLL**-models of Γ. At the end of this section, Section 6.2.2 I mean, we shall see that **C**, as defined by (6.1), is not complete with respect to this semantics if the strategy is Minimal Abnormality.

By Reassurance (Corollary 5.2.2), Γ has **C**-models if it has **LLL**-models. This follows by transitivity: if $\mathcal{M}^0_\Gamma \neq \emptyset$, then $\mathcal{M}^1_\Gamma \neq \emptyset$; so $\mathcal{M}^2_\Gamma \neq \emptyset$; and so on up to the set of **C**-models.

I shall present the rules and marking definition for *combined* proofs. Before getting there, let us first consider the tiresome but completely straightforward construction that proceeds in terms of separate proofs. We first write an **AL**1-proof at a stage s_1. The formulas of unmarked lines are used as premises in a separate **AL**2-proof. Suppose we continue this proof to stage s_2. The formulas of unmarked lines of this proof are the premises of a separate **AL**3-proof, which we bring to a stage s_3. And so on. The marks in all these proofs are determined by the marking definitions of the standard format. Of course, the proof format is tiresome. If the **AL**1-proof is brought to the next stage, say s'_1, a previously unmarked line may be marked. If the formula of that line was introduced in the **AL**2-proof as a premise, that premise has to be removed together with all lines that directly or indirectly refer to it. This may in turn have effects on the **AL**3-proof, and so on. However, this tiresome construction enables us to figure out in which way the marking should proceed when all those proofs are melted together into a single combined proof. It will also enable one to see that the combined proofs presented below are correct.

Combined proofs are novel with respect to those of simple adaptive logics. The conditions of lines of combined proofs may comprise members of the sets of abnormalities of several simple adaptive logics that define the logic **C**. Incidentally, we have seen these proofs at work in Section 3.5, viz. in the case of logics of the **C**-group and of the **S**-group. The difficult bit is to introduce a single marking definition.

As we have seen before, the fascinating aspect of combined proofs is this: notwithstanding the non-constructive character of (6.1), **C**-consequences are derived in a single proof, the rules are simple and the marking definitions algorithmic. As the lower limit is common, the unconditional rule RU is common to all composing logics. The conditional rules, RC^1, RC^2, etc. are different because they refer to the different sets of abnormalities Ω^1, Ω^2, etc. All rules may be applied in the same proof and no restriction on the formulas on which the rules are applied is required.

To avoid every possible ambiguity, I repeat the rules in Table 6.1. They are astoundingly simple. The only required comment is that, for the rules RC^i, the Δ_j ($1 \leq j \leq n$) may contain members of any Ω^k. To be very explicit: the same Δ_j may comprise abnormalities from an Ω^k with $k \leq i$ as well as abnormalities from an Ω^k with $k \geq i$.

Now to some notational matters. While $Dab^i(\Delta)$ denotes the classical disjunction of the members of a finite $\Delta \subset \Omega^i$, let the classical disjunction of the members of a finite $\Delta \subset \Omega^1 \cup \Omega^2 \cup \ldots$ be denoted by $Dab(\Delta)$. Next, to avoid clutter, let us allow for some notational abuse. Officially, the set of abnormalities that are unreliable with respect to **AL**i should be named $U(Cn_{\mathbf{AL}i-1}(\ldots(Cn_{\mathbf{AL}1}(\Gamma))\ldots))$. In the present context of the combined logic **C**, I shall name that set $U^i(\Gamma)$ and I shall do the same for similar expressions, for example $\Phi^i(\Gamma)$, $U^i_s(\Gamma)$, and $\Phi^i_s(\Gamma)$.

Let us compare the combined proofs with the tiresome separate proofs, supposing that $U^i_s(\Gamma)$ and $\Phi^i_s(\Gamma)$ are given for all i—I shall spell these out in a few paragraphs. Suppose that, in the combined proof, A is derived on the condition Δ at line l. When would it be possible to derive A on an unmarked line of one of

Prem	If $A \in \Gamma$:	$\begin{array}{ll} \ldots & \ldots \\ \hline A & \emptyset \end{array}$
RU	If $A_1, \ldots, A_n \vdash_{\mathbf{LLL}} B$:	$\begin{array}{ll} A_1 & \Delta_1 \\ \ldots & \ldots \\ A_n & \Delta_n \\ \hline B & \Delta_1 \cup \ldots \cup \Delta_n \end{array}$
RC^i	If $A_1, \ldots, A_n \vdash_{\mathbf{LLL}} B \check{\vee} Dab^i(\Theta)$:	$\begin{array}{ll} A_1 & \Delta_1 \\ \ldots & \ldots \\ A_n & \Delta_n \\ \hline B & \Delta_1 \cup \ldots \cup \Delta_n \cup \Theta \end{array}$

Table 6.1: Rules for Sequential Superpositions

the separate proofs? The answer is straightforward, but let me proceed slowly.

That A is derivable on the condition Δ, say at line l, means, in view of Lemma 4.4.1, that $A \check{\vee} Dab(\Delta)$ is **LLL**-derivable from Γ, whence this formula can be obtained in any of the separate proofs on the condition $\emptyset$. Consider a separate **AL***i*-proof. The rule RC^i enables one to derive $A \check{\vee} Dab(\Delta - \Omega^i)$ on the condition $\Delta \cap \Omega^i$ at a line of that proof. If the line is unmarked, $A \check{\vee} Dab(\Delta - \Omega^i)$ may be introduced as a premise in the separate **AL***i*+1-proof as well as in all 'higher numbered' proofs (because the composing adaptive logics are reflexive). In one of these proofs, say the **AL***j*-proof, another part of $Dab(\Delta - \Omega^i)$ may be pushed to the condition, viz. $(\Delta - \Omega^i) \cap \Omega^j$ and if the line at which this is done is again unmarked, $A \check{\vee} Dab(\Delta - (\Omega^i \cup \Omega^j))$ may function as a premise in still 'higher numbered' proofs. So line l of the combined proof should be unmarked just in case there is an ordered partition[9] $\langle \Delta_1, \ldots, \Delta_n \rangle$ of Δ for which the following holds: $\Delta_1 \subset \Omega^1 - U^1_1(\Gamma)$ whence $A \check{\vee} Dab(\Delta_2 \cup \ldots \cup \Delta_n)$ can be derived on the condition Δ_1 at an unmarked line of the separate **AL**1-proof; $\Delta_2 \subset \Omega^2 - U^2_s(\Gamma)$ whence $A \check{\vee} Dab(\Delta_3 \cup \ldots \cup \Delta_n)$ can be derived on the condition Δ_2 at an unmarked line of the **AL**2-proof; and so on. In the sequel of this Section, the
NAZIEN terms "partition" will always refer to an ordered partition.

Let us push this one step further. It is at once clear which formulas of the combined proof are derivable at an unmarked line of the separate proofs. For **AL**1, these are the formulas of the unmarked lines (of the combined proof) that have a condition $\Delta \subset \Omega^1$. For **AL**2, the formulas of the unmarked lines that have a condition $\Delta \subset \Omega^1 \cup \Omega^2$. And so on.

The insight from the previous paragraph settles at once which are the minimal Dab^i-formulas in a combined proof at stage s. The minimal Dab^1-formulas are those derived on the condition $\emptyset$. The minimal Dab^2-formulas are those derived on a condition $\Delta \subset \Omega^1$ at an unmarked line. And so on. From the minimal Dab^i-formulas at stage s, $U^i_s(\Gamma)$ and $\Phi^i_s(\Gamma)$ are defined as they are for adaptive logics in standard format—just keep the i apart.

A brief interruption before we proceed. In Chapter 3 we have only met

[9] I simply mean a partition of which the members are put in some order. So the set of sets $\langle \Delta_1, \ldots, \Delta_n \rangle$ is an ordered partition of Δ iff $\Delta = \Delta_1 \cup \ldots \cup \Delta_n$ and $\Delta_i \cap \Delta_j = \emptyset$ for all different i and j ($1 \leq i, j \leq n$). I do not exclude that some Δ_i are empty.

special cases of sequential superpositions, viz. those in which $\Omega^i \subset \Omega^{i+1}$, as in the logics of the **C**-group, and those for which $\Omega^i \cap \Omega^j = \emptyset$ for all different i and j, as in the logics of the **S**-group. The general case, however, is much more complicated. Some sets Ω^i and Ω^j may have a non-empty intersection even if none of them is a subset of the other; other such sets may have an empty intersection.

Reliability Let us now turn to the marking definitions for the combined proofs. As might be expected, Reliability is simple even in the general case. A line with condition $\Delta \subset \Omega^1$ is unmarked at stage s iff $\Delta \subset \Omega^1 - U^1_s(\Gamma)$. In general, a line that has the condition Δ is *unmarked* iff there is a partition $\langle \Delta_1, \ldots, \Delta_n \rangle$ of Δ such that, for every Δ_i $(1 \leq i \leq n)$, $\Delta_i \subset \Omega^i - U^i_s(\Gamma)$. So this reduces to something very simple. Let $R^{\{1,\ldots,n\}}_s(\Gamma) = (\Omega^1 - U^1_s(\Gamma)) \cup \ldots \cup (\Omega^n - U^n_s(\Gamma))$. A line of the combined proof that has the condition Δ is *unmarked* iff there is a n such that $\Delta \subset \Omega^1 \cup \ldots \cup \Omega^n$ and $\Delta \subset R^{\{1,\ldots,n\}}_s(\Gamma)$.

Definition 6.2.1 *Marking for Reliability: A line l is* marked *at stage s iff, where Δ is its condition, there is no $n \subset \mathbb{N}$ such that $\Delta \subset \Omega^1 \cup \ldots \cup \Omega^n$ and $\Delta \subset R^{\{1,\ldots,n\}}_s(\Gamma)$.*

Here is an alternative but equivalent marking definition.

Definition 6.2.2 *Marking for Reliability: A line l is* unmarked *at stage s iff, where Δ is its condition, there is a partition $\langle \Delta_1, \ldots, \Delta_n \rangle$ of Δ such that, for every i $(1 \leq i \leq n)$, $\Delta_i \subset \Omega^i - U^i_s(\Gamma)$.*

With this definition, one may proceed stepwise. The minimal Dab^1-formulas are formulas of lines that have $\emptyset$ as their condition. These define $U^1_s(\Gamma)$. So one defines Δ_1 as $\Delta \cap (\Omega^1 - U^1_s(\Gamma))$. The result gives one the Dab^2-formulas and hence $U^2_s(\Gamma)$. So one defines Δ_2 as $(\Delta - \Delta^1) \cap (\Omega^2 - U^2_s(\Gamma))$. And so on. If the remainder is empty at some point, the line is unmarked; if the remainder contains formulas that do not belong to any 'higher numbered' Ω^i, one knows that the line is marked. As we shall see below, it is essential that the Dab^{i+1}-formulas are determined by the **AL**i-marks and not by the marks of any 'higher numbered' logic.

Note Definition 6.2.2 states which lines are *unmarked.* So all lines that do not fulfil the condition are marked. This should be kept in mind in case one proceeds as in the previous paragraph in order to determine which lines are marked. At the start all lines are considered as being marked; then some lines are found to be unmarked in view of **AL**1; then some lines are found to be unmarked in view of **AL**1 and **AL**2; and so on. Please do not confuse the fact that a line is marked or unmarked according to the definition with the procedure for finding out whether a line is marked.

Before moving to marking for Minimal Abnormality, let us have a look at the two special cases that figure in Chapter 3. First, the case where $\Omega^i \cap \Omega^j = \emptyset$ for all different i and j as in the logics of the **S**-group. The previous marking definition then comes to the one given in Chapter 3, which I rephrase here in generic form and with **AL**1 the 'innermost' logic.

Definition 6.2.3 *Marking for Reliability (special case $\Omega^i \cap \Omega^j = \emptyset$ for all different i and j): Starting from $i = 1$, a line is* marked *at stage s iff, where Δ is its condition, $\Delta \cap U^i_s(\Gamma) = \emptyset$.*

The application of this definition also interacts with the definition of the minimal Dab^i-formulas in the combined proof. The minimal Dab^1-formulas at stage s determine the marks for $i = 1$. The minimal Dab^2-formulas depend on these and determine the marks for $i = 2$. And so on. Central to the interaction is that the minimal Dab^i-formulas are determined in terms of the marks for $i - 1$. In the two previous definitions, we had to take into account that a line is marked for a logic $\mathbf{AL}i$ but not for the logic $\mathbf{AL}i{+}k$. In view of the special case handled by Definition 6.2.3, a line marked for $\mathbf{AL}i$ is marked for all logics $\mathbf{AL}i{+}k$.

The second special case we have to consider is where $\Omega^i \subset \Omega^{i+1}$ for all i, as in the logics of the **C**-group from Chapter 3. This may seem like a complex case, but actually it is not because two interesting facts simplify the combined proofs for this special case.

Fact 6.2.1 *If $A \in \Omega^i - U^i(\Gamma)$, then, for all $j > i$, $A \notin U^j(\Gamma)$.*

Indeed, suppose that $A \in \Omega^i - U^i(\Gamma)$ whereas $A \in U^j(\Gamma)$ for a $j > i$.[10] By Item 6 of Theorem 5.6.7, $A \in \Omega^i - U^i(\Gamma)$ gives us $\check{\neg} A \in Cn_{\mathbf{AL}i}(\ldots(Cn_{\mathbf{AL}1}(\Gamma))\ldots)$, whence $\check{\neg} A \in Cn_{\mathbf{AL}j-1}(\ldots(Cn_{\mathbf{AL}1}(\Gamma))\ldots)$ by the reflexivity of the combining logics. By the definition of $U(\Gamma)$, $A \in U^j(\Gamma)$ entails that there is a $\Delta \subset \Omega^j$ such that $Dab^j(\Delta \cup \{A\})$ is a minimal Dab^j-consequence of $Cn_{\mathbf{AL}j-1}(\ldots(Cn_{\mathbf{AL}1}(\Gamma))\ldots)$. However, as $Dab^j(\Delta \cup \{A\}), \check{\neg} A \vdash_{\mathbf{LLL}} Dab^j(\Delta)$, Theorem 5.6.1 gives us that $Dab^j(\Delta) \in Cn_{\mathbf{AL}j-1}(\ldots(Cn_{\mathbf{AL}1}(\Gamma))\ldots)$, which contradicts that $Dab^j(\Delta \cup \{A\})$ is a minimal Dab^j-consequence of $Cn_{\mathbf{AL}j-1}(\ldots(Cn_{\mathbf{AL}1}(\Gamma))\ldots)$.

The next fact is a counterpart to Fact 6.2.1: if a condition of the combined proof is unreliable with respect to **AL**i, then it is unreliable with respect to all **AL**$i+k$.

Fact 6.2.2 *If $A \in U^i(\Gamma)$, then $A \in U^{i+1}(\Gamma)$.*

Indeed, suppose that $A \in U^i(\Gamma)$ whereas $A \notin U^{i+1}(\Gamma)$. It follows that there is a $\Delta \subset \Omega^i$ such that (i) $Dab^i(\Delta \cup \{A\})$ is a *minimal* Dab^i-consequence of $Cn_{\mathbf{AL}i-1}(\ldots(Cn_{\mathbf{AL}1}(\Gamma))\ldots)$ whereas (ii) $Dab^{i+1}(\Delta)$ is a Dab^{i+1}-consequence of $Cn_{\mathbf{AL}i}(\ldots(Cn_{\mathbf{AL}1}(\Gamma))\ldots)$. But then, as $\Delta \subset \Omega^i$, $Dab^i(\Delta)$ is a Dab^i-consequence of $Cn_{\mathbf{AL}i}(\ldots(Cn_{\mathbf{AL}1}(\Gamma))\ldots)$. So $Dab^i(\Delta)$ is a Dab^i-consequence of $Cn_{\mathbf{AL}i-1}(\ldots(Cn_{\mathbf{AL}1}(\Gamma))\ldots)$ by Theorem 5.6.2. But this contradicts (i).

In view of the two facts, Definition 6.2.3 may also be applied to the special case in which $\Omega^i \subset \Omega^{i+1}$ for all i.

There is a point worth mentioning concerning this special case. Suppose that $Dab^i(\Delta)$ is derivable on the empty condition in a **C**-proof from Γ and that there is a $\Delta' \subset \Delta$ for which $\Delta' \subset \Omega^{i-1}$. This means that $Dab^i(\Delta - \Delta')$ is derivable on the condition Δ' in the same proof. If the line on which $Dab^i(\Delta - \Delta')$ is so derived is unmarked, $Dab^i(\Delta)$ is not a minimal Dab^i-formula.

Minimal Abnormality Let us turn to the Minimal Abnormality strategy. The rules are obviously the same. One might expect the marking definition to be terribly complex, but actually it is not. We just need some preparation.

A partition of Δ that has n members will be called a n-partition of Δ. By an *A-function* I shall mean a partial function f that maps a finite $\Delta \subset \Omega^1 \cup \Omega^2 \cup \ldots$ to a n-partition of Δ for a specific $n \in \{1, 2, \ldots\}$ and has the following properties: (i) if $\Delta \subset \Omega^1 \cup \Omega^2 \cup \ldots \cup \Omega^n$, then $f(\Delta) = \langle \Delta_1, \ldots, \Delta_n \rangle$ is such that $\Delta_1 \subset \Omega^1, \ldots, \Delta_n \subset \Omega^n$ and (ii) otherwise $f(\Delta)$ is undefined. Where $f(\Delta) = \langle \Delta_1, \ldots, \Delta_n \rangle$ some Δ_i may obviously be empty. Moreover, as it is not excluded that, for example, $\Omega^1 \cap \Omega^5 \neq \emptyset$, there may be many A-functions f each of which maps Δ to a different n-partition of Δ, there may even be more A-functions f each of which maps Δ to a different $(n+1)$-partition of Δ, and so on.

Definition 6.2.4 *Where $\Sigma \subset \wp(\Omega^1 \cup \Omega^2 \cup \ldots)$ and f is an A-function for which $f(\Delta)$ is defined for all $\Delta \in \Sigma$, a $\Delta \in \Sigma$ is* i-safe with respect to Σ and f *iff where $f(\Delta) = \langle \Delta_1, \ldots, \Delta_n \rangle$, (i) there is a $\varphi \in \Phi^i_s(\Gamma)$ such that $\Delta_i \cap \varphi = \emptyset$, and (ii) for every $\varphi' \in \Phi^i_s(\Gamma)$ there is a $\Delta' \in \Sigma$ such that, where $f(\Delta') = \langle \Delta'_1, \ldots, \Delta'_n \rangle$, $\Delta'_1 = \Delta_1, \ldots, \Delta'_{i-1} = \Delta_{i-1}$, $\Delta'_i \cap \varphi' = \emptyset$, $\Delta'_{i+1} = \Delta_{i+1}, \ldots,$ and $\Delta'_n = \Delta_n$.*

[10] In the rest of this paragraph, I do not add the superscripted r to the names of the combining logics. This makes the matter a bit more readable and we know that we are dealing with Reliability anyway.

The definition may look somewhat complicated, but is not. Let $\Delta^1, \ldots, \Delta^m \in \Sigma$ and consider

$$\begin{array}{rcl} f(\Delta^1) & = & \langle \Delta^1_1, \ldots, \Delta^1_i, \ldots, \Delta^1_n \rangle \\ \vdots & \vdots & \vdots \\ f(\Delta^m) & = & \langle \Delta^m_1, \ldots, \Delta^m_i, \ldots, \Delta^m_n \rangle\,. \end{array}$$

All these Δ^k are i-safe with respect to f and Σ if (i) for every Δ^k_i there is a $\varphi \in \Phi^i_s(\Gamma)$ for which $\Delta^k_i \cap \varphi = \emptyset$, (ii) for every $\varphi \in \Phi^i_s(\Gamma)$ there is a Δ^k_i for which $\Delta^k_i \cap \varphi = \emptyset$, and (iii) $\Delta^k_j = \Delta^l_j$ for all $k, l \in \{1, \ldots, m\}$ and for all $j \in \{1, \ldots, i-1, i+1, \ldots, n\}$.

Definition 6.2.5 *Marking for Minimal Abnormality: A line l that has A as its formula and Δ as its condition is* unmarked *at stage s iff, where $\Sigma \subset \wp(\Omega^1 \cup \Omega^2 \cup \ldots)$ comprises the conditions on which A is derived at stage s, there is a $n \in \mathbb{N}$, and there is an A-function f such that Δ is i-safe with respect to Σ and f for all $i \in \{1, \ldots, n\}$.*

If A is derived at an unmarked line of one of the separate proofs, it can be derived in the combined proof at a line that is unmarked in view of Definition 6.2.5. This is easily seen if the combined proof from Γ is defined as follows in terms of the separate proofs. All lines of the **AL**1-proof are simply copied in the combined proof, with occurrences of "RC" replaced by "RC^1". Next, we consider the **AL**2-proof and transform it as follows. First we transform the line numbers and references to them by prefixing them with "2." and we replace every occurrence of RC in a justification by RC^2. We furthermore transform the lines of the proof as follows. *Case 1.* Line l has the justification Prem, its formula is A and $A \in \Gamma$. We do nothing to this line. *Case 2.* Line l has the justification Prem, its formula is A and $A \notin \Gamma$. So A is the formula of n ($n \geq 1$) unmarked lines of the **AL**1-proof. Let $i_1, \ldots, i_n$ be the line numbers of these lines and let $\Delta_1, \ldots, \Delta_n$ be their conditions. We remove line l. Every line l' of the **AL**2-proof that refers to l in its justification is replaced by n copies. In the first copy of l' the reference to l in the justification is replaced by i_1 and the condition Θ of line l' is replaced by $\Theta \cup \Delta_1$; in the second copy of l' the reference to l in the justification is replaced by i_2 and the condition Θ of line l' is replaced by $\Theta \cup \Delta_2$; and so on for each of the n copies. Moreover, every line of the **AL**2-proof that refers to a line of which n copies are made, is replaced by n copies and the justifications and conditions are adjusted as described (every copy of the line refers to one of the copies of l' and to its condition we add $\cup\Delta_j$ for the right j. This process is clearly longwinded but leads to a finite sequence of lines, which is appended to the combined proof. Next we proceed in in exactly the same way for the **AL**3-proof, and so on until we reach the proof in which A was derived at an unmarked line.

A simple example will further clarify the matter. Suppose (i) that A was derived in the **AL**1-proof on a set of unmarked lines that have respectively the conditions $\Delta_1, \ldots,$ and Δ_n and that (ii) that from the premise A in the **AL**2-proof the formula B is derived on a set of unmarked lines that have respectively the conditions $\Theta_1, \ldots,$ and Θ_m. Then B will be derived in the combined proof on $n \times m$ conditions, viz. on every condition $\Delta_i \cup \Theta_j$ for which $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, m\}$.

The combined proof is obviously correct with respect to the rules from Table 6.1. Moreover, every *unmarked* line from the separate proofs corresponds to one or more *unmarked* lines of the combined proof. This is easily seen for the lines of the combined proof that correspond to lines of the **AL**2-proof. The A-function f assigns to a condition in the combined proof the partition $\langle \Delta, \Theta \rangle$ in which Θ is the condition of the original line of the **AL**2-proof and Δ is the condition that was taken from one of the lines of the **AL**1-proof. If both lines were unmarked, then the condition of the line in the combined proof is obviously 1-safe as well as 2-safe with respect to $\{\Delta_1, \ldots, \Delta_n\} \times \{\Theta_1, \ldots, \Theta_m\}$ and with respect to the A-function f for which $f(\Delta_i \cup \Theta_j) = \{\Delta_i, \Theta_j\}$, which are the respective conditions that occur in the **AL**1-proof and in the **AL**2-proof. This insight is easily generalized to an arbitrary line of an arbitrary separate proof, whence it leads to a proof by induction.

So if A is derivable on an unmarked line of a separate proof, then it is derivable on an unmarked line in the combined proof. Rather than showing the converse directly, I shall show it indirectly by first presenting a simpler marking definition.

However, let us pause for a moment to see that the application of Definition 6.2.5 to a finite combined proof is a recursive matter. Remember that $\Phi^1_s(\Gamma)$ is defined in terms of the minimal Dab^1-formulas and that these are derived on an empty condition in the combined proof. So it is easy enough to find out which are the members of $\Phi^1_s(\Gamma)$. The marks of lines that have a non-empty subset of Ω^1 as their condition depend only on $\Phi^1_s(\Gamma)$. So it is easy to locate the unmarked ones. Next, $\Phi^2_s(\Gamma)$ is defined in terms of the minimal Dab^2-formulas and that these are derived on a condition which is a (possibly empty) subset of Ω^1. So, again, it is easy to find out which are the members of $\Phi^2_s(\Gamma)$. The marks of lines with a subset of $\Omega^1 \cup \Omega^2$ as their condition depend only on $\Phi^1_s(\Gamma)$ and $\Phi^2_s(\Gamma)$. And so on. geen herhaling?

Please take into account that the Ω^i may have common members. For example, it is possible that $\Delta \subset \Omega^2 \cap \Omega^5$. If a line has A as its formula and *this* Δ as its condition, it may be marked for **AL**2 and unmarked for **AL**5. This, however, is not a problem for what was said in the previous paragraph. It means that A is a **AL**5-consequence but not a **AL**2-consequence. So, even if A is a classical disjunction of members of Ω^3, it will *not* be a Dab^3-formula and will play no role in $\Phi^3(\Gamma)$.[11]

Now we come to the promised alternative alternative marking definition. By a choice set of the *ordered* set $\langle \Phi^1_s(\Gamma), \ldots, \Phi^n_s(\Gamma) \rangle$ I shall mean an ordered set every member of which is an element of the corresponding $\Phi^i_s(\Gamma) \in \langle \Phi^1_s(\Gamma), \ldots, \Phi^n_s(\Gamma) \rangle$. To simplify the notation, let $\langle \varphi_1, \ldots, \varphi_n \rangle$ function as a *variable* for choice sets of $\langle \Phi^1_s(\Gamma), \ldots, \Phi^n_s(\Gamma) \rangle$.

Definition 6.2.6 *n-marking for Minimal Abnormality: A line is n-unmarked at stage s iff, where A is its formula, Δ its condition, there is an A-function f such that (i) $f(\Delta) = \langle \Delta_1, \ldots, \Delta_n \rangle$ and $\Delta_1 \cap \varphi_1 = \emptyset$, ..., and $\Delta_n \cap \varphi_n = \emptyset$ for a choice set $\langle \varphi_1, \ldots, \varphi_n \rangle$ of $\langle \Phi^1_s(\Gamma), \ldots, \Phi^n_s(\Gamma) \rangle$, and (ii) for every choice set $\langle \varphi'_1, \ldots, \varphi'_n \rangle$ of $\langle \Phi^1_s(\Gamma), \ldots, \Phi^n_s(\Gamma) \rangle$, A is derived at stage s on a condition Δ' such that $f(\Delta') = \langle \Delta'_1, \ldots, \Delta'_n \}$, $\Delta'_1 \cap \phi'_1(\Gamma) = \emptyset$, ..., and $\Delta'_n \cap \phi'_n(\Gamma) = \emptyset$.*

[11]Incidentally, this illustrates that some sequential superpositions are not fixed points, a matter to which I return in the text.

A	$\Delta_1 \cup \ldots \cup \Delta_n$
$A \check{\vee} Dab^n(\Delta_n)$	$\Delta_1 \cup \ldots \cup \Delta_{n-1}$
$\vdots$	$\vdots$
$A \check{\vee} Dab^n(\Delta_n) \check{\vee} \ldots \check{\vee} Dab^3(\Delta_3)$	$\Delta_1 \cup \Delta_2$
$A \check{\vee} Dab^n(\Delta_n) \check{\vee} \ldots \check{\vee} Dab^2(\Delta_2)$	Δ_1
$A \check{\vee} Dab^n(\Delta_n) \check{\vee} \ldots \check{\vee} Dab^1(\Delta_1)$	$\emptyset$

Table 6.2: Condition Analysing Table

This definition has two important consequences. First, a line with condition Δ can only be n-unmarked at stage s if $\Delta \subset \Omega^1 \cup \ldots \cup \Omega^n$. Next, a line that is n-unmarked at stage s is $(n+1)$-unmarked at stage s.

Definition 6.2.7 *Marking for Minimal Abnormality: A line l that has A as its formula and Δ as its condition is* unmarked *at stage s iff it is n-unmarked at stage s for an A-function f and for a $n \in \mathbb{N}$.*

Applying this definition to a finite combined proof may be longwinded but is a decidable matter. Calculate $\Phi^1_s(\Gamma)$ and consider n-marking for $n = 1$. This gives one $\Phi^2_s(\Gamma)$. And so on.

If a line of a combined proof is unmarked on Definition 6.2.5, then it is unmarked on Definition 6.2.7. According to the first definition, the line must i-safe with respect to all $i \in \{1, \ldots, n\}$ for some $n \in \mathbb{N}$ and this entails that it is n-unmarked for that n.

Let us now close the circle. Suppose that there is a combined proof from Γ in which A is derived at an unmarked line (according to Definition 6.2.7). Is there a sequence of separate proofs such that A is derived at an unmarked line of one of them?

At a first glance, the answer seems to be positive. Suppose that A was derived on the condition Δ in the combined proof and that the reason why the line is i-unmarked is that $f(\Delta) = \langle \Delta_1, \ldots, \Delta_n \rangle$. Relying on the Conditions Lemma 4.4.1, we know it is possible to derive every formula in the left column of Table 6.2 on the condition occurring in the right column of the same row of the table. Of these, the last line may be derived in the **AL**1-proof. The next to last line is derivable in the **AL**2-proof provided we remove Δ_1 from its condition. And so on.

Constructing the separate proofs in this way will not in itself deliver the required result. The problem is that Definition 6.2.7 *seems* to allow for too many unmarked lines. The (small) problem may be illustrated by a combined proof containing unmarked lines with the following formulas and conditions—I list the justifying choice sets of $\langle \Phi^1_s(\Gamma), \Phi^2_s(\Gamma) \rangle$ in the third column.

A	$\Delta^1_1 \cup \Delta^2_1$	$\langle \varphi^1_1, \varphi^2_1 \rangle$
A	$\Delta^1_2 \cup \Delta^2_2$	$\langle \varphi^1_1, \varphi^2_2 \rangle$
A	$\Delta^1_3 \cup \Delta^2_3$	$\langle \varphi^1_2, \varphi^2_1 \rangle$
A	$\Delta^1_4 \cup \Delta^2_4$	$\langle \varphi^1_2, \varphi^2_2 \rangle$

So the idea is that $\Phi^1_s(\Gamma) = \{\varphi^1_1, \varphi^1_2\}$, $\Phi^2_s(\Gamma) = \{\varphi^2_1, \varphi^2_2\}$, $(\Delta^1_1 \cup \Delta^1_2) \cap \varphi^1_1 = \emptyset$, $(\Delta^1_3 \cup \Delta^1_4) \cap \varphi^1_2 = \emptyset$, $(\Delta^2_1 \cup \Delta^2_3) \cap \varphi^2_1 = \emptyset$, and $(\Delta^2_2 \cup \Delta^2_4) \cap \varphi^2_2 = \emptyset$. Now let us look at the result of 'analysing' these four lines according to Table 6.2. The following formulas and conditions should end up in the **AL**1-proof:

$A \check{\vee} Dab^2(\Delta_1^2) \check{\vee} Dab^1(\Delta_1^1)$	$\emptyset$
$A \check{\vee} Dab^2(\Delta_2^2) \check{\vee} Dab^1(\Delta_2^1)$	$\emptyset$
$A \check{\vee} Dab^2(\Delta_3^2) \check{\vee} Dab^1(\Delta_3^1)$	$\emptyset$
$A \check{\vee} Dab^2(\Delta_4^2) \check{\vee} Dab^1(\Delta_4^1)$	$\emptyset$

and this is quite all right. It is warranted by the combined proof that these lines are derivable in the **AL**1-proof. Moreover, the **AL**1-proof should contain lines with the following formulas and conditions:

$A \check{\vee} Dab^2(\Delta_1^2)$	Δ_1^1
$A \check{\vee} Dab^2(\Delta_2^2)$	Δ_2^1
$A \check{\vee} Dab^2(\Delta_3^2)$	Δ_3^1
$A \check{\vee} Dab^2(\Delta_4^2)$	Δ_4^1

and obviously such lines can be obtained in the **AL**1-proof in view of the Conditions Lemma, but the conditions of the lines do not warrant that the lines are unmarked. Indeed, four *different* formulas are derived and each of them on a condition that overlaps with some choice set of $\Phi_s^1(\Gamma)$.

And yet the matter can be easily repaired. By applying Addition, we can obtain in the **AL**1-proof lines with the following formulas and conditions.

$A \check{\vee} Dab^2(\Delta_1^2 \cup \Delta_3^2)$	Δ_1^1
$A \check{\vee} Dab^2(\Delta_2^2 \cup \Delta_4^2)$	Δ_2^1
$A \check{\vee} Dab^2(\Delta_1^2 \cup \Delta_3^2)$	Δ_3^1
$A \check{\vee} Dab^2(\Delta_2^2 \cup \Delta_4^2)$	Δ_4^1

As $(\Delta_1^1 \cup \Delta_2^1) \cap \varphi_1^1 = \emptyset$ and $(\Delta_3^1 \cup \Delta_4^1) \cap \varphi_2^1 = \emptyset$, the conditions of these lines warrant that all four lines are unmarked.

Given this insight, the better way is to first modify the combined proof. Suppose that A is derived on the conditions $\Delta^1 \ldots, \Delta^n$ at lines that are unmarked in view of Definition 6.2.7. Let the lines be unmarked in view of the A-function f for which $f(\Delta^i) = \langle \Delta_1^i, \ldots, \Delta_k^i \rangle$ and in view of the choice sets of $\langle \Phi_s^1(\Gamma), \ldots, \Phi_s^k(\Gamma) \rangle$. We now 'thicken' the Δ^i as follows. Start with Δ_1^i. This will not overlap with one or more members of Φ_s^1; let these be $\varphi_1^1, \ldots,$ and φ_5^1. In this case, we add four copies of the line at which A is derived on the condition Δ^i. We associate the first line with φ_1^1 and locate all Δ^m for which the first member of $f(\Delta^m)$, viz. Δ_1^m does not overlap with φ_1^1. Let the union of all Δ_1^m be Σ. We replace $\Delta^i = \Delta_1^i \cup \ldots \cup \Delta_k^i$ by $(\Delta_1^i \cup \Sigma) \cup \ldots \cup \Delta_k^i$. Next, we do the same for the other four copies, which are respectively associated with $\varphi_2^1, \ldots, \varphi_5^1$. Having done so for every first element of $f(\Delta^1) \ldots, f(\Delta^n)$, we do the same for every *second* element of an $f(\Delta)$ on which A is derived in the thus transformed proof—note that there may be already many more then n. Next, we move to the third elements of the $f(\Delta)$ on which A is derived in the further transformed proof. And so we continue to the end.

After having thus 'thickened' (and multiplied) all conditions on which A was derived in the original combined proof, we 'analyse' the lines according to Table 6.2. This time we are home. Let me illustrate the matter in terms of the example. We started from lines with the following formulas and conditions:

A	$\Delta^1_1 \cup \Delta^2_1$
A	$\Delta^1_2 \cup \Delta^2_2$
A	$\Delta^1_3 \cup \Delta^2_3$
A	$\Delta^1_4 \cup \Delta^2_4\}$

where it was understood that $\Phi^1_s(\Gamma) = \{\varphi^1_1, \varphi^1_2\}$, $\Phi^2_s(\Gamma) = \{\varphi^2_1, \varphi^2_2\}$, $(\Delta^1_1 \cup \Delta^1_2) \cap \varphi^1_1 = \emptyset$, $(\Delta^1_3 \cup \Delta^1_4) \cap \varphi^1_2 = \emptyset$, $(\Delta^2_1 \cup \Delta^2_3) \cap \varphi^2_1 = \emptyset$, and $(\Delta^2_2 \cup \Delta^2_4) \cap \varphi^2_2 = \emptyset$. So the 'thickening' gives us:[12]

A	$(\Delta^1_1 \cup \Delta^1_2) \cup (\Delta^2_1 \cup \Delta^2_3)$
A	$(\Delta^1_1 \cup \Delta^1_2) \cup (\Delta^2_2 \cup \Delta^2_4)$
A	$(\Delta^1_3 \cup \Delta^1_4) \cup (\Delta^2_1 \cup \Delta^2_3)$
A	$(\Delta^1_3 \cup \Delta^1_4) \cup (\Delta^2_2 \cup \Delta^2_4)\}$

which we next 'analyse' according to Table 6.2. I skip the 'last lines' of then table (the lines going in **AL**1) because this is too obvious. The lines that go into the separate **AL**1-proof have the following formulas and conditions:

$A \check{\vee} Dab^2(\Delta^2_1 \cup \Delta^2_3)$	$\Delta^1_1 \cup \Delta^1_2$
$A \check{\vee} Dab^2(\Delta^2_2 \cup \Delta^2_4)$	$\Delta^1_1 \cup \Delta^1_2$
$A \check{\vee} Dab^2(\Delta^2_1 \cup \Delta^2_3)$	$\Delta^1_3 \cup \Delta^1_4$
$A \check{\vee} Dab^2(\Delta^2_2 \cup \Delta^2_4)$	$\Delta^1_3 \cup \Delta^1_4$

which are all justified by RC and unmarked in the separate **AL**1-proof.

The lines that go into the separate **AL**2-proof have obviously the following formulas and conditions:

$A \check{\vee} Dab^2(\Delta^2_1 \cup \Delta^2_3)$	$\emptyset$
$A \check{\vee} Dab^2(\Delta^2_2 \cup \Delta^2_4)$	$\emptyset$
$A \check{\vee} Dab^2(\Delta^2_1 \cup \Delta^2_3)$	$\emptyset$
$A \check{\vee} Dab^2(\Delta^2_2 \cup \Delta^2_4)$	$\emptyset$
A	$\Delta^2_1 \cup \Delta^2_3$
A	$\Delta^2_2 \cup \Delta^2_4$
A	$\Delta^2_1 \cup \Delta^2_3$
A	$\Delta^2_2 \cup \Delta^2_4$

of which the first four lines are justified by Prem and the four subsequent ones are justified by RC. All eight lines are unmarked in view of the standard format's marking definition for Minimal Abnormality.

The example is simple, but illustrates every possible complication that has to be taken into account for the general proof, whence I rest my case.

So, we have two marking definitions for minimal abnormality in the general case. I have outlined the proof that a line which is unmarked on Definition 6.2.5 is also unmarked on Definition 6.2.7. The latter definition is obviously the simpler one. The former definition was required to show both definitions correct.

For the Reliability strategy I presented a simplified marking definition for two special cases. Given the simplicity of Definition 6.2.7, there is no point in looking for a further simplification.

[12]Thickening with respect to φ^1 is actually superfluous, but I nevertheless do it here to illustrate the general mechanism (and because it is harmless anyway).

A comment So **C**-proofs at a stage are simple. An application of Prem, RU or any of the RC^i brings the proof to its next stage. The marking definitions determine which lines are marked at that stage. If a new stage is arrived at by deriving a new Dab^1-formula, most marks may change in comparison to the previous stage. Indeed, a new Dab^1-formula may change the insights in the premises with respect to all combining logics. But at every stage, the marking is algorithmic. Moreover, unlike what is the case in the separate proofs, we do not have to delete any lines at any point; marking is sufficient.

I have announced this result and it is worth commenting upon it. Most adaptive logics are very complex (the computational complexity of their consequence sets is high). Moreover, (6.1) defines **C** as an infinite superposition of simple adaptive logics. Nevertheless, **C**-proofs are simple. This is important because **C**-proofs explicate the way in which humans reason towards **C**-consequences. Two different questions are whether conclusions about final derivability can be drawn from finite proofs and whether the insights provided by finite proofs form a basis for sensible decision making. These questions are considered in Chapter 10.

6.2.3 Examples and Metatheoretic Problems

Examples: $\mathbf{SG}^r$ and $\mathbf{SG}^m$

In Section 3.5, on page 96, I promised to present the proof that illustrates the difference between $\mathbf{SG}^r$ and $\mathbf{SG}^m$ in terms of the premise set $\Gamma_{10} = \{Pa, Qa, \neg Pb \vee \neg Qb\}$. The most clarifying way to proceed is to write a proof with two columns of marks, those for $\mathbf{SG}^m$ with superscripted m and those for $\mathbf{SG}^r$ with superscripted r, as they appear at stage 16 of the proof. Remember that the 'innermost' combining logics of $\mathbf{SG}^r$ and $\mathbf{SG}^m$ were called $\mathbf{G}^r_0$ and $\mathbf{G}^r_0$.

1	Pa	Prem	$\emptyset$		
2	Qa	Prem	$\emptyset$		
3	$\neg Pb \vee \neg Qb$	Prem	$\emptyset$		
4	$\forall x Px$	1; RC^0	$\{\pm Px\}$	$\checkmark^m$	$\checkmark^r$
5	$\forall x Qx$	2; RC^0	$\{\pm Qx\}$	$\checkmark^m$	$\checkmark^r$
6	$\forall x Px \vee \forall x Qx$	4; RU	$\{\pm Px\}$		$\checkmark^r$
7	$\forall x Px \vee \forall x Qx$	5; RU	$\{\pm Qx\}$		$\checkmark^r$
8	$\pm Px \vee \pm Qx$	1, 2, 3; RU	$\emptyset$		
9	$\forall x(Px \supset Qx)$	1, 2; RC^1	$\{Px \wedge \pm Qx\}$	$\checkmark^m$	
10	$\forall x(Qx \supset Px)$	1, 2; RC^1	$\{Qx \wedge \pm Px\}$	$\checkmark^m$	
11	$\pm Px \vee (Px \wedge \pm Qx)$	1, 2, 3; RU	$\emptyset$		
12	$Px \wedge \pm Qx$	11; RC^0	$\{\pm Px\}$	$\checkmark^m$	$\checkmark^r$
13	$\pm Qx \vee (Qx \wedge \pm Px)$	1, 2, 3; RU	$\emptyset$		
14	$Qx \wedge \pm Px$	13; RC^0	$\{\pm Qx\}$	$\checkmark^m$	$\checkmark^r$
15	$(Px \wedge \pm Qx) \vee (Qx \wedge \pm Px)$	12; RU	$\{\pm Px\}$		$\checkmark^r$
16	$(Px \wedge \pm Qx) \vee (Qx \wedge \pm Px)$	14; RU	$\{\pm Qx\}$		$\checkmark^r$

Note that the formulas of lines 11 and 13 are neither Dab^0-formulas nor Dab^1-formulas because the first disjunct is a member of Ω^0 but not of Ω^1, whereas the second disjunct is a member of Ω^1 but not of Ω^0. The only Dab^0-consequence of the premises is 8. So $U^0_{16}(\Gamma) = U^0(\Gamma) = \{\pm Px, \pm Qx\}$ and $\Phi^0_{16}(\Gamma) = \Phi^0(\Gamma) = \{\{\pm Px\}, \{\pm Qx\}\}$. All marks for $\mathbf{SG}^r$ are caused by $U^0_{16}(\Gamma)$. Next $U^1_{16}(\Gamma) =$

$U^1(\Gamma) = \emptyset$. This is why, in the $\mathbf{SG}^r$-proof, lines 9 and 10 are unmarked and will remain unmarked in every extension of the proof. $\Phi^0_{16}(\Gamma)$ causes lines 4, 5, 12, and 14 to be marked and lines 6, 7, 15, and 16 to be unmarked. These marks remain unchanged in every extension of the proof. Moreover, $\Phi^1_{16}(\Gamma) = \Phi^1(\Gamma) = \{\{Px \wedge \pm Qx\}, \{Qx \wedge \pm Px\}\}$ and this causes lines 9 and 10 to be marked in the $\mathbf{SG}^m$-proof and to remain so in every extension of it.

That lines 9 and 10 are marked in view of Minimal Abnormality and not in view of Reliability does not contradict Corollary 5.3.3. Indeed, that corollary concerns simple adaptive logics, not combined ones, whereas $\mathbf{SG}^r$ and $\mathbf{SG}^m$ are combined logics. As explained on page 96, the examples illustrates that $Cn_{\mathbf{SG}^r}(\Gamma)$ and $Cn_{\mathbf{SG}^m}(\Gamma)$ are incommensurable. $\mathbf{SG}^m$ has more consequences at degree 0 than $\mathbf{SG}^m$, viz. $\forall x Px \vee \forall x Qx$. The mechanism causing this, also causes lines 15 and 16 to be unmarked—these lines are marked for Reliability. As a result, $\Phi^1_{16}(\Gamma) = \Phi^1(\Gamma) = \{\{Px \wedge \pm Qx\}, \{Qx \wedge \pm Px\}\}$ whereas $U^1_{16}(\Gamma) = U^1(\Gamma) = \emptyset$.

The example also illustrates the way in which a finite premise set restricts the combined logic to a finite combination of simple adaptive logics. Obviously, Γ_{10} is a somewhat extreme case in that only two predicates occur in the data, whence only (non-redundant) generalizations of degree 0 and degree 1 can be considered.

Examples: $\mathbf{CG}^r$ and $\mathbf{CG}^m$

The difference between these logics and $\mathbf{SG}^r$ and $\mathbf{SG}^m$ is that the sets of abnormalities of the combining logics contain the abnormalities from degree zero up to their own degree. In Chapter 3, the sets of abnormalities were given a subscript between parentheses and were defined by $\Omega^{(i)} = \Omega^0 \cup \Omega^1 \cup \ldots \cup \Omega^i$. I shall stick to this notation in order to avoid confusion.

Let us consider again Γ_{10} and the same stage 16 as for $\mathbf{SG}^r$ and $\mathbf{SG}^m$. The marks are drastically different.

1	Pa	Prem	$\emptyset$		
2	Qa	Prem	$\emptyset$		
3	$\neg Pb \vee \neg Qb$	Prem	$\emptyset$		
4	$\forall x Px$	1; RC^0	$\{\pm Px\}$	$\checkmark^m$	$\checkmark^r$
5	$\forall x Qx$	2; RC^0	$\{\pm Qx\}$	$\checkmark^m$	$\checkmark^r$
6	$\forall x Px \vee \forall x Qx$	4; RU	$\{\pm Px\}$		$\checkmark^r$
7	$\forall x Px \vee \forall x Qx$	5; RU	$\{\pm Qx\}$		$\checkmark^r$
8	$\pm Px \vee \pm Qx$	1, 2, 3; RU	$\emptyset$		
9	$\forall x(Px \supset Qx)$	1, 2; RC^1	$\{Px \wedge \pm Qx\}$	$\checkmark^m$	$\checkmark^r$
10	$\forall x(Qx \supset Px)$	1, 2; RC^1	$\{Qx \wedge \pm Px\}$	$\checkmark^m$	$\checkmark^r$
11	$\pm Px \vee (Px \wedge \pm Qx)$	1, 2, 3; RU	$\emptyset$		
12	$Px \wedge \pm Qx$	11; RC^0	$\{\pm Px\}$	$\checkmark^m$	$\checkmark^r$
13	$\pm Qx \vee (Qx \wedge \pm Px)$	1, 2, 3; RU	$\emptyset$		
14	$Qx \wedge \pm Px$	13; RC^0	$\{\pm Qx\}$	$\checkmark^m$	$\checkmark^r$
15	$(Px \wedge \pm Qx) \vee (Qx \wedge \pm Px)$	12; RU	$\{\pm Px\}$		$\checkmark^r$
16	$(Px \wedge \pm Qx) \vee (Qx \wedge \pm Px)$	14; RU	$\{\pm Qx\}$		$\checkmark^r$

The difference in marks is caused by the fact that, for these logics, 11 and 13 are $Dab^{(1)}$-formulas. As in the case of $\mathbf{SG}^r$ and $\mathbf{SG}^m$, $U^0_{16}(\Gamma) = U^0(\Gamma) = \{\pm Px, \pm Qx\}$ and $\Phi^0_{16}(\Gamma) = \Phi^0(\Gamma) = \{\{\pm Px\}, \{\pm Qx\}\}$. This causes lines 6 and 7 to be unmarked for Minimal Abnormality but marked for Reliability. As $U^1_{16}(\Gamma) = U^1(\Gamma) = \{\pm Px, \pm Qx, Px \wedge \pm Qx, Qx \wedge \pm Px\}$, whereas $\Phi^1_{16}(\Gamma) = \Phi^1(\Gamma) = \{\{\pm Px, Qx \wedge \pm Px\}, \{\pm Qx, Px \wedge \pm Qx\}\}$. To see the effect of the unmarked lines 15 and 16 on $\Phi^1_{16}(\Gamma)$, note that $\{\pm Px, \pm Qx\} \in \Phi^1_{14}(\Gamma)$.

All $\mathbf{CG}^r$-consequences that occur in the proof are **CL**-consequences. That there are further $\mathbf{CG}^r$-consequences is shown by the following extension of the proof. I go slowly to arrive at 19 and then take a shortcut for 20.

17	$\forall x(Px \supset Qx) \vee \exists x(Px \wedge \neg Qx)$	RU	$\emptyset$
18	$\forall x(\neg Px \supset Qx) \vee \exists x(\neg Px \wedge \neg Qx)$	RU	$\emptyset$
19	$\forall x(Px \supset Qx) \vee \forall x(\neg Px \supset Qx)$	17, 18; RC	$\{\neg Qx \wedge \pm Px\}$
20	$\forall x(Qx \supset Px) \vee \forall x(\neg Qx \supset Px)$	RC	$\{\neg Px \wedge \pm Qx\}$

Note that 19 and 20 are already $\mathbf{G}^r$-consequences of Γ_{10}.

Examples: Handling Degrees of Plausibility

In Section 3.6 we have seen the logic $\mathbf{K}^r$ at work and $\mathbf{K}^m$ was mentioned. These are combined logics obtained by the sequential superposition of an infinity of simple adaptive logics $\mathbf{K}^x_i$, with $x \in \{r, m\}$—with apologies for the notational abuse. The latter are defined by the lower limit logic **K** (a bi-modal predicative version of this modal logic, semantically defined in Section 3.6), the set of abnormalities $\Omega^i = \{\exists(\Diamond^i A \wedge \neg A) \mid A \in \mathcal{A}\} \cup \{\exists(\diamond^i B \wedge \neg B) \mid B \in \mathcal{W}_s\}$, and the strategy Reliability for $\mathbf{K}^r_i$ and minimal abnormality for $\mathbf{K}^m_i$.

I mentioned in Section 3.6 that I evaded a problem—I supposed that all background entities had a different degrees of plausibility—and promised to consider the general case in this section. Remember that $\Diamond^i$ was used to express the plausibility of background entities that are handled pragmatically, whereas $\diamond^i$ was used to express the plausibility of background entities that are handled strictly, which means that these are rejected entirely when they contradict the data.

So let us look into the problem arising when different background entities have the same degree of plausibility. The problem only lies with background entities that are handled strictly. Background entities that are handled pragmatically may obviously lead to mixed *Dab*-formulas, but this is as expected and as desired. Suppose, for example, that T_1 and T_2 have both plausibility i. The date may contradict one of the theories, or both of them, or may jointly contradict them. So, while all consequences of T_1 and T_2 are still plausible to at least degree i, some consequences of T_1 or T_2 (or both) will not be considered as true. If some consequences of T_1 are plausible but cannot in themselves be considered as true, this will have no effect the consequences of T_2.

The trouble resides with background entities that are handled strictly. Suppose that both T_1 and T_2 have both plausibility i and are handled strictly, and that the data contradict T_1. So a Dab^i-formula is **K**-derivable from the premise set. By the reasoning starting with equation 3.7 at page 102, it follows that line that has a condition $\Delta \subset \Omega^i$ will be marked for Reliability as well as for Minimal Abnormality. Whether Δ contains abnormalities of of the form $\Diamond^i A \wedge \neg A$ or of the form $\diamond^i A \wedge \neg A$ or of both is immaterial. This is of course utterly wrong. That T_1 is falsified by the data does obviously not justify that T_2 is also rejected. Incidentally, the problem cannot be removed by moving from a bi-modal version of **K** to a multi-modal version that has a countable infinity of different diamonds $\diamond_1$, $\diamond_2$, and so on—the idea would be to assign a different diamond to every background entity. This move does not resolve the problem. If a Dab^i-formula for one $\diamond_j$ is derivable, all lines that are derived on a condition $\Delta \subset \Omega^i$ will still be marked. I return to this problem in Section 6.2.4.

By defining, for each $i \in \{1, 2, \ldots\}$, $\Omega^{(i)} = \Omega^1 \cup \ldots \cup \Omega^i$, obvious variants for $\mathbf{K}^r$ and $\mathbf{K}^m$ are devised. They are just like the previous logics, except that the set of abnormalities of composing logics are $\Omega^{(1)}$, $\Omega^{(2)}$, and so on.

Another way to realize a similar effect is by keeping the the original sets of abnormalities Ω^i and replacing the modal logic $\mathbf{K}$ by $\mathbf{T}$. The logic $\mathbf{T}$ is semantically defined by the semantics of $\mathbf{K}$ from Section 3.6, except that the relation R is required to be reflexive (for all $w \in W$, Rww). The reflexivity of R warrants that $\Diamond A \vdash_{\mathbf{T}} \Diamond\Diamond A$ (and obviously also $\diamond A \vdash_{\mathbf{T}} \diamond \diamond A$). The resulting combined adaptive logics $\mathbf{T}^r$ and $\mathbf{T}^m$ are applied to diagnosis (in their mono-modal form) in [BMPV03].

There are very different ways to express degrees of plausibility. Some very instructive ones proceed in terms of a logic containing a paraconsistent negation.[13] All known (and minimally adequate) ones require classical negation as well as the **CLuN**-negation in the consistent case—the inconsistent case is commented upon below. I shall briefly present one of the possible approaches here.

Let us extend the language with the symbol $\sim$ for classical negation. The reader may wonder why I do not use $\check{\neg}$ for that purpose. First of all, $\check{\neg}$ is superimposed on $\mathcal{L}_s$ and the present context requires an intertwined classical negation. Moreover, it is handy to still have $\check{\neg}$ around with its specific function. Premises that are considered as certain, and not just as preferred or plausible, will be expressed in terms of the classical negation. So the negation $\neg$ will not occur in them. Let $\mathcal{W}^{\not\neg}$ denote the set of formulas in which $\neg$ does not occur (but $\sim$ may occur). Premises that express a plausibility will consist of a formula of $\mathcal{W}^{\not\neg}$ preceded by a sequence of symbols, depending on the degree of plausibility.

Let us now move to the combined logic, which is astoundingly simple. It uses a double paraconsistent negation to express plausibility. The idea is that $\neg\neg$ is read as "it is plausible that", $\neg\neg\neg\neg$ as "it is plausible that it is plausible that", which is a weaker plausibility, and so on. To keep things readable, one may write $\neg^i$ instead of an iteration of negation signs. Remember that $\neg\neg A \nvdash_{\mathbf{CLuN}} A$. So the double negation may be used to express plausibility.

As we want a combined logic, we need a sequence of sets of abnormalities. So we define, for every $i \in \{1, 2, \ldots\}$, $\Omega^i = \{\neg^{2i} A \wedge {\sim} A \mid A \in \mathcal{W}^{\not\neg}\}$. We then define a sequence of simple adaptive logics, which we shall call $\mathbf{P}^x_i$ (with $i \in \{1, 2, \ldots\}$ and with $x \in \{r, m\}$ according as the strategy is Reliability or Minimal Abnormality). Every $\mathbf{P}^x_i$ is defined by the lower limit logic **CLuN** (adjusted to handle the intermingled $\sim$), the set of abnormalities Ω^i, and Reliability or Minimal Abnormality. The logics $\mathbf{P}^x_i$ are then combined along the recipe of (6.1). Keep in mind that even $\mathbf{P}^x_1$ is closed under the lower limit logic **CLuN**. This was adjusted for the classical negation $\sim$. So the certain consequences (those that do not have $\neg$ as their first symbol) are closed under **CL** and all derived formulas that belong to $\mathcal{W}^{\not\neg}$ are equally closed under **CL**. Let the

[13] Many paraconsistent logics are closely related to modal logics; see, for example, [Bat02b, Béz02, Béz06]. In [Odi03] Sergei Odintsov points to a similarity between Łukasiewicz's modalities to a contradiction operator. This is helpful to understand the relation.

resulting combined logic be called $\mathbf{P}^m$ if Minimal Abnormality is the strategy and $\mathbf{P}^r$ if Reliability is the strategy.

To make the matter more concrete, I present an extremely simple example of a $\mathbf{P}^m$-proof—the $\mathbf{P}^r$-proof is identical. Let $\Gamma_4 = \{p \supset q, \neg^4 p, \neg^2{\sim}q\}$.

1	$p \supset q$	Prem	$\emptyset$	
2	$\neg^4 p$	Prem	$\emptyset$	
3	$\neg^2{\sim}q$	Prem	$\emptyset$	
4	p	2; RC^2	$\{\neg^4 p \wedge {\sim}p\}$	$\checkmark^8$
5	q	1, 4; RU	$\{\neg^4 p \wedge {\sim}p\}$	$\checkmark^8$
6	${\sim}q$	3; RC^1	$\{\neg^2{\sim}q \wedge {\sim}{\sim}q\}$	
7	${\sim}p$	1, 6; RU	$\{\neg^2{\sim}q \wedge {\sim}{\sim}q\}$	
8	$\neg^4 p \wedge {\sim}p$	2, 7; RU	$\{\neg^2{\sim}q \wedge {\sim}{\sim}q\}$	

Note that $\neg^4 p \wedge {\sim}p$ is a minimal Dab^2 formula, that $U^1(\Gamma) = U^1_s(\Gamma) = \emptyset$, $\Phi^1(\Gamma) = \Phi^1_s(\Gamma) = \{\emptyset\}$, $U^2(\Gamma) = U^2_s(\Gamma) = \{\neg^4 p \wedge {\sim}p\}$, and $\Phi^2(\Gamma) = \Phi^2_s(\Gamma) = \{\{\neg^4 p \wedge {\sim}p\}\}$. So ${\sim}p$ and ${\sim}q$ are finally derived from Γ_4 by the combined logic. Whether the strategy is Reliability or minimal Abnormality makes no difference in this case.

Nothing changes to this proof if sets of abnormalities are defined in such a way that they contain their predecessors ($\Omega^i \subset \Omega^{i+1}$ for all $i \in \{1, 2, \ldots\}$). A different way to realize a similar result is by extending the lower limit **CLuN** with the axiom schema $A \supset \neg\neg A$. Note that this is natural. What is certain is also plausible, and what is plausible is also plausibly plausible.

Instead of expressing plausibility by $\neg\neg$, one may use $\neg{\sim}$ for this purpose—a line I shall follow in Section 9.7. If one reads the classical negation as "is false" and the paraconsistent one as "not", that A is plausible is then expressed by stating that it is not false (in the paraconsistent sense of "not", so that A's not being false does not exclude its being false nevertheless).

Let us now have a look at the paraconsistent case. That the premises expressing a plausibility are inconsistent causes no problem for $\mathbf{P}^m$ and $\mathbf{P}^r$. There also is no problem if A and $\neg A$ have the same plausibility. This is easily seen by considering a premise set that contains $\neg\neg p$ as well as $\neg\neg{\sim}p$. From these, $(\neg\neg p \wedge {\sim}p) \check{\vee} (\neg\neg{\sim}p \wedge {\sim}{\sim}p)$ is **CLuN**-derivable. If neither p nor ${\sim}p$ is **CLuN**-derivable from the certain premises, $\neg\neg p$ and $\neg\neg{\sim}p$ annihilate each other's effect on the consequence set. If p is **CLuN**-derivable from the certain premises, this causes an abnormality with $\neg\neg{\sim}p$, whereas $\neg\neg p$ is then a redundant premise. The opposite situation obtains if ${\sim}p$ is derivable from the certain premises.

A completely different situation arises when the certain premises form an inconsistent set. The previous construction is clearly not sensible in this case because it causes triviality. Still, it is obvious in which way the construction may be modified. We keep everything as it is, except that we define the new negation ${\sim}$ as a *paraconsistent* one. In other words, we give ${\sim}$ the same meaning as $\neg$, but use it for a different purpose. The premise set will be closed under this version of **CLuN**. This means that all **CLuN**-consequences of the certain premises are derivable from them and have precedence over the statements expressing plausibility. The premise $\neg^{2i} A$ (with $A \in \mathcal{W}^{\not\sim}$) will cause A to be added to the consequence set in as far as this causes no contradiction with the **CLuN**-consequences of the certain premises. To see what I mean consider the premise set $\{p, \neg\neg{\sim}p\}$, in which both negations have the same meaning, viz. the meaning of the **CLuN**-negation. Although **CLuN** does not prevent p and $\sim p$ from being both true, the combined logic will still prevent $\sim p$ from being derivable.

And yet, the situation is far from optimal. Consider again the premise set Γ_4. On the present version of **CLuN**, $q \nvdash_{\mathbf{CLuN}} \sim\sim q$ and $p \supset q, \sim q \nvdash_{\mathbf{CLuN}} \sim p$. Moreover, $\Gamma_4 \nvdash_{\mathbf{CLuN}} (\neg^4 p \wedge \sim p) \check{\vee} (\neg^2 \sim q \wedge \sim\sim q)$. So lines 7–8 have to be removed from the proof, whence lines 4, 5, and 6 are unmarked. The upshot is that the consequence set is more $\sim$-inconsistent than the premise set.

What is going on here? Whether $\sim$ is the negation of **CL** or of **CLuN**, $\Gamma_4 \vdash_{\mathbf{CLuN}} q \check{\vee} (\neg^4 p \wedge \sim p)$—compare this to line 5—and $\Gamma_4 \vdash_{\mathbf{CLuN}} \sim q \check{\vee} (\neg^2 \sim q \wedge \sim\sim q)$—compare this to line 6. Note that I am talking about two variants of **CLuN**, one extended with the symbol $\sim$ denoting classical negation and one extended with the symbol $\sim$ denoting the **CLuN**-negation. The two derivability statements hold for both variants and so does $\Gamma_4 \vdash_{\mathbf{CLuN}} (q \wedge \sim q) \check{\vee} (\neg^4 p \wedge \sim p) \check{\vee} (\neg^2 \sim q \wedge \sim\sim q)$. If $\sim$ is classical negation $(q \wedge \sim q) \check{\vee} (\neg^4 p \wedge \sim p) \check{\vee} (\neg^2 \sim q \wedge \sim\sim q)$ is equivalent to $(\neg^4 p \wedge \sim p) \check{\vee} (\neg^2 \sim q \wedge \sim\sim q)$. If $\sim$ is the **CLuN**-negation, these are not equivalent. So although the **CLuN**-consequences of the *certainties* prevent their $\sim$-negations from being $\mathbf{P}^m$-derivable, not all consequences of the plausible statements prevent their negations from being $\mathbf{P}^m$-derivable.[14]

For most if not all purposes, one will want to require the following. Where Γ is a premise set containing formulas that express plausibility, the consequence set should not be more $\sim$-inconsistent than Γ. The logic we are considering is inadequate in view of this requirement. A logic that fulfils this requirement is readily available. It is obtained by first closing the premise set under $\mathbf{CLuN}^r$ or $\mathbf{CLuN}^m$, and only then start the sequential superposition of the logics handling plausibility. So, if the chosen strategy is Minimal Abnormality, the innermost logic in (6.1) is $\mathbf{CLuN}^m$, the next one the simple adaptive logic handling $\neg^2$, the next one the simple adaptive logic handling $\neg^4$, and so on. Call the resulting logics $\mathbf{PP}^m$ and $\mathbf{PP}^r$—the first "P" refers to the paraconsistent context.

Let us apply these logics to Γ_4. Since this is a consistent premise set, the result should be the same as before, viz. should be the same as for $\mathbf{P}^m$ and $\mathbf{P}^r$. Here is what becomes of the previous proof, followed by a bit of explanation. I write "RC" for the conditional rule of $\mathbf{CLuN}^m$ as the other conditional rules have a superscript.

1	$p \supset q$	Prem	$\emptyset$	
2	$\neg^4 p$	Prem	$\emptyset$	
3	$\neg^2 \sim q$	Prem	$\emptyset$	
4	p	2; RC^2	$\{\neg^4 p \wedge \sim p\}$	✓^{9}
5	q	1, 4; RU	$\{\neg^4 p \wedge \sim p\}$	✓^{9}
6	$\sim q$	3; RC^1	$\{\neg^2 \sim q \wedge \sim\sim q\}$	
7	$\sim q \supset \sim p$	1; RC	$\{q \wedge \sim q\}$	
8	$\sim p$	6, 7; RU	$\{\neg^2 \sim q \wedge \sim\sim q, q \wedge \sim q\}$	
9	$\neg^4 p \wedge \sim p$	2, 8; RU	$\{\neg^2 \sim q \wedge \sim\sim q, q \wedge \sim q\}$	

So the central difference with $\mathbf{P}^m$ and $\mathbf{P}^r$ is that $q \wedge \sim q$ was a logical falsehood of the involved variant of **CLuN** there, whereas it is logically contingent with respect to the variant of **CLuN** that is the lower limit of $\mathbf{PP}^m$ and $\mathbf{PP}^r$. So we

[14] Some do, however. If both $\neg^{2i} A$ and $\neg^{2j} \sim A$ are premises, then the **CLuN**-theorem $\sim A \check{\vee} A$ will cause $(\neg^{2i} A \wedge \sim A) \check{\vee} (\neg^{2j} \sim A \wedge A)$ to be a **LLL**-consequence of the premise set. So if $i < j$, $\neg^{2j} \sim A \wedge A$ is derivable on the condition $\{\neg^{2i} A \wedge \sim A\}$.

have to rely on the falsehood of $q \wedge \sim q$ in order to obtain $\sim p$. I mentioned before that, on the present version of **CLuN**, with $\sim$ paraconsistent, $\Gamma_4 \nvdash_{\mathbf{CLuN}} (\neg^4 p \wedge \sim p) \check{\vee} (\neg^2 \sim q \wedge \sim\sim q)$. However, $\Gamma_4 \vdash_{\mathbf{CLuN}} (\neg^4 p \wedge \sim p) \check{\vee} (\neg^2 \sim q \wedge \sim\sim q) \check{\vee} q \wedge \sim q$. Given the way in which the simple adaptive logics are combined and given that Γ_4 is $\sim$-consistent, this leads to $\neg^4 p \wedge \sim p$ being a minimal Dab^4-formula, as is witnessed by line 8 of the last proof.

As we covered quite some ground on degrees of plausibility, let me summarize. I first presented a simple solution that applies only to premise sets with consistent certainties. This proceeds in terms of the combined adaptive logics $\mathbf{P}^m$ and $\mathbf{P}^r$, in which the meaning of the new negation $\sim$ is that of the **CL**-negation. Next (and after a digression about an inadequate solution), I presented a solution for the general case, in which the premise set may contain inconsistent certainties. This was provided by the combined adaptive logics $\mathbf{PP}^m$ and $\mathbf{PP}^r$.

Before moving to the general (possibly inconsistent) case, I mentioned that the approach in terms of $\neg\neg$ may be replaced by an approach in terms of $\neg\sim$. For the general case, this approach has nothing new to offer because $\neg$ and $\sim$ have the same meaning in $\mathbf{PP}^m$ and $\mathbf{PP}^r$.

Before leaving the matter, it is useful to add a comment on the application of these logics and a connected comment on notation. Sometimes one wants to add plausibility statements (or preferences) to an adaptive logic that it has **CL** as its lower limit, for example a logic of inductive generalization. In such cases, it is more handy to opt for $\mathbf{P}^m$ and $\mathbf{P}^r$. As $\neg$ is the standard negation in such cases, one better uses $\sim$ for the paraconsistent negation. In other words, plausibility is then better expressed by $\sim\sim$. Needless to say, it is also possible to reformulate the logics of inductive generalization in such a way that the premises are first interpreted by an inconsistency-adaptive logic, say $\mathbf{CLuN}^m$, and to choose **CLuN** as the lower limit of the selected logic of inductive generalization. If the premises are consistent, the outcome will be identical to the one obtained by the logics presented in Chapter 3—this is a general point that deserves to be repeated. Even with this reformulation, one obviously expresses plausibility by $\sim\sim$ because $\neg$ is then already used as the standard paraconsistent negation.

Example: Preferential selection of abnormalities

This section concerns the mechanism that was described in Section 3.7 under the heading "Conjectures". The mechanism may be combined with all adaptive logics. When a *Dab*-formula that counts more than one disjunct is derived, one may have *extra-logical* reasons to consider some of the disjuncts as false, whence a shorter *Dab*-formula becomes derivable. The extra-logical reasons are introduced into the proof by new premises. Needless to say, the effect of new premises should be defeasible. For example the new premise may have the form $\Diamond A$. Its normal effect will be that A is derivable, but the premise set may block the normal effect.

The preferential selection, or de-selection, of abnormalities may proceed in terms of any combined adaptive logic that is able to handle plausibility statements. We came across several such systems, proceeding in terms of a sequence of diamonds or a sequence of double negations. Here I shall consider the approach from [Bat06b], which may be combined with $\mathbf{CLuN}^m$ and similar inconsistency-adaptive logics that have a weak lower limit logic.

The approach illustrates several new features. The first is a still other way in which a paraconsistent negation may be invoked to express plausibility. The matter is not only technical. It also demonstrates once more how ill-advised, to say it politely, was the strong resistance of classical logicians with respect to paraconsistency. Another new feature is that more complex expressions will correspond to a higher plausibility—they correspond to lower plausibility in all logics we met so far. The approach also does not require the introduction of any new logical symbol—more new features will turn out to be involved.

Let us first have a look at the underlying idea. Suppose that, in a **CLuN**^{r}-proof, one derived

$$(p \wedge \neg p) \check{\vee} (q \wedge \neg q) \tag{6.2}$$

and that this actually is a minimal *Dab*-formula at the stage. Suppose that one introduces $\neg(q \wedge \neg q)$ as a new premise. The formula (6.2) states that either $p \wedge \neg p$ or $q \wedge \neg q$ is true but does not specify which of them is true. The new premise moreover denies that $q \wedge \neg q$ is true. Can this be taken as a reason to consider $p \wedge \neg p$ as 'the real problem'? In other words, is it possible to devise a logic which leads to the defeasible conclusion that the shorter *Dab*-formula, $p \wedge \neg p$, is the real problem? If this question is answered in the positive, one is able to express the defeasible denial of an abnormality within the standard language. Unlikely as it may appear—it appeared impossible to me when I first tried out the approach—this approach is feasible.

Inconsistency-adaptive logics like **CLuN**^{r} *isolate* contradictions. For example, $p \wedge \neg p$ is not derivable from q and $\neg(p \vee q)$. The formula

$$(p \wedge \neg p) \check{\vee} ((q \wedge \neg q) \wedge \neg(q \wedge \neg q)) \tag{6.3}$$

is **CLuN**-derivable from (6.2) together with the new premise $\neg(q \wedge \neg q)$, and (6.2) is **CLuN**-derivable from (6.3). In other words, to consider the new premise as a good reason to defeasibly derive $p \wedge \neg p$ from (6.2) comes to considering contradictions of contradictions as more likely false than simple contradictions. But is this justifiable? What is so special about $\neg(A \wedge \neg A)$, or about $(A \wedge \neg A) \wedge \neg(A \wedge \neg A)$?

There is an answer to these questions, and it is convincing with respect to the intended applications. The original theory was meant as consistent and **CL** was taken as its underlying logic. But the negation of a contradiction is a theorem of **CL**. So it makes no sense that it occurs in the original premises. Put differently, if one goes out of one's way to affirm the negation of a contradiction, then one is affirming something special, in our case, the fact that the contradiction is false—defeasibly false: unless and until proven otherwise. Some people might see trouble coming from the fact that $\neg(A \wedge \neg A)$ is **CLuN**^{r}-derivable from Γ whenever $A \wedge \neg A \notin U(\Gamma)$.[15] However, this reinforces my point rather than weakening it. Indeed, it shows that, even in the context of **CLuN**^{r}, it does not make sense to introduce $\neg(A \wedge \neg A)$ as a new premise unless $A \wedge \neg A$ is a disjunct of a minimal *Dab*-formula of the proof. Given all this, $\neg(A \wedge \neg A)$ can sensibly be taken as a defeasible rejection of $A \wedge \neg A$.

The application context requires that we reject abnormalities in a defeasible way, viz. with a certain plausibility only. The idea was to consider a contradic-

[15] Indeed, as $(A \wedge \neg A) \check{\vee} \neg(A \wedge \neg A)$ is a **CLuN**-theorem, $\neg(A \wedge \neg A)$ is derivable in any proof from any Γ on the condition $\{A \wedge \neg A\}$. So if $A \wedge \neg A \notin U(\Gamma)$, then $\neg(A \wedge \neg A)$ is finally **CLuN**^{r}-derivable from Γ.

tion of contradictions as more likely false than a simple contradiction. So we defeasibly reject $q \wedge \neg q$ by adding the new premise $\neg(q \wedge \neg q)$, which caused (6.3) to be derivable. In view of this, a contradiction of a contradiction of contradictions, such as $((q \wedge \neg q) \wedge \neg(q \wedge \neg q)) \wedge \neg((q \wedge \neg q) \wedge \neg(q \wedge \neg q))$, should be taken as even more likely false and the new premise $\neg((q \wedge \neg q) \wedge \neg(q \wedge \neg q))$ expresses a stronger rejection of $q \wedge \neg q$ than $\neg(q \wedge \neg q)$. Put differently, it expresses that $q \wedge \neg q$ is rejected with a *higher* plausibility. At face value, however, it does not. Suppose that (6.2) is derived from the premises, and that $\neg((q \wedge \neg q) \wedge \neg(q \wedge \neg q))$ is the *only* new premise. Apparently this premise cannot be assigned a sensible effect on the derivability of disjunctions of contradictions. All that would follow is

$$(p \wedge \neg p) \check{\vee} ((q \wedge \neg q) \wedge \neg((q \wedge \neg q) \wedge \neg(q \wedge \neg q)))$$

and the second disjunct is not even a contradiction. Moreover, apart from (6.2) itself, no *Dab*-formula is derivable from (6.2) together with *this* new premise.

There is a way out. Although priorities cannot be expressed by negations of contradictions

$$\neg(A \wedge \neg A),\ \neg((A \wedge \neg A) \wedge \neg(A \wedge \neg A)),\ \ldots$$

they can be expressed by conjunctions of negations of contradictions thus:

$$\neg(A \wedge \neg A),\ \neg(A \wedge \neg A) \wedge \neg((A \wedge \neg A) \wedge \neg(A \wedge \neg A)), \ldots$$

Indeed, if (6.2) is derived from the premises, then $\neg(q \wedge \neg q) \wedge \neg((q \wedge \neg q) \wedge \neg(q \wedge \neg q))$ is sufficient to put the blame on $p \wedge \neg p$ because it warrants the derivability of

$$(p \wedge \neg p) \vee (((q \wedge \neg q) \wedge \neg(q \wedge \neg q)) \wedge \neg((q \wedge \neg q) \wedge \neg(q \wedge \neg q)))\,.$$

Even if $\neg(p \wedge \neg p)$ were also added as a new premise, it would still follow by **CLuN** that

$$((p \wedge \neg p) \wedge \neg(p \wedge \neg p)) \vee (((q \wedge \neg q) \wedge \neg(q \wedge \neg q)) \wedge \neg((q \wedge \neg q) \wedge \neg(q \wedge \neg q)))$$

and this would be sufficient to eliminate the second disjunct as the 'more complex' contradiction, whence the blame would still be on $p \wedge \neg p$.

Lest the formulas run out of the margins, let $!A$ abbreviate $\exists(A \wedge \neg A)$, whence $!!A$ abbreviates $\exists(A \wedge \neg A) \wedge \neg\exists(A \wedge \neg A)$, etc.[16] Next let $!^i A$ abbreviate whatever is abbreviated by i exclamation marks followed by A. Finally, let $¡^i A$ abbreviate $\neg !^1 A \wedge \neg !^2 A \wedge \ldots \wedge \neg !^i A$.[17] The *complexity* of a contradiction $!A$ will be said to be i if $!A$ is identical to $!^i B$ in which B is not a contradiction.

So we arrived at an approach that is justifiable. Introducing the new premise $¡^1 A$ is a sensible way of rejecting $!A$ if $!A$ is a disjunct of a minimal *Dab*-formula at the stage of the proof. Now consider a minimal *Dab*-formula $Dab(\Delta)$ of a proof at a stage. It is obvious that from $Dab(\Delta)$ a *Dab*-formula $Dab(\Theta)$ is **CLuN**-derivable, possibly $Dab(\Delta)$ itself, of which each disjunct (each member of Θ) is a contradiction of complexity 1. If $!A \in \Theta$,[18] stating $¡^1 A$ comes to

[16] As $!A$ is a closed formula, $!!A$ does not start with an existential quantifier.

[17] As I pointed out in [Bat09], the structure of these formulas are strikingly similar to the consistency statements in Newton da Costa's $\mathbf{C}_n$-logics. This strengthens the idea that Newton was close to inventing inconsistency-adaptive logics.

[18] To keep things simple, suppose moreover that the proof does not contain another minimal *Dab*-formula of which $!A$ is a disjunct.

denying $!A$ with plausibility 1. This means that $!A$ is (defeasibly) stated to be false unless all other members of Θ are also denied to degree 1—that is, unless $Dab(\{!^2B \mid B \in \Theta\})$ is **CLuN**-derivable from the present premises. Stating ¡^2A comes to denying $!A$ with plausibility 2; this means that $!A$ is (defeasibly) stated to be false unless all other members of Θ are also denied with plausibility 2—that is, unless $Dab(\{!^3B \mid B \in \Theta\})$ is **CLuN**-derivable from the present premises. And so on.[19]

This approach has an advantage over approaches that express the highest plausibility by the least complex formulas. Suppose that one introduces the new premise $\Diamond^1\neg(A \wedge \neg A)$ and that one later wants to introduce a new premise to deny $B\wedge\neg B$ even more strongly than $B\wedge\neg B$. One then has to introduce as a new premise, for example, $\Diamond^1\neg(B \wedge \neg B)$, to *replace* $\Diamond^1\neg(A \wedge \neg A)$ by $\Diamond^2\neg(A \wedge \neg A)$, and to revise the proof in view of the replacement. Such a revision is never required by the present approach. It is always possible to revise one's views by *adding* further premises. So while adaptive logics are non-monotonic, the logics we are after here are 'monotonic with respect to premises'.

Let us call these logics $\mathbf{PCLuN}^r$ and $\mathbf{PCLuN}^m$. They are sequential superpositions of simple adaptive logics, which I shall call $\mathbf{PCLuN}^r_i$ and $\mathbf{PCLuN}^m_i$ after their strategy. The combining logics are as expected. Their lower limit logic be **CLuN**. Their strategies Reliability or Minimal Abnormality. What is special is their sets of abnormalities: Ω^1, Ω^2, ... They are defined by $\Omega^i = \{!^iA \mid A \in \mathcal{F}\}$. Note that Ω^1 is just the set of abnormalities of $\mathbf{CLuN}^r$ and $\mathbf{CLuN}^m$, which are here renamed to $\mathbf{CLuN}^r_1$ and $\mathbf{CLuN}^m_1$.

Several combining logics in the section had the property that $\Omega^i \subset \Omega^{i+1}$ for all $i \in \{1, 2, \ldots\}$. For $\mathbf{PCLuN}^r$ or $\mathbf{PCLuN}^m$, this relation is reversed: $\Omega^{i+1} \subset \Omega^i$ for all $i \in \{1, 2, \ldots\}$. So $\Omega^1 \supset \Omega^2 \supset \ldots$. This connected to the discussed advantage that the highest plausibility is expressed by the least complex formulas. Alas, this advantage is mirrored by a disadvantage. First of all, (6.1) has to be reformulated as follows.

$$Cn^{\mathcal{L}}_{\mathbf{C}}(\Gamma) = \mathcal{W} \cap (Cn^{\mathcal{L}_+}_{\mathbf{AL}1}(Cn^{\mathcal{L}_+}_{\mathbf{AL}2}(Cn^{\mathcal{L}_+}_{\mathbf{AL}1}(Cn^{\mathcal{L}_+}_{\mathbf{AL}3}(\ldots(\Gamma)\ldots))))) \tag{6.4}$$

This clearly does not look nice, but it is nevertheless clear what is meant. There is, however, a worse problem. Some (infinite and decidable) premise sets cannot sensibly be approached in terms of dynamic proofs.

Nevertheless, the applications of (6.4) that are intended in the present context proceed as exactly desired. The idea is to start from a premise set that was intended as consistent and does not contain any statement expressing a plausibility. As explained before, such premise sets do not contain formulas of the form $!^iA$ because these are **CL**-tautologies. Denials of contradictions (of any degree) are added in view of present insights, viz. in view of minimal *Dab*-formulas that occur in the proof at a stage. So there is never a need to add new premises ¡^nA for more than a finite number of $n \in \mathbb{N}$. As, for the intended applications, there will be a maximal n, (6.4) may be rephrased as follows.

$$Cn^{\mathcal{L}}_{\mathbf{C}}(\Gamma) = \mathcal{W} \cap (Cn^{\mathcal{L}_+}_{\mathbf{AL}1}(Cn^{\mathcal{L}_+}_{\mathbf{AL}2}(Cn^{\mathcal{L}_+}_{\mathbf{AL}1}(Cn^{\mathcal{L}_+}_{\mathbf{AL}3}(\ldots(Cn^{\mathcal{L}_+}_{\mathbf{AL}n}(\Gamma))\ldots))))) \tag{6.5}$$

[19]The "defeasibly" is between parentheses because it has nothing to do with the plausibility with which the other disjuncts are denied, but rather with the question whether the minimal *Dab*-formula is a minimal *Dab*-consequence of the premises. If, for example, $!A$ is derived at a later stage of the proof, $!A$ is a minimal *Dab*-formula and the only effect of the new premise ¡A is that $!!^2A$ is also a minimal *Dab*-formula.

This means that even the dynamic proofs are simple. The rules are as usual and so is the marking definition, except that marking starts with n and moves down to 1.

To exemplify the dynamic proofs, consider a simple propositional premise set: $\Gamma_5 = \{p, \neg q, t, r \supset q, \neg p \vee s, \neg p \vee q, \neg t \vee u\}$. New premises will be introduced as a reaction to *Dab*-formulas on the basis of non-logical grounds, which are not discussed here. The logic is $\mathbf{PCLuN}^r$ or $\mathbf{PCLuN}^m$.

1	p	Prem	$\emptyset$	
2	$\neg q$	Prem	$\emptyset$	
3	t	Prem	$\emptyset$	
4	$r \supset q$	Prem	$\emptyset$	
5	$\neg p \vee s$	Prem	$\emptyset$	
6	$\neg p \vee q$	Prem	$\emptyset$	
7	$\neg t \vee u$	Prem	$\emptyset$	
8	$\neg r$	2, 4; RC	$\{!^1 q\}$	$\checkmark^{20}$
9	s	1, 5; RC	$\{!^1 p\}$	
10	u	3, 7; RC	$\{!^1 t\}$	
11	$(!^1 p) \vee (!^1 q)$	1, 2, 6; RU	$\emptyset$	
12	$¡^1 p$	Prem	$\emptyset$	
13	$(!^2 p) \vee (!^1 q)$	11, 12; RU	$\emptyset$	
14	$!^1 q$	13; RC	$\{!^2 p\}$	
15	$¡^1 q$	Prem	$\emptyset$	
16	$(!^2 p) \vee (!^2 q)$	13, 15; RU	$\emptyset$	
17	$¡^2 p$	Prem	$\emptyset$	
18	$(!^3 p) \vee (!^2 q)$	16, 17; RU	$\emptyset$	
19	$!^2 q$	18; RC	$\{!^3 p\}$	
20	$!^1 q$	19; RU	$\{!^3 p\}$	

The proof proceeds very slowly. The marks at stage 20 of the proof are exactly as they would be if plausibility were expressed by the $\Diamond$ of **T**. The reader may easily check that the same holds for previous stages—run through the stages of the proof from stage 11 on.

Metatheoretic Problems

Not all sequential superpositions have the fixed point property and there is a more severe problem as well. An abstract example is sufficient to illustrate the lack of the fixed point property. Let $\Omega^1 = \{A^1_1, A^1_2, \ldots\}$ be the set of abnormalities of $\mathbf{AL}1^r$ and let $\Omega^2 = \{A^2_1, A^2_2, \ldots\}$ the set of abnormalities of $\mathbf{AL}2^r$, with $A^1_i \neq A^2_j$ for all $i, j \in \{1, 2, \ldots\}$. Let $Cn_{\mathbf{C}^r}(\Gamma) = Cn_{\mathbf{AL}2^r}(Cn_{\mathbf{AL}1^r}(\Gamma))$ and consider the premise set $\Gamma_6 = \{A^1_1 \vee A^1_2, A^1_1 \vee A^2_1, B \vee A^1_2\}$, supposing that disjunction is classical and that B is different from every A^i_j.

1	$A^1_1 \vee A^1_2$	Prem	$\emptyset$	
2	$A^1_1 \vee A^2_1$	Prem	$\emptyset$	
3	$B \vee A^1_2$	Prem	$\emptyset$	
4	B	3; RC^1	$\{A^1_2\}$	$\checkmark^1$
5	A^1_1	2; RC^2	$\{A^2_1\}$	

Obviously $\Phi^1(\Gamma_6) = \{\{A^1_1\}, \{A^1_2\}\}$ and $\Phi^2(\Gamma_6) = \{\emptyset\}$. So B is not a $\mathbf{C}^r$-consequence of Γ_6, whereas A^1_1 is. This, however, means that $\mathbf{C}^r$ is not a fixed point. Indeed, if one applies $\mathbf{C}^r$ to $Cn_{\mathbf{C}^r}(\Gamma_6)$, $\Phi^1(Cn_{\mathbf{C}^r}(\Gamma_6)) = \{\{A_1^1\}\}$, whence $B \in Cn_{\mathbf{C}^r}(Cn_{\mathbf{C}^r}(\Gamma_6))$.

The proof exemplifies the problem for Reliability, but not for Minimal Abnormality. The reason is that the Deduction Theorem 5.6.8 holds for Minimal Abnormality. If a formula B is a $\mathbf{AL}i^m$-consequence of a premise set Γ together with a finite set of formulas $\{C_1, \ldots, C_n\}$, then $C_1 \check{\supset} (\ldots \check{\supset} (C_n \check{\supset} B) \ldots)$ is a $\mathbf{AL}i^m$-consequence of Γ. So whenever all members of $\{C_1, \ldots, C_n\}$ are finally derived by a 'higher numbered' logic, then so is B.

Let me illustrate this for the premise set Γ_6, the combined logic now being defined by $Cn_{\mathbf{C}^m}(\Gamma) = Cn_{\mathbf{AL2}^m}(Cn_{\mathbf{AL1}^m}(\Gamma))$. I proceed slowly to make the proof self-explanatory.

1	$A_1^1 \vee A_2^1$	Prem	$\emptyset$	
2	$A_1^1 \vee A_1^2$	Prem	$\emptyset$	
3	$B \vee A_2^1$	Prem	$\emptyset$	
4	B	3; RC^1	$\{A_2^1\}$	$\checkmark^1$
5	A_1^1	2; RC^2	$\{A_1^2\}$	
6	$A_1^1 \check{\supset} B$	4; RU	$\{A_2^1\}$	
7	$\check{\neg} A_1^1$	RC^1	$\{A_1^1\}$	$\checkmark^1$
8	$A_1^1 \check{\supset} B$	7; RU	$\{A_1^1\}$	
9	B	5, 6; RU	$\{A_2^1, A_1^2\}$	
10	B	5, 8; RU	$\{A_1^1, A_1^2\}$	

For a while, it was hoped that this argument could be generalized and that all sequential superpositions that have Minimal Abnormality as their strategy are a fixed point. This, however, revealed a problem.

Consider the same (abstract) logic $\mathbf{C}^m$. The premise set is a variation on Γ_3 from page 137. Let $\Gamma_7 = \{A_i^1 \vee A_j^1 \mid i, j \in \mathbb{N}; i \neq j\} \cup \{B \vee A_i^1 \vee A_i^2 \mid i \in \mathbb{N}\}$. Note that $\Phi^2(\Gamma_7) = \{\emptyset\}$, whereas $\Phi^1(\Gamma_7) = \{\Omega^1 - \{A_i^1\} \mid i \in \mathbb{N}\}$; so $\Phi^1(\Gamma_7)$ contains all sets that comprise all but one member of Ω^1. So $B \notin Cn_{\mathbf{C}^m}(\Gamma_7)$. But $\{B \vee A_i^1 \mid i \in \mathbb{N}\} \in Cn_{\mathbf{C}^m}(\Gamma_7)$, whence $B \in Cn_{\mathbf{C}^m}(Cn_{\mathbf{C}^m}(\Gamma_7))$. So $\mathbf{C}^m$ is not a fixed point.

With this example, the Deduction Theorem cannot come to the rescue because of the involved infinities. The situation is such that $\Gamma_7 \cup \{B \vee A_i^1 \mid i \in \mathbb{N}\} \vdash_{\mathbf{C}^m} B$, but as $\{B \vee A_i^1 \mid i \in \mathbb{N}\}$ is an infinite set, one cannot push it into the antecedent of an implication. Note also that $\mathbf{AL}1^m$ does not deliver anything useful to be combined with $\mathbf{AL}2^m$-consequences. For example, it is possible to derive in the combined proof, for every $i \in \mathbb{N}$, the formula $B \vee A_i^2$ on the condition $\{A_i^1\}$, but this condition varies itself with i, and so the line on which the formula is derived on that condition is marked.

The reader may complain at this point. Consider indeed the semantics of $\mathbf{C}^m$, described in the sixth paragraph following (6.1). This semantics obviously defines a logic that is a fixed point. So what is going on here?

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Enough turmoil. Here are the answers. If the strategy is Reliability, $\mathbf{C}$ is sound and complete with respect to its semantics and is not a Fixed Point. If the strategy is Minimal Abnormality, the proof theory is adequate with respect to the definition of $\mathbf{C}$, viz. (6.1), but neither the definition nor the proof theory is complete with respect to that semantics. The reason for this is that, by selecting models, one retains information that cannot possibly be transferred to the 'higher numbered' logic by formulas of the language $\mathcal{L}$ or $\mathcal{L}_+$. The $\mathbf{AL}1^m$-models contain the information that one of the A_i^1 is false. At the syntactic level, this information can only be expressed by the *infinite* disjunction $\bigvee\{\check{\neg}A^1_i \mid i \in \mathbb{N}\}$.

This leads to two subsidiary problems. Is it possible to upgrade the logics in such a way that they are sound and complete with respect to the semantics? The answer is: no, unless by allowing for infinite formulas. Is it possible to downgrade the semantics in such a way that it is adequate for the definition, viz. (6.1), and the proof theory? Here the answer is positive but leads to a model-theoretically ugly construction. We again start from the **LLL**-models of the premise set Γ. From these we select the $\mathbf{AL}1^m$-models. Let Γ′ be the set of all *formulas* verified by all these models. Next we extend this set of models to all **LLL**-models of Γ′. From these models we select the $\mathbf{AL}2^m$-models. And so on.

I have shown and illustrated that sequential superpositions that have Reliability or Minimal Abnormality as their strategy are not fixed points in general. Of course, some such logics are fixed points. It is very likely, but was not proved, that the combined logics of which the combining logics are such that $\Omega^i \subseteq \Omega^{i+1}$ for all $i \in \{1, 2, \ldots\}$ are fixed points. This is very likely because in this case no Dab^i-formula can be derived on a condition that is a subset of Ω^{i+1}. This means that the Dab^i-consequences of $Cn_{\mathbf{C}}(\Gamma)$ are identical to the Dab^i-consequences of Γ. Examples of such logics are $\mathbf{CLI}^r$, $\mathbf{CIL}^r$ and $\mathbf{CG}^r$ from Chapter 3, and the logic $\mathbf{T}^r$ from the present chapter. Obviously, the restriction is heavy.

What if a sequential superposition is not a fixed point? The answer depends on the context in which the logic is applied. Sometimes this is all right because of the way in which the combined logic functions in a formal problem solving process—see some paragraphs before the start of Section 6.2.1. Where a fixed point is desired, it is better to apply the ordered fusions from Section 6.2.5 instead of the sequential superposition.

6.2.4 Closed Unions of Consequence Sets

We now come to a very easy and transparent way to combine simple adaptive logics. We shall see that the rules and marking definitions are less complex than for sequential superpositions—“complex” here refers to human understanding; the difference in computational complexity is hardly significant. Curious as it may seem to be, this approach is viable as soon as certain weak conditions are fulfilled.

Consider the adaptive logics, **AL**1, **AL**2, etc. and let **C** be defined by

$$Cn_{\mathbf{C}}(\Gamma) = Cn_{\mathbf{LLL}}(Cn_{\mathbf{AL1}}(\Gamma) \cup Cn_{\mathbf{AL2}}(\Gamma) \cup \ldots)\,. \tag{6.6}$$

Again, I consider the combination of infinitely many logics because it is more general than the finite case. As in previous sections, a specific combination of a premise set and an intended conclusion may limit the number of combined logics to a finite set. As “closed union of consequence sets” is an awfully long name for a kind a logics, let us replace it by “CUC”.

The need for closing the union under **LLL** is obvious. Suppose that conjunction is classical, A is an **AL**1-consequence of Γ, and B is an **AL**2-consequence of Γ. The conjunction $A \wedge B$ is, in general, only obtained by closing the union under **LLL**.

I shall only consider the situation in which **AL**1, **AL**2, etc. share their lower limit logic as well as their strategy. So the combining logics differ only with respect to their set of abnormalities.

As was the case for sequential superpositions, we sometimes will want to modify (6.6) to

$$Cn_{\mathbf{C}}(\Gamma) = Cn_{\mathbf{LLL}}(Cn_{\mathbf{AL}}(\Gamma) \cup Cn_{\mathbf{AL1}}(\Gamma) \cup Cn_{\mathbf{AL2}}(\Gamma) \cup \ldots)\,.$$

in which **AL** is the adaptive logic that shares its lower limit and strategy with the **AL***i* and has as its set of abnormalities the union of the sets of abnormalities of the **AL***i*.

It is quite obvious that we shall have to add a further restriction. Indeed, one will want to require that Reassurance holds for CUCs, and one will consider Strong Reassurance highly desirable. However, let us first have a look at the combined proofs and the semantics.

The first three rules for CUCs are those from Table 6.1, but require a different, actually simpler, interpretation than for sequential superpositions. Indeed, the rules RU and RC^i may *only* be applied in case all Δ_j in the local premises belong to the same Ω^i. Moreover, there is a further unconditional rule which was mentioned already in Section 3.5. I here present the general form in terms of **LLL** instead of **CL**.

$$\text{RU}^* \quad \text{If } A_1, \ldots, A_n \vdash_{\mathbf{LLL}} B: \quad \begin{array}{ll} A_1 & \Delta_1 \\ \ldots & \ldots \\ A_n & \Delta_n \\ \hline B & * \end{array}$$

The function of RU^* is obviously to govern the steps that are required to close the union of consequence sets under **LLL**. Note the $*$ in the condition of the line written by application of RU^*.[20] As lines of the combined proofs have two kinds of conditions, we need two marking definitions. In Section 3.5 I presented a single marking definition for Reliability. In the present context, it is easier to state two separate marking definitions. I first mention that for a condition $*$, even though it refers to the definition for conditions that are sets of abnormalities.

Definition 6.2.8 *A line l, which has $*$ as its condition, is marked at stage s iff a line mentioned in the justification of l is marked at stage s. (Marking for $*$.)*

Incidentally, it is worth noting that rule RU^* is derivable in all adaptive logics, simple as well as combined. If the rule is applied, Definition 6.2.8 adequately governs the marks of lines with a condition $*$. However, let us return to the adaptive logics under discussion.

In the previous section, I used the term "separate proofs" to refer to **AL***i*-proofs from the right premise set. The notion of separate is most handy in the present context, because it allows for a very simple statement of the marking definition for lines that have a set as their condition. Remember that every

[20]The paragraph following rule RU^* in Section 3.5 suggests a different approach, in terms of a set of sets. The approach is followed neither there nor here because it is a complication. Fundamentalist adaptive logicians, however, would prefer that approach.

such condition Δ is a subset of one or more Ω^i and that every such Ω^i is the set of abnormalities of the logic $\mathbf{AL}i$. Consider a combined proof from Γ. If, from this proof, we select the lines that have a condition $\Delta \subset \Omega^i$, we obtain an $\mathbf{AL}i$-proof from Γ. Let us call this selection of lines the $\mathbf{AL}i$-*reduction* of the combined proof. As $\mathbf{AL}i$ is a simple adaptive logic, the marking definitions for the standard format specify which lines of an $\mathbf{AL}i$-reduction are marked and which unmarked.

Definition 6.2.9 *A line l, which has a set as its condition, is* unmarked *at stage s of a CUC-proof iff there is a combining* $\mathbf{AL}i$ *such that l is unmarked in the* $\mathbf{AL}i$*-reduction of the CUC-proof. (Marking for CUCs.)*

Needless to say, that "l is unmarked in the $\mathbf{AL}i$-reduction of the CUC-proof" entails that l occurs in the $\mathbf{AL}i$-reduction. Note that this marking definition applies independently of the strategy because it refers to the standard format marking definitions of respectively Reliability and Minimal Abnormality.

At this point, I return to the Reassurance problem for CUCs. For sequential superpositions, Reassurance and Strong Reassurance are obvious in view of the construction. It is essential to prove Reassurance for CUCs however. If it does not hold, these logics turn a premise set that has a sensible **LLL**-consequence set into triviality. It is worth noting that the possible absence of Reassurance need not derive from the deductive closure under **LLL**. To consider an extreme example, let **AL**1 and **AL**2 be such that $Cn_{\mathbf{AL2}}(\Gamma)$ comprises all formulas that are not members of $Cn_{\mathbf{AL1}}(\Gamma)$. So the union of both consequence sets is obviously trivial. The example is awkward, but can easily be realized. Let **AL**1 and **AL**2 have **LP** as their lower limit logic—**LP** is described in Section 7.2. Let their sets of abnormalities be respectively $\Omega^1 = \{\neg A \mid A \in \mathcal{F}_s^p\}$ and $\Omega^2 = \{A \mid A \in \mathcal{F}_s^p\}$, where $\mathcal{F}_s^p$ is the set of primitive formulas of the language $\mathcal{L}_s$. Note that $Cn_{\mathbf{AL1}}(\emptyset) \cup Cn_{\mathbf{AL2}}(\emptyset) = \mathcal{W}_s$. This shows at once that even a lower limit logic that has no (definable) detachable binary connectives cannot save CUCs for triviality. Of course, this does not show that Reassurance fails. Indeed, $\mathcal{W}_s$ has **LP**-models. But if one replaces $\neg$ by $\check{\neg}$ in Ω^1, one obtains $\check{\neg}$-triviality, which has no models.

As Strong Reassurance holds for the combining logics, there is a way to warrant Reassurance for CUCs. All one has to require is that $\Omega^i \subseteq \Omega^{i+1}$ for all $i \in \{1, 2, \ldots\}$. Let me show this correct. I warn the reader that the next two paragraphs are complicated. They are also very instructive, however, because they show in a concise way what is going on.

I shall prove Reassurance for logics that have Minimal Abnormality as their strategy. It follows from this that Reassurance also holds for logics that have Reliability as their strategy.

Let **C** be a CUC that has Minimal Abnormality as its strategy[21] and Γ a premise set. Every composing logic $\mathbf{AL}i$ defines a set of minimal Dab^i-consequences of Γ. As $\Omega^i \subseteq \Omega^{i+1}$ for all $i \in \{1, 2, \ldots\}$, every minimal Dab^i-consequence of Γ is also a minimal Dab^{i+1}-consequence of Γ.[22] So, by Fact 5.1.4, there is, for every $\varphi \in \Phi^{i+1}(\Gamma)$ a $\psi \in \Phi^i(\Gamma)$ such that $\psi \subseteq \varphi$. So, for

[21] I skip the superscript m in the names for the composing logics to simplify the notation.

[22] It is obviously a Dab^{i+1}-consequence of Γ. It is a minimal one because every Dab^{i+1}-consequence of Γ that is not a Dab^i-consequence of Γ has a disjunct in $\Omega^{i+1} - \Omega^i$.

every $\varphi_k \in \Phi^k(\Gamma)$, there are $\varphi_1 \in \Phi^1(\Gamma)$, $\varphi_2 \in \Phi^2(\Gamma)$, ..., $\varphi_{k-1} \in \Phi^{k-1}(\Gamma)$ such that $\varphi_1 \subseteq \varphi_2 \subseteq \ldots \subseteq \varphi_{k-1} \subseteq \varphi_k$.

We can, however, prove a much stronger fact. Consider an arbitrary composing adaptive logic $\mathbf{AL}k$. A $\varphi \subset \Omega^k$ will be called *orderly* iff, for all $i \in \{1, \ldots, k\}$, $\varphi \cap \Omega^i \in \Phi^i(\Gamma)$. In order to understand the subsequent argument, it is important to realize that this comes to the following. If $\varphi_k \in \Phi^k(\Gamma)$, there are $\varphi_1 \in \Phi^1(\Gamma)$, $\varphi_2 \in \Phi^2(\Gamma)$, ..., $\varphi_{k-1} \in \Phi^{k-1}(\Gamma)$ such that $\varphi_1 \subseteq \varphi_2 \subseteq \ldots \subseteq \varphi_{k-1} \subseteq \varphi_k$ *and* $\varphi_1 \subseteq \Omega^1$, $\varphi_2 - \varphi_1 \subseteq \Omega^2 - \Omega^1$, $\varphi_3 - \varphi_2 \subseteq \Omega^3 - \Omega^2$, ..., and $\varphi_k - \varphi_{k-1} \subseteq \Omega^k - \Omega^{k-1}$—remember here and below that $\Omega^3 - \Omega^2 = \Omega^3 - (\Omega^1 \cup \Omega^2)$ etc. I shall now outline the proof that, for every Γ and k, there is an orderly $\varphi \in \Phi^k(\Gamma)$.

If $k = 1$, then φ is orderly because $\varphi \in \Phi^1(\Gamma)$ and $\bigcup \Phi^1(\Gamma) \subseteq \Omega^1$. So suppose that $k > 1$ and that there are orderly $\varphi \in \Phi^{k-1}(\Gamma)$. Remember that φ contains an element of every minimal Dab^{k-1} consequence of Γ and that φ is a minimal choice set of Σ^{k-1} which contains every Δ for which $Dab^{k-1}(\Delta)$ is a minimal Dab^{k-1}-consequence of Γ.

Consider a list $L = \langle \Delta_1, \Delta_2, \ldots \rangle$ of all Δ_i for which $Dab^k(\Delta_i)$ is a minimal Dab^k-consequence of Γ but not a Dab^{k-1}-consequence of Γ. Every such Δ_i has the form $\Delta_i^1 \cup \Delta_i^2 \cup \Delta_i^3$ with $\Delta_i^1 \subseteq \bigcup \Phi^{k-1}(\Gamma)$, $\Delta_i^2 \subseteq \Omega^k - \Omega^{k-1}$, and $\Delta_i^3 \subseteq \Omega^{k-1} - \bigcup \Phi^{k-1}(\Gamma)$.[23] So Δ_i^1 comprises only abnormalities that already occur in members of $\Phi^{k-1}(\Gamma)$; Δ_i^2 comprises abnormalities that did not occur in any member of $\Phi^{k-1}(\Gamma)$ because they are not members of Ω^{k-1} but only of Ω^k; Δ_i^3 comprises members of Ω^{k-1} that do not occur in any member of $\Phi^{k-1}(\Gamma)$. It is possible that Δ_i^1 and Δ_i^3 are empty, but not that Δ_i^2 is empty because $Dab^k(\Delta_i)$ is not a Dab^{k-1}-consequence of Γ.

Starting with Δ_1, we stepwise add a disjunct of each Δ_i from L to φ as follows: if $\Delta_i^1 \cup \Delta_i^2 \cap \varphi \neq \emptyset$, we add nothing; otherwise we add to φ a member of $\Delta_i^1 \cup \Delta_i^2$. The result is a choice set of $\Sigma^{k-1} \cup \{\Delta_1, \ldots \Delta_i\}$. Moreover, if φ is a minimal choice set of those sets, it is orderly. If φ is not a minimal choice set of those sets, we know from Fact 5.1.7 that $\Delta_i \cap \varphi = \emptyset$ and that, for every $A \in \Delta_i$, there is a minimal choice set ψ of $\Sigma^{k-1} \cup \{\Delta_1, \ldots \Delta_i\}$ for which $A \in \psi$, and $\psi - \{A\} \supset \varphi$. So one of these ψ is such that $A \in \psi \cap \Delta_i^3$ and as $\psi - \{A\} \supset \varphi$, ψ is an orderly minimal choice set of $\Sigma^{k-1} \cup \{\Delta_1, \ldots \Delta_i\}$. Choose these ψ instead of φ and continue to Δ_{i+1}. Doing this for all members of the list L, we obtained an orderly $\varphi \in \Phi^k(\Gamma)$. It follows by induction that there is an orderly $\varphi \in \Phi^k(\Gamma)$ for every Γ and k.

Given this somewhat complex preparation, the proof that Reassurance holds for **C** is easy.Suppose that Γ has **LLL**-models whereas $Cn_{\mathbf{C}}(\Gamma)$ does not. This means that $Cn_{\mathbf{AL}1}(\Gamma) \cup Cn_{\mathbf{AL}2}(\Gamma) \cup \ldots$ has no **LLL**-models—have another look at (6.6) and remember that **LLL** is a Tarski logic. As **LLL** is compact, there are $A_1, \ldots, A_n \in Cn_{\mathbf{AL}1}(\Gamma) \cup Cn_{\mathbf{AL}2}(\Gamma) \cup \ldots$ such that $\{A_1, \ldots, A_n\}$ has no **LLL**-model. Let $A_1 \in Cn_{\mathbf{AL}i_1}(\Gamma)$, ..., $A_n \in Cn_{\mathbf{AL}i_n}(\Gamma)$—some of these $\mathbf{AL}i_j$ may be identical and if A_j belongs to several consequence sets, just choose one of them. As every $\mathbf{AL}i_j$ is in standard format and hence sound and complete with respect to its semantics, there are CUC-proofs from Γ in which every A_j is finally derived by $\mathbf{AL}i_j$. Let k the maximum of $\{i_1, \ldots, i_n\}$. In other words, the highest numbered composing logic invoked to derive each member of $\{A_1, \ldots, A_n\}$ is $\mathbf{AL}k$.

Choose an *orderly* $\varphi \in \Phi^k(\Gamma)$. In view of the supposition, the CUC-proof

[23] Remember that $\Omega^{k-1} = \Omega^1 \cup \ldots \cup \Omega^{k-1}$ and $\bigcup \Phi^{k-1}(\Gamma) = \bigcup \Phi^1(\Gamma) \cup \ldots \cup \bigcup \Phi^{k-1}(\Gamma)$.

contains unmarked lines with the following formulas and conditions

A_1	Δ_1
A_2	Δ_2
$\vdots$	$\vdots$
A_n	Δ_n

in which $\Delta_j \subset \Omega^{i_j} - \varphi = \emptyset$ for all $j \in \{1, \ldots, n\}$. Note that it is possible to choose such Δ. Remember indeed that $\varphi \cap \Omega^{i_j} \in \Phi^{i_j}(\Gamma)$ and that, for every member of $\Phi^{i_j}(\Gamma)$, A_j is derivable in the CUC-proof on a condition that has no element in common with that member.

In view of the Conditions Lemma 4.4.1, $\Gamma \vdash_{\mathbf{LLL}} A_j \check{\vee} Dab(\Delta_j)$ for every j $(1 \leq j \leq n)$. It follows that, as Γ has **LLL**-models, so does $\{A_1 \check{\vee} Dab(\Delta_1), \ldots, A_n \check{\vee} Dab(\Delta_n)\}$. But $\{A_1, \ldots, A_n\}$ has no **LLL**-model. So every **LLL**-model of Γ verifies at least one member of $\Delta_1 \cup \ldots \cup \Delta_n$. But this is impossible because $\Delta_1 \cup \ldots \cup \Delta_n \subset \Omega^k$, $(\Delta_1 \cup \ldots \cup \Delta_n) \cap \varphi = \emptyset$ and, in view of Lemma 5.2.1, there is a **LLL**-model M of Γ for which $Ab(M) \cap \Omega^k = \varphi$.

Reassurance being established, what about Strong Reassurance? Actually, this is warranted by the combining logics. Indeed, all comparisons of models proceed in terms of the separate composing logics—models of different composing logics are never compared. As Strong Reassurance holds for the composing logics, 'infinitely descending sequences of models' cannot occur.

The previous paragraph was written in terms of the most obvious semantics for CUCs: each combining logic separately selects its models from the **LLL**-models of the premise set. Next, one considers all **LLL**-models of the semantic consequences of the combining logics. This is an ugly and awkward construction, and not a unified semantics for the CUCs. However, it is the best that is available at the moment.[24] cite in fn

Summing up: requiring that $\Omega^i \subseteq \Omega^{i+1}$ for all $i \in \{1, 2, \ldots\}$ warrants Reassurance as well as Strong Reassurance for the CUCs. There may be other conditions that warrant Reassurance and possibly Strong Reassurance, but they are not known at this moment.

We are not home yet. A property we should certainly check is whether **C** is a fixed point. Even requiring $\Omega^i \subseteq \Omega^{i+1}$ for all $i \in \{1, 2, \ldots\}$ is not sufficient to warrant that property. Here is an example, with thanks to Frederik Van De Putte—I also had some instructive discussions with him on the rest of this Section. Consider the CUC-logic $\mathbf{T}^r_u$, in which the u refers to CUC—the combining logics $\mathbf{T}^r_i$ are as in the previous section. Consider $\Gamma_8 = \{\neg p \vee \neg q, \neg q \vee \neg r, \Diamond p, \Diamond\Diamond q, \Diamond\Diamond r\}$.

1	$\neg p \vee \neg q$	Prem	$\emptyset$
2	$\neg q \vee \neg r$	Prem	$\emptyset$
3	$\Diamond p$	Prem	$\emptyset$
4	$\Diamond\Diamond q$	Prem	$\emptyset$
5	$\Diamond\Diamond r$	Prem	$\emptyset$
6	p	3; RC^1	$\{\Diamond p \wedge \neg p\}$
7	$\neg q$	1, 6; RU	$\{\Diamond p \wedge \neg p\}$

[24] In [VDPng], Frederik Van De Putte considers the semantics obtained by selecting the **LLL**-models of a premise set that are selected by all composing logics. He shows that this intersection is not empty. The CUC is sound (but not complete) with respect to this semantics. So this is a different way for proving Strong Reassurance.

8	q	4; RU	$\{\Diamond\Diamond q \wedge \neg q\}$	$\checkmark^{11}$
9	r	5; RU	$\{\Diamond\Diamond r \wedge \neg r\}$	$\checkmark^{11}$
10	$(\Diamond\Diamond q \wedge \neg q) \check{\vee} (\Diamond\Diamond p \wedge \neg p)$	1, 3, 4; RU	$\emptyset$	
11	$(\Diamond\Diamond q \wedge \neg q) \check{\vee} (\Diamond\Diamond r \wedge \neg r)$	1, 4, 5; RU	$\emptyset$	

So r is not a member of $Cn_{\mathbf{T}^r_u}(\Gamma_8)$. However, p and $\neg q$ are. So, let us consider the $\mathbf{T}^r_u$-proof from $\Gamma_8 \cup \{p, \neg q\}$, or actually from $Cn_{\mathbf{T}^r_u}(\Gamma_8)$.

1	$\neg p \vee \neg q$	Prem	$\emptyset$
2	$\neg q \vee \neg r$	Prem	$\emptyset$
3	$\Diamond p$	Prem	$\emptyset$
4	$\Diamond\Diamond q$	Prem	$\emptyset$
5	$\Diamond\Diamond r$	Prem	$\emptyset$
6	p	Prem	$\emptyset$
7	$\neg q$	Prem	$\emptyset$
8	$\Diamond\Diamond q \wedge \neg q$	3, 7; RU	$\emptyset$
9	r	5; RU	$\{\Diamond\Diamond r \wedge \neg r\}$

In the proof from these premises, line 9 is unmarked and will remain unmarked in any extension of the proof. So $\mathbf{T}^r_u$ is not a fixed point.

Before we proceed, a further comment is desirable. The reader may have seen that the *first* proof, where the logic is $\mathbf{T}^r_u$, may be continued as follows.

12	$(\Diamond\Diamond q \wedge \neg q) \check{\vee} (\Diamond p \wedge \neg p)$	1, 3, 4; RU	$\emptyset$
13	$\neg(\Diamond p \wedge \neg p)$	RU	$\{\Diamond p \wedge \neg p\}$
14	$\Diamond\Diamond q \wedge \neg q$	12, 13; RU	$\{\Diamond p \wedge \neg p\}$

This, however, does not make $\Diamond\Diamond q \wedge \neg q$ into a minimal Dab^2-formula. The logic $\mathbf{T}^r_u$ is a CUC, not a sequential superposition. Dab^2-formulas have to be $\mathbf{T}^r_2$-derived from Γ_8, and so have to be $\mathbf{T}$-derived from Γ_8. This means that they have to be derived on the empty condition in the combined proofs. Actually, lines 12–14 do not teach us anything new. Indeed, 13 is a consequence of 7, and line 14 may also be justified by RU from lines 4 and 7. All this teaches us is that $\Diamond\Diamond q \wedge \neg q$ is a $\mathbf{T}^r_1$-consequence of Γ_8, not that it is a Dab^2-formula. Precisely this causes $\mathbf{T}^r_u$ not to be a fixed point.

Again, the example does not illustrate the absence of the Fixed Point property for logics that have Minimal Abnormality as their strategy. The reason is again that the Deduction Theorem holds for such logics. So $\neg q \check{\supset} r$ is $\mathbf{T}^m_2$-derivable from the premises and hence so is r.

Even Minimal Abnormality cannot warrant that the combined logic is a fixed point. Here is an example, varying on a theme we met before. To keep it simple, I present a concrete propositional example. Let, just for this section, $\mathbf{CLuN}^m_u$ be defined by $Cn_{\mathbf{CLuN}^m_u}(\Gamma) = Cn^{\mathcal{L}_s}_{\mathbf{CLuN}}(Cn^{\mathcal{L}_S}_{\mathbf{CLuN}^m_1}(\Gamma) \cup Cn^{\mathcal{L}_S}_{\mathbf{CLuN}^m_2}(\Gamma))$—the logics $\mathbf{CLuN}^m_i$ were defined in the previous section (in the paragraphs preceding (6.4) and their sets of abnormalities are $\Omega^i = \{!^i A \mid A \in \mathcal{F}\}$. Keep in mind for the next paragraphs that $\Omega^2 \subset \Omega^1$, which is the inverse of the usual order in this section. I shall also skip the reference to the language for the consequence sets; the definition of the logic makes that explicit.

Let $\Gamma_9 = \{!^2 q_i \vee !^2 q_j \mid i, j \in \mathbb{N}; i \neq j\} \cup \{!^1 p_i \vee !^1 p_j \vee !^2 q_k \mid i, j, k \in \mathbb{N}; i \neq j\} \cup \{r \vee !^1 p_i \vee !^2 q_j \mid i, j \in \mathbb{N}\}$.[25] The members of $\Phi^2(\Gamma_9)$ are the sets that

[25] There is no typo here; $i \neq j$ occurs only twice.

comprise all but one member of $\{!^2q_i \mid i \in \mathbb{N}\}$; $\Phi^1(\Gamma_9)$ contains two kinds of members, on the one hand $\{!^2q_i \mid i \in \mathbb{N}\}$, on the other hand the sets that comprise all but one member of $\{!^1p_i \mid i \in \mathbb{N}\}$ as well as all but one member of $\{!^2q_i \mid i \in \mathbb{N}\}$.

So $r \notin Cn_{\mathbf{CLuN}_2^m}(\Gamma_9)$: r can only be derived from a formula of the form $r \vee !^1p_i \vee !^2q_j$ and $!^1p_i$ cannot be pushed to the condition by $\mathbf{CLuN}_2^m$. Next, $r \notin Cn_{\mathbf{CLuN}_1^m}(\Gamma_9)$. To see this consider again the infinite proof. It is possible to insert a line at which r is derived on the condition $\{!^1p_i, !^2q_j\}$ and it is possible to do so for every i and j, but it is not possible to derive r on a different condition. However, whatever j, $!^2q_j \in \{!^2q_i \mid i \in \mathbb{N}\}$ and $\{!^2q_i \mid i \in \mathbb{N}\} \in \Phi^1(\Gamma_9)$. So all those lines are marked.

What about the closure under the lower limit logic **CLuN**? Suppose that r is **CLuN**-derivable from $Cn_{\mathbf{CLuN}_1^m}(\Gamma_9) \cup Cn_{\mathbf{CLuN}_2^m}(\Gamma_9)$. So it is derivable from finitely many members of that union, because **CLuN** is compact. It follows that there are $A_1, \ldots, A_n \in Cn_{\mathbf{CLuN}_1^m}(\Gamma_9)$ and $B_1, \ldots, B_m \in Cn_{\mathbf{CLuN}_2^m}(\Gamma_9)$ such that $A_1, \ldots, A_n, B_1, \ldots, B_m \vdash_{\mathbf{CLuN}} r$. **CLuN** contains the classical symbols and the Deduction Theorem for $\check{\supset}$. So $A_1, \ldots, A_n, \vdash_{\mathbf{CLuN}} \check{\neg}B_1 \check{\vee} \ldots \check{\vee} \check{\neg}B_m \check{\vee} r$, whence $\check{\neg}B_1 \check{\vee} \ldots \check{\vee} \check{\neg}B_m \check{\vee} r \in Cn_{\mathbf{CLuN}_1^m}(\Gamma_9)$. We can safely suppose that all A_i and B_i contain only propositional letters that occur in Γ_9. We can moreover safely suppose that all A_i and B_i are members of $\mathcal{W}_s$—it becomes clear in a minute why this is so. It follows that the $\check{\neg}B_i$ are classical negations of members of $\mathcal{W}_s$. However, it can be shown by a lengthy induction that all formulas that contain a classical negation and occur in $Cn_{\mathbf{CLuN}_1^m}(\Gamma_9)$ (or in $Cn_{\mathbf{CLuN}_2^m}(\Gamma_9)$) are equivalent to a member of $\mathcal{W}_s$.[26] So it is impossible that $\check{\neg}B_1 \check{\vee} \ldots \check{\vee} \check{\neg}B_m \check{\vee} r \in Cn_{\mathbf{CLuN}_1^m}(\Gamma_9)$. So $r \notin Cn_{\mathbf{CLuN}_u^m}(\Gamma_9)$.

However, $r \in Cn_{\mathbf{CLuN}_u^m}(Cn_{\mathbf{CLuN}_u^m}(\Gamma_9))$. Indeed, $\{!^1p_i \vee !^1p_j \mid i, j \in \mathbb{N}; i \neq j\} \cup \{r \vee !^1p_i \mid i \in \mathbb{N}\} \subset Cn_{\mathbf{CLuN}_u^m}(\Gamma_9)$. So $\Phi^2(Cn_{\mathbf{CLuN}_u^m}(\Gamma_9)) = \Phi^2(\Gamma_9)$; it contains the sets that comprise all but one member of $\{!^2q_i \mid i \in \mathbb{N}\}$. However, $\Phi^1(Cn_{\mathbf{CLuN}_u^m}(\Gamma_9))$ contains only the sets that comprise all but one member of $\{!^1p_i \mid i \in \mathbb{N}\}$ as well as all but one member of $\{!^2q_i \mid i \in \mathbb{N}\}$.

Incidentally, the example shows that it is a mistake to identify, in general, the members of $Cn_{\mathbf{C}_u^m}(\Gamma_9)$ with the formulas verified by the **LLL**-models that are minimally abnormal with respect to all composing simple adaptive logics. Indeed, in the example under discussion, all those models verify r. The **CLuN**-models of Γ_9 that are minimally abnormal with respect to $\mathbf{CLuN}_2^m$ all verify $\{!^1p_i \vee !^1p_j \mid i, j \in \mathbb{N}; i \neq j\} \cup \{r \vee !^1p_i \mid i \in \mathbb{N}\}$. Those *of these* **CLuN**-models that are moreover minimally abnormal with respect to $\mathbf{CLuN}_1^m$ all verify r.

Time to proceed. We have seen that, even if $\Omega^{i+1} \subset \Omega^i$ for all $i \in \{1, 2, \ldots\}$ (or the inverse as in the case of $\mathbf{CLuN}_u^m$), the CUC-combination need not be a fixed point. The comments are as at the end of the previous section. Sometimes the way in which an adaptive logic functions in a formal problem-solving process does not require that the logic is a fixed point. In other contexts, one better applies an ordered fusion of the composing logics. nazien of ord.fus. OK

[26]The induction is lengthy, but it is easy to understand why the property holds. Consider all formulas composed by letters that occur in the premises. All we can say about the falsehood of such formulas in all $\mathbf{CLuN}_1^m$-models of Γ_9 comes to what $\Phi^1(\Gamma_9)$ tells us: that every such model falsifies all members of $\{!^2q_i \mid i \in \mathbb{N}\}$ or falsifies one member of $\{!^1p_i \mid i \in \mathbb{N}\}$ and one member of $\{!^2q_i \mid i \in \mathbb{N}\}$. From this nothing follows that can be expressed by a formula of $\mathcal{L}_S$.

In Chapter 3, we have seen three examples of CUC-combinations for which $\Omega^{i+1} \subset \Omega^i$ for all $i \in \{1, 2, \ldots\}$: **HLI**, **HIL**, and **HG**. It is very well possible, even likely, that these are fixed points or that they are for the premise sets to which they are intended to apply, which are finite sets of singular statements, also called data. It is even likely that they are fixed point when they are combined with background knowledge—see Section 3.6. Indeed, it is unlikely that any set of background knowledge has the odd structure of Γ_9. However, this is a book on logic and good books on logic do not make claims that are not proven.

This seems to be the best place to illustrate a feature of the logics of the **H**-group that may be somewhat unexpected. Consider an $\mathbf{HG}^r$-proof from the premise set $\Gamma_{10} = \{Pa, \neg Pb, Qc, \neg Qd, Re, \neg Rd \vee Qa\}$ from Section 3.5. Note that the conditional rules of the different combining logics are written as RC$^{(i)}$ as in Chapter 3, where the Ω^i are defined as disjoint sets and the composing adaptive logics have $\Omega^{(i)} = \Omega^0 \cup \Omega^1 \cup \ldots \cup \Omega^i$ as sets of abnormalities.

1	Pa	premise	$\emptyset$	
2	$\neg Pb$	premise	$\emptyset$	
3	Qc	premise	$\emptyset$	
4	$\neg Qd$	premise	$\emptyset$	
5	Re	premise	$\emptyset$	
6	$\neg Rd \vee Qa$	premise	$\emptyset$	
7	$\forall x Rx$	5; RC$^{(0)}$	$\{\pm Rx\}$	
8	Qa	6, 7; RU	$\{\pm Rx\}$	
9	$\forall x(Px \supset Qx)$	1, 8; RC$^{(1)}$	$\{\pm Rx, (Px \wedge \pm Qx)\}$	$\checkmark^{12}$
10	$\pm Px$	1, 2; RU	$\emptyset$	
11	$\pm Qx$	3, 4; RU	$\emptyset$	
12	$\pm Rx \vee (Rx \wedge \pm Px)$	1, 2, 5; RU	$\emptyset$	
13	$\pm Rx \vee (Rx \wedge \pm Qx)$	3, 4, 5; RU	$\emptyset$	

All minimal *Dab*-consequences (of every degree) of the premises are derived in this proof at lines 10–13. So $U^{(0)} = \{\pm Px, \pm Qx\}$ and $U^{(1)} = \{\pm Rx, Rx \wedge \pm Px, Rx \wedge \pm Qx\}$. Line 9 is marked because its condition is of degree 1. So it seems that $\forall x(Px \supset Qx)$ is not finally derivable from Γ_{10}.

However, $\forall x(Px \supset Qx)$ is finally derivable from Γ_{10} because the $\mathbf{HG}^r$-consequences of a premise set are closed under **CL**—Frederik Van De Putte first pointed out the mechanism invoked below. Continuing the proof, I first extend the proof with lines 14–16, which are the unconditional lines corresponding to lines 7–9. Next, I derive two negations of abnormalities, one of degree 0 and one of degree 1. Finally, I apply RU*.

14	$\forall x Rx \vee \pm Rx$	5; RU	$\emptyset$
15	$Qa \vee \pm Rx$	6, 14; RU	$\emptyset$
16	$\forall x(Px \supset Qx) \vee \pm Rx \vee (Px \wedge \pm Qx)$	1, 15; RU	$\emptyset$
17	$\neg(\pm Rx)$	RC$^{(0)}$	$\{\pm Rx\}$
18	$\neg(Px \wedge \pm Qx)$	RC$^{(1)}$	$\{Px \wedge \pm Qx\}$
19	$\forall x(Px \supset Qx)$	16, 17, 18; RU*	$*$

It is worth mentioning that $\neg(\pm Rx)$ is **CL**-equivalent to $\forall x Rx \vee \forall x \neg Rx$ and that $\neg(Px \wedge \pm Qx)$ is **CL**-equivalent to $\forall x(Px \supset Qx) \vee \forall x(Px \supset \neg Qx)$. Lines 17 and 18 are unmarked and hence so is line 19.

6.2.5 Ordered Fusions of Adaptive Logics

I shall use this label to denote combined adaptive logics defined as follows, where **AL**1 $*$ **AL**1 denotes the *fusion* of **AL**1 and **AL**1 and similarly for continuous expressions.

$$Cn_{\mathbf{C}}(\Gamma) = Cn_{\mathbf{AL1}*\mathbf{AL2}*\ldots}(\Gamma) \tag{6.7}$$

Given the complex character of combined adaptive logics, we better restrict our attention to composing logics that share their lower limit as well as their strategy.

As was the case for sequential superpositions, we sometimes will want to modify (6.7) to

$$Cn_{\mathbf{C}}(\Gamma) = Cn_{\mathbf{AL}*\mathbf{AL1}*\mathbf{AL2}*\ldots}(\Gamma)$$

in which **AL** is the adaptive logic that shares its lower limit and strategy with the **AL***i* and has as its set of abnormalities the union of the sets of abnormalities of the **AL***i*.

The whole point is what is meant by the fusion **AL**1 $*$ **AL**2 $*$ ….[27] Let me start by following the standard recipe for fusions of Tarski logics: $Cn_{\mathbf{C}}(\Gamma)$ is the smallest Γ' fulfilling, for any fused logic **AL***i*, (i) $Cn_{\mathbf{AL}i}(\Gamma) \subseteq \Gamma'$ and (ii) $Cn_{\mathbf{AL}i}(\Gamma') \subseteq \Gamma'$. This kind of combination is simply a no-go, except in a special case. Let me first show that, in general, no sensible combination results. Consider the fusion of two adaptive logics **AL**1$*$**AL**2, let Ω^1 and Ω^2 respectively be the sets of abnormalities and let the members of Ω^1 be abbreviated by $!A$ and the members of Ω^2 by $?A$. Suppose that some 'mixed' abnormalities are **LLL**-derivable from the premise set. The simplest possible example would be that $!A \check{\vee} ?B$ is **LLL**-derivable from Γ and that no other disjunction of abnormalities is **LLL**-derivable from Γ. Note that $?B$ is an **AL**1-final consequence of Γ, whereas $!A$ is an **AL**2-final consequence of Γ.

But consider the following proof, in which RC^1 is the conditional rule of $\mathbf{AL}_1$ and RC^2 that of $\mathbf{AL}_2$.

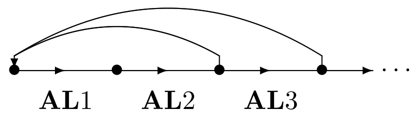
1	$!A \check{\vee} ?B$	Prem	
2	$!A$	1; RC^1	$\{?B\}$
3	$?B$	1; RC^2	$\{!A\}$

In view of the fact that $?B$ is a **AL**1-final consequence of Γ and that $!A$ is a **AL**2-final consequence of Γ, lines 2 and 3 should be unmarked. And indeed, lines 1 and 2 form a **AL**1-proof and line 2 is unmarked in it, whereas lines 1 and 3 form a **AL**2-proof and line 3 is unmarked in that one. However, precisely because the lines are unmarked, the combined proof gives us: $!A \in U^1_3(\Gamma)$ and $?B \in U^2_3(\Gamma)$. So lines 2 and 3 are marked on the marking definition for Reliability. But as the lines are marked, $U^1_3(\Gamma) = \emptyset$ and $U^2_3(\Gamma) = \emptyset$. But then the marking definition decrees that lines 2 and 3 are unmarked. And so on. In other words, (6.7) causes a circularity that the marking definition is unable to handle—similarly for the marking definition for Minimal Abnormality. It is not difficult to see that the situation does not clear up in case more premises, whether *Dab*-formulas or not, are added.

[27] Note that "fusion" is used with a different meaning in [CCG+08]. In that book, which is on Tarski Logics, the present construction corresponds best to what is there called the fibring of Hilbert calculi with careful reasoning.

There are special cases in which such fusions of adaptive logics are sensible. The most striking ones are where, either because of properties of the logics or because of properties of the premise set, it holds in general that $Dab(\Delta)$ (with $\Delta \subset \Omega^1 \cup \Omega^2 \cup \ldots$) is **LLL**-derivable from Γ iff there is a Ω^i such that $\Delta \subset \Omega^i$. Of course, this special case is extremely exceptional.

Ordered fusions While only some exceptional fusions of adaptive logics seem to deliver sensible results, the picture drastically changes if we consider *ordered fusions*. As this is a notion not documented in the literature, I shall be very explicit. The way to interpret $\mathbf{AL}1 * \mathbf{AL}2 * \ldots$ in terms of ordered fusions is made clear by the following picture.



The idea is that we start at the leftmost bullet with the set of formulas Γ and subsequently close this set under a sequence of logics. We first close Γ by **AL**1. Next we close the result by **AL**2. This brings us to a node at which there are two paths, one returning to the first node and one moving on to the right. If we arrive at such a node for the first time or if the present consequence set is different from the one with which we arrived here previously, we move back to the leftmost node; otherwise we proceed to the right. So, as we arrive here for the first time, we move back to the first node, close the present consequence set by **AL**1 and close the result by **AL**2. If the result is different from the one with which we first arrived at this node, we move again to the leftmost node, otherwise we proceed to the right.

If the number of combining adaptive logics is finite, the last line should still have an arrow to the right, indicating that it leads to the end stage of the increasing consequence set.

The advantages of the present construction are obvious. An ordered fusion of (finitely or infinitely many) simple adaptive logics has all the properties we desire for an adaptive logic. Strong Reassurance, Fixed Point, . . . are all obvious by the construction. So from a definitional point of view, we have reached an ideal combination. The definition (6.7) is clear and so are the metatheoretic properties. One might expect trouble for the combined proofs, but there is none.

Ordered fusions are of recent vintage and were not much studied. Also the Ghent ideas on the proof format are presumably not completely stable. For this reason I shall keep the further discussion informal. Let us first reconsider the abstract proof from Γ_6 (page 227). The logic will be **C**, as defined by (6.7) with $*$ interpreted as an ordered fusion. Let $\Omega^i = \{A^i_1, A^i_2, \ldots\}$. I again 'forget' to introduce $\check{\vee}$ for *Dab*-formulas and add a new complication in comparison to the former proof from these premises.

1	$A^1_1 \vee A^1_2$	Prem	$\emptyset$		
2	$A^1_1 \vee A^2_1$	Prem	$\emptyset$		
3	$B \vee A^1_2$	Prem	$\emptyset$		
4	A^2_1	2; RC^2	$\{A^1_1\}$	$\checkmark^1_1$	$\checkmark^6_{1,2,1}$
5	B	3; RC^1	$\{A^1_2\}$	$\checkmark^1_1$	$-_{1,2,1}$

6	A_1^1	2; RC^2	$\{A_1^2\}$		$-_{1,2}$	

The marks are accurate for stage 6 of the proof. The first column of marks contains the marks determined by the **AL**1-marking definition because **AL**1 is the first logic in the ordered fusion. At this point, Dab^1-formulas are derived on the empty condition. So $U_6^1(\Gamma) = \{A_1^1, A_2^1\}$, $\Phi_6^1(\Gamma) = \{\{A_1^1\}, \{A_2^1\}\}$, whence lines 4 and 5 are marked—the superscript of the mark refers to the stage from which the line would be marked if it were present at that stage, the subscript to the fact that we are applying **AL**1 to Γ. The second column of marks depends on the **AL**2-marking definition. Dab^2-formulas have to be derived at an **AL**1-unmarked line on a condition $\Delta \subset \Omega^1$ because, at this point, **AL**2 applies to $Cn_{\mathbf{AL1}}(\Gamma)$. As line 4 is **AL**1-marked, $U_6^2(Cn_{\mathbf{AL1}}(\Gamma)) = \{\emptyset\}$ and $\Phi_6^2(Cn_{\mathbf{AL1}}(\Gamma)) = \{\emptyset\}$. So line 6 is unmarked. Note that $-$ functions an the 'unmarked'-sign and that its subscript indicates that **AL**2 applies to $Cn_{\mathbf{AL1}}(\Gamma)$. Moreover, that line 6 is unmarked in this proof at stage 6 is final. Subsequent considerations cannot overrule this: as far as this stage is concerned, A_1^1 is a **AL**2-consequence of $Cn_{\mathbf{AL1}}(\Gamma)$. The next step concerns the application of **AL**1 to $Cn_{\mathbf{AL2}}(Cn_{\mathbf{AL1}}(\Gamma))$. The effect of this is represented in the third column of marks. Line 6 is unmarked in the second column. As a result $U_6^1(Cn_{\mathbf{AL2}}(Cn_{\mathbf{AL1}}(\Gamma))) = \{A_1^1\}$ and $\Phi_6^1(Cn_{\mathbf{AL2}}(Cn_{\mathbf{AL1}}(\Gamma))) = \{\{A_1^1\}\}$. So at this point line 4 is still marked but line 5 is unmarked: B is derived as desired.

The signs indicating that a line is marked or unmarked are tiresome but completely transparent. Note that these signs illustrate marking at stage 6 of the proof. If the proof is extended even with one line, we have to erase all the signs and start applying the **AL**1-marking definition in view of Dab^1-formulas derived on the empty condition. So, what I indicated in the previous paragraph is the marking in view of one stage (stage 6) of the proof. The complications, indicated by the different columns of marks, derive from the way in which ordered fusions combine simple adaptive logics.

A second example is the ordered fusions proof from Γ_8 (page 233). The lower limit logic is **T**; let the strategy be again Reliability. I again add sings indicating that a line is unmarked (for the present stage).

1	$\neg p \vee \neg q$	Prem	$\emptyset$		
2	$\neg q \vee \neg r$	Prem	$\emptyset$		
3	$\Diamond p$	Prem	$\emptyset$		
4	$\Diamond\Diamond q$	Prem	$\emptyset$		
5	$\Diamond\Diamond r$	Prem	$\emptyset$		
6	p	3; RC^1	$\{\Diamond p \wedge \neg p\}$	$-_1$	
7	$\neg q$	1, 6; RU	$\{\Diamond p \wedge \neg p\}$	$-_1$	
8	q	4; RU	$\{\Diamond\Diamond q \wedge \neg q\}$		$\checkmark^{12}_{1,2}$
9	r	5; RU	$\{\Diamond\Diamond r \wedge \neg r\}$		$-_{1,2}$
10	$(\Diamond\Diamond q \wedge \neg q) \check{\vee} (\Diamond\Diamond p \wedge \neg p)$	1, 3, 4; RU	$\emptyset$		
11	$(\Diamond\Diamond q \wedge \neg q) \check{\vee} (\Diamond\Diamond r \wedge \neg r)$	1, 4, 5; RU	$\emptyset$		
12	$\Diamond\Diamond q \wedge \neg q$	1, 7; RU	$\{\Diamond p \wedge \neg p\}$	$-_1$	

So with the ordered fusions logic r is a final consequence of Γ_8 because line 9 is unmarked in view of the application of $\mathbf{T}_2^u$ to $Cn_{\mathbf{T}_1^r}(\Gamma)$. This is a clear gain over the CUC that was applied to this premise set before. The corresponding sequential superposition would obviously lead to the same result for this premise set but not for others because ordered fusions are a fixed point while sequential superpositions are not.

Some readers may notice that, with the present logic, it is possible to derive $(\Diamond\Diamond q \wedge \neg q) \check{\vee} (\Diamond p \wedge \neg p)$ from Γ_8 by the lower limit **T**. They may wonder whether doing so would not change the marks in the preceding proof. The answer is negative. Indeed, $(\Diamond\Diamond q \wedge \neg q) \check{\vee} (\Diamond p \wedge \neg p)$ is not a Dab^1-formula. So it does not affect the lines unmarked at the first step (the first column of 'marks', viz. signs $-_1$). So it really makes no difference whether the composing logics are $\mathbf{K}_i^u$ and the Ω^i are defined in such a way that $\Omega^i \subset \Omega^{i+1}$ for all i, or whether the composing logics are $\mathbf{T}_i^u$ and the Ω^i are defined as they are in Section 6.2.4.

In conclusion The reader will have noticed that there are many open problems relating to combined adaptive logics. These relate especially to many properties that were proved for adaptive logic in standard format in Chapter 5. It is to be hoped that all this will be cleared up soon. Indeed, many applications, for example formal problem-solving processes require combined adaptive logics. The most urgent task seems to be the study of ordered fusions. These have all the required metatheoretic properties, and this is what matters most, but lack a decent study and even a stable formulation at the object level. I hope that this chapter illustrates the importance of combined adaptive logics and their great promise, and may convince young logicians to tackle the problems.