Chapter 7

More Ways to Handle Inconsistency

This chapter contains some results on variants for inconsistency handling. First, I consider some more regular paraconsistent logics, all interesting for historical reasons and most of them also for systematic reasons. In Section 7.3 I present adaptive logics that have these regular paraconsistent logics as their lower limit. A very different result is presented in Section 7.4. By varying the set of abnormalities and combining two simple inconsistency-adaptive logics that have **CLuN** as their lower limit logic, one obtains consequence sets that are in general richer than those provided by $\mathbf{CLuN}^r$ and $\mathbf{CLuN}^m$. The consequence sets are even very attractive in comparison to those of inconsistency-adaptive logics that have stronger lower limit logics. A further strengthening of $\mathbf{CLuN}^r$ and $\mathbf{CLuN}^m$ is presented in Section 7.5: objects that are inconsistent with respect to the same properties are identified whenever this does not generate new abnormalities.

7.1 The Need for Variants

The reader may wonder why twenty-four different logics of inductive generalization were presented in Chapter 3, whereas the only inconsistency-adaptive logics that occur in the previous pages are $\mathbf{CLuN}^r$ and $\mathbf{CLuN}^m$. When, somewhere in 1980, I wrote [Bat89] in which $\mathbf{CLuN}^r$ was first presented, I had the impression (and argued) that no other paraconsistent lower limit logic would deliver more adequate results. This was a mistake. For one thing, the Minimal Abnormality strategy was unknown in those days—its semantics was first presented in [Bat86a] and the matter was only clarified in [Bat99b]. Moreover, it became clear that someone may have reasons to choose a different paraconsistent logic as the lower limit. In the context of mathematical theories, for example, one may have good reasons to require that Replacement of Identicals holds unconditionally. A different reason will be adduced by a dialetheist like Graham Priest, for whom **LP** is the true logic and hence the standard of deduction. For Priest the only sensible inconsistency-adaptive logics have **LP** as their lower limit. These adaptive logics are all *ampliative* for him, because they lead to a richer consequence set than his standard of deduction—see also Section 1.1.

In this chapter, variants for $\mathbf{CLuN}^r$ and $\mathbf{CLuN}^m$ will be presented. Before doing so, I introduce some more paraconsistent logics.

7.2 Some More Regular Paraconsistent Logics

There are infinitely many regular paraconsistent logics, all stronger than **CLuN** and weaker than **CL**. I shall only mention a few that are important either for systematic reasons or for historical reasons. All these logics are arrived at by enriching the negation of **CLuN**. Some of them are strictly paraconsistent, which means that they validate no instance of Ex Falso Quodlibet. Some are maximally paraconsistent, which means that every extension of them is either **CL** or the trivial logic **Tr**.

Some enrichments cause **CLuN** to collapse into **CL**. This is the case when one of the following is added as an axiom schema to **CLuN**: $(\neg A \vee B) \supset (A \supset B)$, $((A \vee B) \wedge \neg A) \supset B$, $((\neg A \supset B) \wedge \neg B) \supset A$, $(\neg A \supset B) \supset (\neg B \supset A)$, and $(\neg A \supset B) \supset ((\neg A \supset \neg B) \supset A)$. All of the following are theorems of some regular paraconsistent logics: $\neg(A \wedge \neg A)$, $\neg\neg(A \vee \neg A)$, $\neg A \supset (\neg\neg A \supset B)$, $(A \wedge B) \supset (\neg(A \wedge B) \supset C)$, $\neg\neg A \supset A$, $\neg(A \supset B) \supset (A \wedge \neg B)$, $\neg(A \wedge B) \supset (\neg A \vee \neg B)$, $\neg(A \vee B) \supset (\neg A \wedge \neg B)$, $\neg\forall x A \supset \exists x \neg A$, $\neg\exists x A \supset \forall x \neg A$, and the converse of the last six formulas. Adding certain sets of these to **CLuN** results in **CL**. Note that some of these formulas are special cases of Ex Falso Quodlibet, whereas others introduce negation properties that are not related to Ex Falso Quodlibet.

A maximal paraconsistent logic that is not strictly paraconsistent is the predicative version of Ayda Arruda's Vasil'ev system from [Arr77], which I shall call **CLuNv**. The propositional version is obtained by adding to propositional **CLuN** the axiom schema $A \supset (\neg A \supset B)$ with the proviso $A \notin \mathcal{S}$. The natural upgrading to the predicative level is obtained by extending **CLuN** with $A \supset (\neg A \supset B)$, restricted to $A \notin \mathcal{F}^p_s$.[1] To obtain the **CLuNv**-semantics, replace the clause for negation, C¬, in the **CLuN**-semantics by the following two clauses.

C¬1 where $A \in \mathcal{F}^p_s$, $v_M(\neg A) = 1$ iff $v_M(A) = 0$ or $v(\neg A) = 1$
C¬2 where $A \notin \mathcal{F}^p_s$, $v_M(\neg A) = 1$ iff $v_M(A) = 0$

Like all maximal paraconsistent logics, **CLuNv** has a three-valued semantics in which all logical symbols are truth-functions. The interested reader may easily devise this by upgrading [Bat80] to the predicative level, using the techniques from [BDC04].

A maximal paraconsistent logic that is strictly paraconsistent is the logic which in Ghent is called **CLuNs** because Kurt Schütte presented its propositional version in [Sch60]. This logic, which is truth-functional with respect to a three-valued semantics, was rediscovered by many and is probably the most popular paraconsistent logic—see [BDC04] for a study and references. I here present a particular version of it, which validates Replacement of Identicals.

An axiom system for (this version of) **CLuNs** is obtained from that for **CLuN** by removing the restriction in A=2—thus returning to the unrestricted **CL**-version of A=2—and adding the following negation reducing axioms: $\neg\neg A \equiv$

[1] It is obviously possible to avoid restricted axiom schemas, as Arruda does herself, by adding $\neg A \supset (\neg\neg A \supset B)$, $(A \supset C) \supset (\neg(A \supset C) \supset B)$, etc.

A, $\neg(A \supset B) \equiv (A \wedge \neg B)$, $\neg(A \wedge B) \equiv (\neg A \vee \neg B)$, $\neg(A \vee B) \equiv (\neg A \wedge \neg B)$, $\neg(A \equiv B) \equiv ((A \vee B) \wedge (\neg A \vee \neg B))$, $\neg\forall\alpha A \equiv \exists\alpha\neg A$, and $\neg\exists\alpha A \equiv \forall\alpha\neg A$.

The inexperienced reader should be warned that Replacement of Equivalents is not validated by **CLuNs** except where it occurs outside the scope of a negation. Thus $\vdash_{\mathbf{CLuNs}} \neg(A \supset B) \equiv (A \wedge \neg B)$ and hence $\vdash_{\mathbf{CLuNs}} \neg(A \supset B) \equiv \neg\neg(A \wedge \neg B)$, but $\nvdash_{\mathbf{CLuNs}} (A \supset B) \equiv \neg(A \wedge \neg B)$ and $\nvdash_{\mathbf{CLuNs}} \neg\neg(A \supset B) \equiv \neg(A \wedge \neg B)$.

CLuNs-models are like **CLuN**-models (and **CL**-models) but have a different valuation function. In view of Replacement of Identicals, we need some preparation. Considering the same models as for **CL**, we define equivalence classes over $\mathcal{W}^p_{\mathcal{O}}$, the set of closed primitive formulas of $\mathcal{L}_{\mathcal{O}}$. This means that we define, for every $A \in \mathcal{W}^p_{\mathcal{O}}$, a set $[\![A]\!]$ of formulas. In the present case, the equivalence classes depend on the **CL**-model and are defined by: (i) $A \in [\![A]\!]$ and (ii) if A is $B(\alpha)$, with $\alpha \in \mathcal{C} \cup \mathcal{O}$, and $v(\beta) = v(\alpha)$, then $B(\beta) \in [\![A]\!]$. Note that $[\![A]\!] = \{A\}$ if $A \in \mathcal{S}$.

A **CLuNs**-model is a couple $M = \langle D, v\rangle$ as for **CL** and **CLuN**. The valuation function $v_M \colon \mathcal{W}_{\mathcal{O}} \to \{0, 1\}$ determined by M is defined as follows:

C$\mathcal{S}$	where $A \in \mathcal{S}$, $v_M(A) = v(A)$
C$\mathcal{P}^r$	$v_M(\pi^r\alpha_1 \ldots \alpha_r) = 1$ iff $\langle v(\alpha_1), \ldots, v(\alpha_r)\rangle \in v(\pi^r)$
C=	$v_M(\alpha = \beta) = 1$ iff $v(\alpha) = v(\beta)$
C$\supset$	$v_M(A \supset B) = 1$ iff $v_M(A) = 0$ or $v_M(B) = 1$
C$\wedge$	$v_M(A \wedge B) = 1$ iff $v_M(A) = 1$ and $v_M(B) = 1$
C$\vee$	$v_M(A \vee B) = 1$ iff $v_M(A) = 1$ or $v_M(B) = 1$
C$\equiv$	$v_M(A \equiv B) = 1$ iff $v_M(A) = v_M(B)$
C$\forall$	$v_M(\forall\alpha A(\alpha)) = 1$ iff $\{v_M(A(\beta)) \mid \beta \in \mathcal{C} \cup \mathcal{O}\} = \{1\}$
C$\exists$	$v_M(\exists\alpha A(\alpha)) = 1$ iff $1 \in \{v_M(A(\beta)) \mid \beta \in \mathcal{C} \cup \mathcal{O}\}$
C$\neg$	where $A \in \mathcal{W}^p_{\mathcal{O}}$, $v_M(\neg A) = 1$ iff $v_M(A) = 0$ or $v(\neg B) = 1$ for a $B \in [\![A]\!]$.
C$\neg\neg$	$v_M(\neg\neg A) = v_M(A)$
C$\neg\supset$	$v_M(\neg(A \supset B)) = v_M(A \wedge \neg B)$
C$\neg\wedge$	$v_M(\neg(A \wedge B)) = v_M(\neg A \vee \neg B)$
C$\neg\vee$	$v_M(\neg(A \vee B)) = v_M(\neg A \wedge \neg B)$
C$\neg\equiv$	$v_M(\neg(A \equiv B)) = v_M((A \vee B) \wedge (\neg A \vee \neg B))$
C$\neg\forall$	$v_M(\neg\forall\alpha A(\alpha)) = v_M(\exists\alpha\neg A(\alpha))$
C$\neg\exists$	$v_M(\neg\exists\alpha A(\alpha)) = v_M(\forall\alpha\neg A(\alpha))$

$M \Vdash A$ iff $v_M(A) = 1$. In view of Section 1.6, this semantics defines $\Gamma \vDash_{\mathbf{CLuNs}} A$ and $\vDash_{\mathbf{CLuNs}} A$ for all $\Gamma \subseteq \mathcal{W}_s$ and $A \in \mathcal{W}_s$. Theorems 2.2.2–2.2.5 and Lemma 2.2.1 are easily adjusted to **CLuNs**.[2] As is the case for **CLuN**, the consistent **CLuNs**-models form a semantics for **CL**.

The above semantics concerns the language $\mathcal{L}_s$. It is easily upgraded to $\mathcal{L}_S$ along the lines of Section 2.5. Completeness and soundness are easily proved by varying on the proofs of Theorem 1.7.4 and of Lemmas 1.7.1 and 1.7.2.

The logic **CLuNs** has a nice property which is expressed by the following lemma. By a $\vee$-$\wedge$-function C of a set of formulas $\{A_1, \ldots A_n\}$ I shall mean a formula composed by members of $\{A_1, \ldots A_n\}$ concatenated by the operators $\vee$ and $\wedge$. Thus $p \vee (q \wedge r)$, $p \vee (q \vee r)$, $p \wedge (q \wedge r)$, $p \wedge (q \vee r)$, $q \vee (p \wedge r)$, ... are $\vee$-$\wedge$-functions of $\{p, q, r\}$.

[2] If in clause C$\neg$ the phrase "for a $B \in [\![A]\!]$" is replaced by "for all $B \in [\![A]\!]$", we obtain a different but equivalent semantics. An indeterministic semantics for **CLuNs** is obtained by extending the indeterministic semantics for **CLuN** with the clauses C$\neg\neg$–C$\neg\exists$.

Lemma 7.2.1 *If* $\Gamma \vdash_{\mathbf{CLuNs}} \exists(A \wedge \neg A)$*, then there are* $B_1, \ldots, B_n \in \mathcal{F}_s^p$ *and there is a* $\vee$-$\wedge$*-function* C *of* $\exists(B_1 \wedge \neg B_1), \ldots, \exists(B_n \wedge \neg B_n)$ *such that* $\Gamma \vdash_{\mathbf{CLuNs}} C$ *(Contradiction Reduction Property).*

Proof. The proof proceeds by an obvious induction on the complexity of A, where the complexity of A is the number of logical symbols different from $=$ that occur in A. The basis is that $A \in \mathcal{F}_s^p$, whence we are done. The induction hypothesis is that the lemma holds whenever the complexity of A is n. We prove that the lemma holds whenever the complexity of A is $n+1$.

Case 1: A is $\neg B$. So $\exists(A \wedge \neg A)$ is $\exists(\neg B \wedge \neg\neg B)$. This is justified by $\exists(\neg B \wedge \neg\neg B) \vdash_{\mathbf{CLuNs}} \exists(B \wedge \neg B)$.

Case 2: A is $(B \supset C)$. So $\exists(A \wedge \neg A)$ is $\exists((B \supset C) \wedge \neg(B \supset C))$. This is justified by $\exists((B \supset C) \wedge \neg(B \supset C)) \vdash_{\mathbf{CLuNs}} \exists(B \wedge \neg B) \vee \exists(C \wedge \neg C)$.

The other cases are safely left to the reader. ■

Graham Priest's **LP** (see for example [Pri79, Pri06]) is obtained from **CLuNs** by removing $\supset$ and $\equiv$ from the language, and reintroducing them by the definitions $A \supset B =_{df} \neg A \vee B$ and $A \equiv B =_{df} (A \supset B) \wedge (B \supset A)$—both connectives are non-detachable. Obtaining the semantics is obvious. In the absence of a detachable implication, the axiomatization consists of the axiom scheme $A \vee \neg A$ and a set of rules.

The $\mathbf{C}_n$ logics of Newton da Costa require some preparation. Let A° abbreviate $\neg(A \wedge \neg A)$. Next, let A^1 abbreviate A°, let A^2 abbreviate $A^{\circ\circ}$, etc., and let $A^{(n)}$ abbreviate $A^1 \wedge A^2 \wedge \ldots \wedge A^n$.[3] Let $A \equiv^c B$ denote that A and B are congruent in the sense of Kleene *or* that one formula results from the other by deleting vacuous quantifiers—Kleene [Kle52, p. 153] summarizes his definition as follows: "two formulas are congruent, if they differ only in their bound variables, and corresponding bound variables are bound by corresponding quantifiers."

Given this, a logic $\mathbf{C}_n$ is obtained by extending **CLuN** with: (i) the axiom schema $\neg\neg A \supset A$, (ii) the rule "if $A \equiv^c B$, then $\vdash A \equiv B$", (iii) the axiom schema $B^{(n)} \supset ((A \supset B) \supset ((A \supset \neg B) \supset \neg A))$, (iv) the axiom schema $(A^{(n)} \wedge B^{(n)}) \supset (A \dagger B)^{(n)}$, where $\dagger \in \{\vee, \wedge, \supset\}$, (v) the axiom schema $\forall x(A(x))^{(n)} \supset (\forall x\, A(x))^{(n)}$, and (vi) the axiom schema $\exists x(A(x))^{(n)} \supset (\exists x\, A(x))^{(n)}$. As appears from (iii), $A^{(n)}$ functions as a consistency statement in that $A, \neg A, A^{(n)} \vdash_{\mathbf{C}_n} B$. It follows that classical negation can be defined within these logics by $\sim A =_{df} \neg A \wedge A^{(n)}$. These logics may also be formulated in terms of a consistency operator as in [CCM07]. Newton da Costa presented $\mathbf{C}_\omega$ as the limit system of the $\mathbf{C}_n$ logics, but $\mathbf{C}_\omega$ is not an extension of **CLuN**.[4] An equally sensible limit—actually a better one according to [CM99]—is **CLuN** extended with axiom schema (i) and rule (ii). Let us call this logic $\mathbf{C}_{\overline{\omega}}$. I shall present a semantics for the $\mathbf{C}_n$ logics in Section 8.6.

The reader may wonder why I write the defined classical negation in the previous paragraph as $\sim$ rather than as $\check{\neg}$. The point is that $\sim$ is definable within the logic, whereas $\check{\neg}$ is external to the logic and is superimposed as

[3]For example A^2 abbreviates $\neg(\neg(A \wedge \neg A) \wedge \neg\neg(A \wedge \neg A))$. While $\neg A \wedge A$ is $\mathbf{C}_1$-equivalent to $A \wedge \neg A$, $\neg(\neg A \wedge A)$ and $\neg(A \wedge \neg A)$ are not $\mathbf{C}_1$-equivalent. Which of the latter two is taken to express the consistency of A in $\mathbf{C}_1$ is a conventional matter.

[4]The logic $\mathbf{C}_\omega$ is obtained by extending positive intuitionistic logic with (i) and (ii) from the text.

explained in Section 4.3. This may sound a slightly theoretical point, but we shall see in Section 7.3 that $\sim$ and $\check{\neg}$ are indeed very different connectives.

The $\mathbf{C}_n$ logics were the first paraconsistent logics ever presented by a direct axiomatization. Earlier, Stanisław Jaśkowski had devised paraconsistent logics (so-called discussive or discursive logics) in terms of modal logics. Only recently, axiomatic systems for the so defined logics (in the language $\mathcal{L}_s$) were obtained, viz. by Janusz Ciuciura in [Ciu], and these paraconsistent logics turn out to be regular.

The idea underlying Jaśkowski's work is to obtain a paraconsistent logic under a translation. One chooses a modal logic, for example **S5** for the logic **D2**,[5] and evaluates formulas A from the propositional fragment of $\mathcal{L}_s$ by evaluating their Jaśkowski-transformation A^d, which is defined by: (i) if $A \in \mathcal{S}$, then $A^d = A$, (ii) $(\neg B)^d = \neg B^d$, (iii) $(B \vee C)^d = B^d \vee C^d$, (iv) $(B \wedge C)^d = B^d \wedge \Diamond C^d$, (v) $(B \supset C)^d = \Diamond B^d \supset C^d$, and (vi) $(B \equiv C)^d = (\Diamond B^d \supset C^d) \wedge \Diamond(\Diamond C^d \supset B^d)$. The logic **D2** is as follows: $\Gamma \vdash_{\mathbf{D2}} A$ iff $\{\Diamond B^d \mid B \in \Gamma\} \vdash_{\mathbf{S5}} \Diamond A^d$. Obviously **D2** is paraconsistent: $p, \neg p \nvdash_{\mathbf{D2}} q$ because $\Diamond p, \Diamond \neg p \nvdash_{\mathbf{S5}} \Diamond q$—note also that $p \wedge \neg p \nvdash_{\mathbf{D2}} q$ because $\Diamond(p \wedge \Diamond \neg p) \nvdash_{\mathbf{S5}} \Diamond q$. I refer to the aforementioned [Ciu] for a direct axiomatization of **D2**.

Although **D2** and similar Jaśkowski logics are paraconsistent, they were not meant to handle inconsistent *theories*, but rather to handle opposing viewpoints in discussions—the name 'discussive' refers to that. Intuitively, the different worlds of a modal model may be seen as representing the position of different participants in a discussion, with the 'real' world, w_0, representing the speaker's position.

The logic **AN** was developed by Joke Meheus in [Meh99b] and published in [Meh00]. It is a very peculiar logic because, although it is paraconsistent, this is realized by weakening disjunction. It shares this property with some logics presented by Paul Weingartner, for example in [Wei00]. There is a difference, however. Weingartner's logics are formulated as **CL** with a 'filter' defined on it—the filter selects certain **CL**-theorems and certain **CL**-derivability statements as appropriate and dismisses others. In contradistinction to this, **AN** is formulated as a distinct logic.

The idea behind **AN** is that one may separate rules of inference in analysing and constructive ones. Thus $\neg(A \vee B)/\neg A \wedge \neg B$ and $A \supset B, \neg B/\neg A$ are analysing, while $A/A \vee B$ and $A/B \supset A$ are constructive. Meheus argues that, in explicating a working scientist's reasoning, it is advisable to rely on a logic that validates all analysing rules at the expense of giving up some constructive rules.

Meheus presents a very elegant three-valued semantics for **AN**, but I shall present the two-valued semantics. Implication and equivalence are handled by the usual explicit definitions $A \supset B =_{df} \neg A \vee B$ and $A \equiv B =_{df} (A \supset B) \wedge (B \supset A)$. For the semantics, a pre-valuation v_M is defined and the valuation V_M is defined in terms of the pre-valuation: $V_M(A) = v_M(B)$, where B is the prenex conjunctive normal form of A. A formula is said to be in prenex conjunctive normal form (PCNF) if it consists of a sequence of quantifiers followed by a formula that is a conjunction of disjunctions of atoms. Thus $\forall x \exists y \forall z((Px \vee \neg Qz \vee Rx) \wedge (Ry \vee Sy) \wedge Qz)$ is in PCNF. A formula A in PCNF is said to be the

[5] A semantics for **S5** is obtained by adding to the **K**-semantics from Section 3.6 the requirement that the relation R is reflexive, symmetric and transitive.

PCNF of a formula B iff B can be obtained from A by application of the above definitions together with relettering,[6] and with the following transformations, in which $\mathsf{A} \approx \mathsf{B}$ expresses the permission to transform a formula by replacing a subformula of the form A by the corresponding form B or *vice versa*, and in which occurrences of † may, in every item, be systematically replaced by $\wedge$ as well as by $\vee$: (i) $(A \dagger B) \approx (B \dagger A)$, (ii) $(A \dagger A) \approx A$, (iii) $\neg\neg A \approx A$, (iv) $\neg(A \wedge B) \approx \neg A \vee \neg B$, (v) $\neg(A \vee B) \approx \neg A \wedge \neg B$, (vi) $\neg\forall\alpha A \approx \exists\alpha\neg A$, (vii) $\neg\exists\alpha A \approx \forall\alpha\neg A$, (viii) $(\forall\alpha A \dagger B) \approx \forall\alpha(A \dagger B)$, (ix) $(\exists\alpha A \dagger B) \approx \exists\alpha(A \dagger B)$, (x) $\bigwedge(\Gamma \cup \bigvee(\Delta \cup \{B \wedge C\})) \approx \bigwedge(\Gamma \cup \bigvee(\Delta \cup \{B\}) \cup \bigvee(\Delta \cup \{C\}))$, with the proviso, in (viii) and (ix), that α does not occur in B.

Let us now turn to the models. The assignment function is the standard one, as for the **CL**-semantics from Section 1.7. The *pre-valuation* $v_M : \mathcal{W}_\mathcal{O} \to \{0, 1\}$, defined by the model $M = \langle D, v\rangle$, is as for **CL**, except that the clauses for implication and equivalence are removed and that the clauses for negation and disjunction are replaced by the following:

C¬ where $A \in \mathcal{W}^p_\mathcal{O}$, $v_M(\neg A) = 1$ iff $v_M(A) = 0$ or $v(\neg A) = 1$.
C∨ $v_M(A \vee B) = 1$ iff ($v_M(A) = 1$ and $v_M(\neg A) = 0$) or ($v_M(B) = 1$ and $v_M(\neg B) = 0$) or ($v_M(A) = v_M(B) = 1$)
C¬¬ $v_M(\neg\neg A) = v_M(A)$
C¬∧ $v_M(\neg(A \wedge B)) = v_M(\neg A \vee \neg B)$
C¬∨ $v_M(\neg(A \vee B)) = v_M(\neg A \wedge \neg B)$
C¬∀ $v_M(\neg\forall\alpha A(\alpha)) = v_M(\exists\alpha\neg A(\alpha))$
C¬∃ $v_M(\neg\exists\alpha A(\alpha)) = v_M(\forall\alpha\neg A(\alpha))$

$M \Vdash A$ iff $V_M(A) = 1$. In view of Section 1.6, this semantics defines $\Gamma \vDash_{\mathbf{AN}} A$ and $\vDash_{\mathbf{AN}} A$ for all $\Gamma \subseteq \mathcal{W}_s$ and $A \in \mathcal{W}_s$. Theorems 2.2.2–2.2.5 and Lemma 2.2.1 are easily adjusted to **AN**. As is the case for **CLuN**, the consistent **AN**-models form a semantics for **CL**.

We have seen that the *valuation* reduces all formulas to their PCNF, and hence eliminates negations of complex formulas. We nevertheless need the clauses for negations of complex formulas (C¬¬ up to C¬∃) because clause C∨ refers to the pre-valuation value of negations of arbitrarily complex formulas.

The difference between **AN** and **CLuNs** lies with the clause C∨. To understand the idea behind **AN**, it is good to remember the well-known **CL**-proof from A and $\neg A$ to B: from A follows $A \vee B$ by Addition and from $A \vee B$ and $\neg A$ follows B by Disjunctive Syllogism. Nearly all paraconsistent logics invalidate Disjunctive Syllogism. They rely on the idea that, if both A and $\neg A$ may be true, then so are $A \vee B$ and $\neg A$ even if B is false. This obviously presupposes the validity of Addition and **AN** gives up that presupposition. The idea behind **AN** is that all analysing inferences should be retained, including Disjunctive Syllogism, and Addition and other constructive rules should be weakened to make this possible. The outcome is C∨. If A and B are both true, then obviously $A \vee B$ is safe: if Disjunctive Syllogism can be applied, that is if $\neg A$ or $\neg B$ is true, then it leads to a true formula, viz. B or A. If A is true and $\neg A$ false, then $A \vee B$ is safe: if Disjunctive Syllogism can be applied, its minor premise must be $\neg B$ and its conclusion A, which is true anyway. Similarly for the case in which B is true and $\neg B$ is false.

[6] Relettering means replacing in a subformula of the form $\forall\alpha A(\alpha)$ (in the quantifier as well as in its scope) every occurrence of α by another variable that does not occur in $\forall\alpha A(\alpha)$.

Although this is not a book on paraconsistency, I consider the matter sufficiently fascinating to add a further result, first mentioned (in a slightly different form) in [Bat00b]. Consider the defined connective $\supset$ in **AN**. In **CLuN** and in most other paraconsistent logics mentioned in this book, $\supset$ warrants Modus Ponens but not Modus Tollens.[7] In **AN**, however, $A \supset B$ warrants both Modus Ponens and Modus Tollens.

By taking $\supset$ as primitive, giving it the meaning it has in the above **AN**-semantics, a nice reformulation is arrived at. Disjunction and conjunction are defined from the implication: $A \vee B =_{df} \neg A \supset B$ (relevant logicians would call this fission) and $A \wedge B =_{df} \neg(A \supset \neg B)$ (relevant logicians would call this fusion). The reformulation is a semantics that looks precisely like the previous one, except that the pre-valuation $v_M \colon \mathcal{W}_\mathcal{O} \to \{0,1\}$ fulfils the following requirements:

C$\mathcal{S}$	where $A \in \mathcal{S}$, $v_M(A) = v(A)$
C$\mathcal{P}^r$	$v_M(\pi^r\alpha_1 \ldots \alpha_r) = 1$ iff $\langle v(\alpha_1), \ldots, v(\alpha_r)\rangle \in v(\pi^r)$
C=	$v_M(\alpha = \beta) = 1$ iff $v(\alpha) = v(\beta)$
C$\neg$	where $A \in \mathcal{W}^p_\mathcal{O}$, $v_M(\neg A) = 1$ iff $v_M(A) = 0$ or $v(\neg A) = 1$.
C$\supset$	$v_M(A \supset B) = 1$ iff ($v_M(A) = 0$ or $v_M(B) = 1$) and ($v_M(\neg B) = 0$ or $v_M(\neg A) = 1$)
C$\forall$	$v_M(\forall\alpha A(\alpha)) = 1$ iff $\{v_M(A(\beta)) \mid \beta \in \mathcal{C} \cup \mathcal{O}\} = \{1\}$
C$\exists$	$v_M(\exists\alpha A(\alpha)) = 1$ iff $1 \in \{v_M(A(\beta)) \mid \beta \in \mathcal{C} \cup \mathcal{O}\}$
C$\neg\neg$	$v_M(\neg\neg A) = v_M(A)$
C$\neg\supset$	$v_M(\neg(A \supset B)) = 1$ iff $v_M(A) = v_M(\neg B) = 1$
C$\neg\forall$	$v_M(\neg\forall\alpha A(\alpha)) = v_M(\exists\alpha\neg A(\alpha))$
C$\neg\exists$	$v_M(\neg\exists\alpha A(\alpha)) = v_M(\forall\alpha\neg A(\alpha))$

Of course, this is only the pre-valuation. In order to obtain the valuation, we still have to give every formula the pre-valuation value of its PCNF. It is obviously possible to replace the PCNF by a prenex form in which occur only primitive symbols of the present semantics. This, however, is a technical matter which should not concern us here.

7.3 The Corresponding Adaptive Logics

In view of the way in which $\mathbf{CLuN}^r$ and $\mathbf{CLuN}^m$ were defined and in view of the standard format, to define an inconsistency-adaptive logic that has **CLuNs** as its lower limit logic seems an obvious exercise: just replace **CLuN** by **CLuNs** in the triples defining $\mathbf{CLuN}^r$ and $\mathbf{CLuN}^m$:

(1) *lower limit logic*: **CLuNs**,
(2) *set of abnormalities*: $\Omega = \{\exists(A \wedge \neg A) \mid A \in \mathcal{F}_s\}$,
(3) *adaptive strategy*: Reliability, respectively Minimal Abnormality.

These logics will be called $\mathbf{CLuNs}^{f,r}$ and $\mathbf{CLuNs}^{f,m}$. The names may be unexpected, but the logics will turn out to have unexpected properties themselves. The upper limit of both adaptive logics is obviously **CL**.

Suppose that $\Gamma \subseteq \mathcal{W}_s$ is a consistent premise set. So no disjunction of abnormalities is **CLuNs**-derivable from it.[8] So no line of a $\mathbf{CLuNs}^{f,r}$-proof from

[7] For example, there are **CLuN**-models that verify B as well as $\neg B$, and hence also $A \supset B$, but falsify $\neg A$.

[8] This is an obvious consequence of the fact that $Cn^{\mathcal{W}_s}_{\mathbf{CLuNs}}(\Gamma) \subseteq Cn^{\mathcal{W}_s}_{\mathbf{CL}}(\Gamma)$ for all Γ.

Γ will ever be marked, whence all **CL**-consequences of Γ are also $\mathbf{CLuNs}^{f,r}$-consequences of Γ. Semantic considerations offer obviously the same outcome. $U(\Gamma) = \emptyset$ and $\Phi(\Gamma) = \{\emptyset\}$, whence the set of $\mathbf{CLuNs}^{f,r}$-models of Γ is identical to the set of $\mathbf{CLuNs}^{f,m}$-models of Γ as well as to the set of consistent **CLuNs**-models of Γ, which may be identified with the **CL**-models of Γ—remember that Lemma 2.2.1 may be adjusted to **CLuNs**.

Some examples will help us to understand the situation for inconsistent premise sets.

Example 7.1 $\Gamma_1 = \{p, \neg(p \vee q), q \vee r\}$. In order to make all aspects clear, let us first attempt to derive r from this premise set by $\mathbf{CLuN}^r$. A typical attempted proof would proceed as follows:

1	p	Premise	$\emptyset$	
2	$\neg(p \vee q)$	Premise	$\emptyset$	
3	$q \vee r$	Premise	$\emptyset$	
4	$\neg p$	2; RC	$\{(p \vee q) \wedge \neg(p \vee q)\}$	$\checkmark^7$
5	$\neg q$	2; RC	$\{(p \vee q) \wedge \neg(p \vee q)\}$	$\checkmark^7$
6	r	3, 5; RC	$\{(p \vee q) \wedge \neg(p \vee q), q \wedge \neg q\}$	$\checkmark^7$
7	$(p \vee q) \wedge \neg(p \vee q)$	1, 2; RU	$\emptyset$	

As r is only derivable from the premises on a condition that is a superset of the condition of line 6, r is not finally $\mathbf{CLuN}^r$-derivable from Γ_1—it is not finally $\mathbf{CLuN}^m$-derivable from Γ_1 either. Seen from the viewpoint of **CLuN**, this is sensible. The premises require $p \vee q$ to be true because they require p to be true. That $\neg(p \vee q)$ is also a premise leads to the inconsistency $(p \vee q) \wedge \neg(p \vee q)$, but does not provide a reason to consider $\neg q$ as true.

The viewpoint of **CLuNs** is different: asserting $\neg(p \vee q)$ literally comes to asserting $\neg p \wedge \neg q$. Indeed, $\vdash_{\mathbf{CLuNs}} \neg(p \vee q) \equiv (\neg p \wedge \neg q)$. So let us check what happens if $\mathbf{CLuNs}^{f,r}$ is applied to Γ_1.

1	p	Premise	$\emptyset$
2	$\neg(p \vee q)$	Premise	$\emptyset$
3	$q \vee r$	Premise	$\emptyset$
4	$\neg p$	2; RU	$\emptyset$
5	$\neg q$	2; RU	$\emptyset$
6	r	3, 5; RC	$\{q \wedge \neg q\}$

Lines 3 and 5 have an empty condition. As a result, 6 has a singleton condition and $q \wedge \neg q$ seems to be reliable: apparently $q \wedge \neg q \notin U(\Gamma_1)$. However, this is only apparently so. I repeat the proof from 6 on:

6	r	3, 5; RC	$\{q \wedge \neg q\}$	$\checkmark^9$
7	$(q \wedge \neg q) \check{\vee} r$	3, 5; RU	$\emptyset$	
8	$(q \wedge \neg q) \check{\vee} (r \wedge p)$	1, 7; RU	$\emptyset$	
9	$(q \wedge \neg q) \check{\vee} ((r \wedge p) \wedge \neg(r \wedge p))$	4, 8; RU	$\emptyset$	

At stage 9 of the proof, line 6 is marked. The reader may easily verify, for example by means of a tableau method, that neither $q \wedge \neg q$ nor $(r \wedge p) \wedge \neg(r \wedge p)$ is a **CLuNs**-consequence of Γ_1, whence 9 is a minimal *Dab*-consequence of Γ_1. So line 6 will remain marked in every extension of the proof—actually r is not a final $\mathbf{CLuNs}^{f,r}$-consequence of Γ_1.

For all we have seen at this point, $\mathbf{CLuNs}^{f,r}$ does not seem to be doing worse than $\mathbf{CLuN}^r$. However, it does, as the following example illustrates.

Example 7.2 Let $\Gamma_2 = \{p, \neg p, \neg q, q \vee r\}$. Consider again a $\mathbf{CLuN}^r$-proof:

1	p	Premise	$\emptyset$
2	$\neg p$	Premise	$\emptyset$
3	$\neg q$	Premise	$\emptyset$
4	$q \vee r$	Premise	$\emptyset$
5	$p \wedge \neg p$	1, 2; RU	$\emptyset$
6	r	3, 4; RC	$\{q \wedge \neg q\}$

The reader may easily verify that $q \wedge \neg q$ is not a disjunct of a minimal *Dab*-consequence of Γ_2—the only minimal *Dab*-consequence of Γ_2 is $p \wedge \neg p$. So r is finally derived at line 6.

Let us now consider the corresponding $\mathbf{CLuNs}^{f,r}$-proof. The first 6 lines are identical to those of the $\mathbf{CLuN}^r$-proof. I repeat only line 6 and then continue:

6	r	3, 4; RC	$\{q \wedge \neg q\}$	$\checkmark^8$
7	$(q \wedge \neg q) \check{\vee} r$	3, 4; RU	$\emptyset$	
8	$(q \wedge \neg q) \check{\vee} ((r \wedge p) \wedge \neg(r \wedge p))$	1, 2, 7; RU	$\emptyset$	

So line 6 is marked. Moreover, neither disjunct of 8 is **CLuNs**-derivable from Γ_2. So r is not a final $\mathbf{CLuNs}^{f,r}$-consequence of Γ_2.

Line 8 can be obtained in this proof because $r, p, \neg p \vdash_{\mathbf{CLuNs}} (r \wedge p) \wedge (\neg r \vee \neg p)$ and $\neg r \vee \neg p \vdash_{\mathbf{CLuNs}} \neg(r \wedge p)$. As $\neg r \vee \neg p \nvdash_{\mathbf{CLuN}} \neg(r \wedge p)$, line 8 cannot be obtained in the $\mathbf{CLuN}^r$-proof. Precisely because **CLuN** is weaker than **CLuNs**, it does not spread inconsistencies. As a result, inconsistencies are better isolated, less *Dab*-formulas are derivable from a premise set, and hence less lines are marked. This does not mean that adaptive logics that have **CLuN** as their lower limit have in general a richer consequence set than adaptive logics that have **CLuNs** as their lower limit. We have seen, for example, that $\Gamma_1 \vdash_{\mathbf{CLuNs}^{f,r}} \neg q$ whereas $\Gamma_1 \nvdash_{\mathbf{CLuN}^r} \neg q$.

Returning to **CLuNs**, the preceding proofs suggest that $\mathbf{CLuNs}^{f,r}$ and $\mathbf{CLuNs}^{f,m}$ are flip-flop logics. This will indeed turn out to be the case: every line that has a non-empty condition, will be marked in an extension of the proof from an abnormal premise set. In other words, the only final $\mathbf{CLuNs}^{f,r}$-consequences of an abnormal premise set are its **CLuNs**-consequences. Incidentally, both preceding $\mathbf{CLuNs}^{f,r}$-proofs are identical to the corresponding $\mathbf{CLuNs}^{f,m}$-proofs.

We have two urgent questions at this point. First, are $\mathbf{CLuNs}^{f,r}$ and $\mathbf{CLuNs}^{f,m}$ indeed flip-flop logics? Next, is it possible to define adaptive logics that are not flip-flops and have **CLuNs** as their lower limit logic? Let me begin with a precise definition.

Definition 7.3.1 *An adaptive logic* **AL** *is a flip-flop logic iff* $Cn_{\mathbf{AL}}(\Gamma) = Cn_{\mathbf{ULL}}(\Gamma)$ *for all normal* Γ *and* $Cn_{\mathbf{AL}}(\Gamma) = Cn_{\mathbf{LLL}}(\Gamma)$ *for all abnormal* Γ.

I first show that $\mathbf{CLuNs}^{f,m}$ is a flip-flop logic. In view of the soundness and completeness of **CLuNs** with respect to its semantics (the adjusted Theorems 2.2.4 and 2.2.5) and the soundness and completeness of $\mathbf{CLuNs}^m$ with respect to its semantics (Theorem 5.3.4), I freely move from the derivability relation to the semantic consequence relation in the proof of the following theorem.

Theorem 7.3.1 $\mathbf{CLuNs}^{f,m}$ *is a flip-flop logic.*

Proof. If Γ is normal, then $Cn_{\mathbf{CLuNs}^{f,m}}(\Gamma) = Cn_{\mathbf{CL}}(\Gamma)$ in view of item 1 of Theorem 5.6.7. So I have to show that, for all abnormal Γ and for all $A \in \mathcal{W}$, if $\Gamma \vdash_{\mathbf{CLuNs}^{f,m}} A$, then $\Gamma \vdash_{\mathbf{CLuNs}} A$.

Suppose that Γ is an abnormal premise set and that there is an $A \in \mathcal{W}$ such that $\Gamma \vdash_{\mathbf{CLuNs}^{f,m}} A$ whereas $\Gamma \nvdash_{\mathbf{CLuNs}} A$. So $M \Vdash A$ holds for every minimal abnormal **CLuNs**-model M of Γ, whereas a **CLuNs**-model of Γ, call it M_1, is such that $M_1 \nVdash A$. By Strong Reassurance (Theorem 5.2.1) there is a minimal abnormal **CLuNs**-model of Γ, call it M_2, such that $Ab(M_2) \subset Ab(M_1)$.

As Γ is abnormal by the supposition, there is a B such that $M_2 \Vdash \exists(B \wedge \neg B)$, whence $M_1 \Vdash \exists(B \wedge \neg B)$. It follows that $M_2 \Vdash \exists((A \wedge B) \wedge \neg(A \wedge B))$ whereas $M_1 \nVdash \exists((A \wedge B) \wedge \neg(A \wedge B))$. But this contradicts $Ab(M_2) \subset Ab(M_1)$. ■

If Γ is normal, $Cn_{\mathbf{CLuNs}^{f,r}}(\Gamma) = Cn_{\mathbf{CL}}(\Gamma)$ by item 1 of Theorem 5.6.7. By Theorem 7.3.1, $Cn_{\mathbf{CLuNs}}(\Gamma) = Cn_{\mathbf{CLuNs}^{f,m}}(\Gamma)$ if Γ is abnormal. Moreover, $Cn_{\mathbf{CLuNs}}(\Gamma) \subseteq Cn_{\mathbf{CLuNs}^{f,r}}(\Gamma) \subseteq Cn_{\mathbf{CLuNs}^{f,m}}(\Gamma)$ in view of Corollary 5.3.3. So we established the following corollary.

Corollary 7.3.1 $\mathbf{CLuNs}^{f,r}$ *is a flip-flop logic.*

This being settled, what about the second question: Is it possible to define decent adaptive logics that are not flip-flops and have **CLuNs** as their lower limit logic? By "decent adaptive logics" I mean logics that are not border cases in the sense of Section 5.9, in other words logics that do not have static proofs. If I meant only this, matters would be easy. One might just replace, in the definition of $\mathbf{CLuNs}^{f,r}$ and $\mathbf{CLuNs}^{f,m}$, the set of abnormalities Ω by $\{p \wedge \neg p\}$. This provably leads to an adaptive logic that is not a flip-flop and does not have static proofs. But I obviously mean something more. The adaptive logics should have **CL** as their upper limit logic. This means that the adaptive logics, which should not be flip-flops, should deliver the same consequence set as **CL** for all normal premise sets.

The answer to the question is positive and devising such logics is actually very simple. Let $\mathcal{F}^p_s$ be the set of primitive formulas of the language $\mathcal{L}_s$ from the **CLuNs**-semantics. The following triples define $\mathbf{CLuNs}^r$ and $\mathbf{CLuNs}^m$:

(1) *lower limit logic*: **CLuNs**,
(2) *set of abnormalities*: $\Omega = \{\exists(A \wedge \neg A) \mid A \in \mathcal{F}^p_s\}$,
(3) *adaptive strategy*: Reliability, respectively Minimal Abnormality.

Let us return to the proof examples from the present section to see what the new logics make of them. The $\mathbf{CLuNs}^r$-proof from $\Gamma_1 = \{p, \neg(p \vee q), q \vee r\}$ proceeds as follows.

1	p	Premise	$\emptyset$
2	$\neg(p \vee q)$	Premise	$\emptyset$
3	$q \vee r$	Premise	$\emptyset$
4	$\neg p$	2; RU	$\emptyset$
5	$\neg q$	2; RU	$\emptyset$
6	r	3, 5; RC	$\{q \wedge \neg q\}$
7	$(q \wedge \neg q) \check{\vee} r$	3, 5; RU	$\emptyset$
8	$(q \wedge \neg q) \check{\vee} (r \wedge p)$	1, 7; RU	$\emptyset$
9	$(q \wedge \neg q) \check{\vee} ((r \wedge p) \wedge \neg(r \wedge p))$	4, 8; RU	$\emptyset$

Line 6 is unmarked because 9 is not a *Dab*-formula of $\mathbf{CLuNs}^r$. Indeed, the formula $(r \wedge p) \wedge \neg(r \wedge p)$ is not a $\mathbf{CLuNs}^r$-abnormality because $r \wedge p \notin \mathcal{F}_s^p$.

The situation is similar in the $\mathbf{CLuNs}^r$-proof from $\Gamma_2 = \{p, \neg p, \neg q, q \vee r\}$:

1	p	Premise	$\emptyset$
2	$\neg p$	Premise	$\emptyset$
3	$\neg q$	Premise	$\emptyset$
4	$q \vee r$	Premise	$\emptyset$
5	$p \wedge \neg p$	1, 2; RU	$\emptyset$
6	r	3, 4; RC	$\{q \wedge \neg q\}$
7	$(q \wedge \neg q) \check{\vee} r$	3, 4; RU	$\emptyset$
8	$(q \wedge \neg q) \check{\vee} ((r \wedge p) \wedge \neg(r \wedge p))$	1, 2, 7; RU	$\emptyset$

Here too line 6 is unmarked because 8 is not a *Dab*-formula of $\mathbf{CLuNs}^r$—the formula $(r \wedge p) \wedge \neg(r \wedge p)$ is not a $\mathbf{CLuNs}^r$-abnormality because $r \wedge p \notin \mathcal{F}_s^p$.

That $\mathbf{CLuNs}^r$ and $\mathbf{CLuNs}^m$ are not flip-flops will be shown in Section 8.6. To show that the upper limit of these logics is **CL** is easy. In view of the Contradiction Reduction Property (Lemma 7.2.1), tying members of Ω to triviality comes to tying all inconsistent formulas to triviality. Some differences between the consequence sets assigned by $\mathbf{CLuNs}^r$ and $\mathbf{CLuNs}^m$ on the one hand and $\mathbf{CLuN}^r$ and $\mathbf{CLuN}^m$ on the other hand will be presented in Section 7.4.

Consider the adaptive logics $\mathbf{LP}^{f,r}$ and $\mathbf{LP}^{f,m}$, which are just like $\mathbf{CLuNs}^{f,r}$ and $\mathbf{CLuNs}^{f,m}$ except that their lower limit logic is **LP**. The proof of Theorem 7.3.1 is easily adjusted to show that $\mathbf{LP}^{f,m}$ is a flip-flop and $\mathbf{LP}^{f,r}$ is a flip-flop because of the reasoning that led to Corollary 7.3.1. Decent adaptive logics that have **LP** as their lower limit are easily obtained. So let $\mathbf{LP}^r$ and $\mathbf{LP}^m$ be defined as $\mathbf{CLuNs}^r$ and $\mathbf{CLuNs}^m$, except that the lower limit logic **CLuNs** is replaced by **LP**. These logics are not flip-flops, as we shall see in Section 8.6.

To devise adaptive logics that have **CLuNv** as their lower limit is rather straightforward. Suppose that one combines **CLuNv** (as the lower limit logic) with the set of abnormalities $\{\exists(A \wedge \neg A) \mid A \in \mathcal{F}_s\}$ and with either reliability or Minimal Abnormality. In view of Fact 5.9.8, the adaptive logic remains unchanged if one filters out all abnormalities $\exists(A \wedge \neg A)$ for which $A \notin \mathcal{F}_s^p$—all those abnormalities are **CLuNv**-falsehoods. So the straightforward adaptive logics that have **CLuNv** as their lower limit are $\mathbf{CLuNv}^r$ and $\mathbf{CLuNv}^m$, obtained by combining **CLuNv** and $\Omega = \{\exists(A \wedge \neg A) \mid A \in \mathcal{F}_s^p\}$ with Reliability and Minimal Abnormality respectively. These logics can be shown not to be flip-flops (by the method of Section 8.6).

Let us now turn to the historically important set of $\mathbf{C}_n$ logics. Remember that they form a hierarchy and that consistency (and classical negation) can be expressed in all of them, except for $\mathbf{C}_{\overline{\omega}}$, by different, increasingly complex, formulas. With hindsight, it seems as if these systems were designed to execute a certain stratagem. If T has **CL** as its underlying logic, but turns out to be trivial, **CL** is replaced by $\mathbf{C}_1$. This results in a theory, T_1, which is too weak in comparison to the original intention, as was explained in Section 2.1. However, the researcher may add consistency statements to T_1, which have the form $A^{(1)}$, in other words $\neg(A \wedge \neg A)$. Remember that $A \wedge A^{(1)}$ entails $\sim\neg A$ whereas $\neg A \wedge A^{(1)}$ entails $\sim A$. The consistency statements strengthen T_1 into T_1', but adding them involves a danger: T_1' may turn out to be trivial. This will happen if a consistency statement $A^{(1)}$ was added while both A and $\neg A$

turn out to be $\mathbf{C}_1$-derivable from T_1'; or also if $A^{(1)}$ and $B^{(1)}$ were added while $(A \wedge \neg A) \check{\vee} (B \wedge \neg B)$ turns out to be $\mathbf{C}_1$-derivable from T_1'. Non-triviality is regained if $\mathbf{C}_1$ is replaced by $\mathbf{C}_2$.[9] Indeed, the formulas $A^{(1)}$ are not consistency statements in $\mathbf{C}_2$.[10] Let the resulting theory be T_2. In view of the insights gained while tinkering with T_1, one may tinker again, adding consistency statements appropriate for $\mathbf{C}_2$, which have the form $A^{(2)}$, in order to build the desired theory T_2'. If triviality would pop up again, one replaces $\mathbf{C}_2$ by $\mathbf{C}_3$. And so on.

I mentioned the stratagem because the adaptive logics which I shall introduce solve this problem for you. To be more precise, the logic $\mathbf{C_1}^m$, for example, will add to T_1 all consistency statements that can be justifiedly added in view of logical considerations. Relying on non-logical grounds, more consistency-statements may be added and an adaptive logic will guide such additions.[11] For now let us consider the adaptive logics that have $\mathbf{C}_n$ logics as their lower limit.

These adaptive logics were only formulated recently. There was a reason for this delay. Some inconsistencies are connected within every $\mathbf{C}_n$ and it was not clear in which way the set of abnormalities has to be devised in order to avoid flip-flops. This problem is meanwhile solved, as we shall see in Section 8.6, and the logics defined in the next paragraph are not flip-flops.

Let, for every n, $\mathbf{C}_n{}^r$ and $\mathbf{C}_n{}^m$ be the adaptive logics defined by the lower limit $\mathbf{C}_n$, the set of abnormalities $\Omega = \{\exists(A \wedge \neg A) \mid A \in \mathcal{F}_s\}$, and Reliability or Minimal Abnormality respectively.

Let us consider a simple proof. Let $\Gamma_3 = \{\neg\neg p, \neg q, \neg p, p \vee r, q \vee s\}$.

1	$\neg\neg p$	Prem	$\emptyset$	
2	$\neg q$	Prem	$\emptyset$	
3	$\neg p$	Prem	$\emptyset$	
4	$p \vee r$	Prem	$\emptyset$	
5	$q \vee s$	Prem	$\emptyset$	
6	r	3, 4; RC	$\{p \wedge \neg p\}$	✓[8]
7	s	2, 5; RC	$\{q \wedge \neg q\}$	
8	$p \wedge \neg p$	1, 3; RU	$\emptyset$	

Whether the lower limit logic is $\mathbf{C}_1$, $\mathbf{C}_{\overline{\omega}}$, or any $\mathbf{C}_n$, the proof remains identical—a matter to which I return in the sequel. In whichever way the proof (from these premises) is extended, the marks of lines 1–8 are stable from this point on; s is a final consequence of the premise set while r is not.

The adaptive logics solve certain problems that arise if the aforementioned stratagem is applied. Thus even the infinite set of consistency statements $\{\neg(A \wedge \neg A) \mid A \in \mathcal{W}\}$ is insufficient to obtain by $\mathbf{C}_n$ all $\mathbf{C}_n{}^m$-final consequences of

[9] An alternative is to withdraw the guilty consistency statements, but the stratagem described in the text makes the hierarchy of logics meaningful.

[10] Moreover, we may take it for granted that consistency statements suitable for $\mathbf{C}_2$, viz. formulas of the form $A^{(2)}$, do not occur in T_1' because A^2 is a $\mathbf{C}_1$-theorem, just like all formulas of the form A^1 are $\mathbf{CL}$-theorems.

[11] The idea is similar to the conjectures introduced in Section 3.7. Abnormalities or negations of abnormalities may be introduced in precisely that way—remember that they will have a certain degree of plausibility. They may also be introduced by the means described under the heading "Preferential selection of abnormalities" in Section 6.2.3. Apart from a reference to methods presented earlier in the same section, a specific method is presented there. Especially this method is suitable for the present context. It is closely linked to ideas underlying da Costa's hierarchy of $\mathbf{C}_n$-logics.

the original premise set—an example is $\{\forall x Px \supset (\exists x \neg Px \supset r)\}$.[12] If a non-recursive set of minimal disjunctions of contradictions[13] is $\mathbf{C}_n$-derivable from the premise set, no recursive set of consistency statements can be added to the original premises to obtain by any $\mathbf{C}_n$ the consequences that ${\mathbf{C}_n}^m$ delivers from the original premise set. So in many cases the adaptive logics do better than anyone can possibly do by following the stratagem.

Let us now return to the feature that all ${\mathbf{C}_n}^m$ deliver the same consequence set for a given premise set. This is obviously related to the fact that the classical symbols are superimposed. As a result, $\check{\neg} q$ is ${\mathbf{C}_n}^m$-derivable from Γ_3 for all n. The theorem $q \check{\vee} \check{\neg} q$ warrants that $\neg q \vdash_{\mathbf{C}_n} \check{\neg} q \check{\vee} (q \wedge \neg q)$, whatever n.

Of course, one may try to stay closer to the structure of the $\mathbf{C}_n$ logics, and derive s from $q \vee s$ and $\neg q$ by first deriving $\sim q$. To clarify the point, consider $\mathbf{C}_2$, in which $\sim q$ is defined as $\neg q \wedge \neg(q \wedge \neg q) \wedge \neg(\neg(q \wedge \neg q) \wedge \neg\neg(q \wedge \neg q))$. Here one sees that $\check{\neg}$ and $\sim$ are indeed two very different logical symbols. As the point is somewhat subtle, let me explain. If $\neg q$ is a premise, one may reason that, in every $\mathbf{C}_2$-model of the premises, either q is true or q is false. The superimposed classical symbols enable one to express this by the theorem $q \check{\vee} \check{\neg} q$. They also enable one to draw from Γ_3 the conclusion that $\check{\neg} q \check{\vee} (q \wedge \neg q)$, in other words that $\check{\neg} q$ obtains or that the abnormality $q \wedge \neg q$ obtains. This reasoning underlies line 7 in the preceding proof. To make this completely clear, let me repeat the previous proof from Γ_3 with the derivation of $\check{\neg} p$ and $\check{\neg} q$ made explicit.

1	$\neg\neg p$	Prem	$\emptyset$	
2	$\neg q$	Prem	$\emptyset$	
3	$\neg p$	Prem	$\emptyset$	
4	$p \vee r$	Prem	$\emptyset$	
5	$q \vee s$	Prem	$\emptyset$	
6	$\check{\neg} p$	3; RC	$\{p \wedge \neg p\}$	$\checkmark^{10}$
7	r	6, 4; RU	$\{p \wedge \neg p\}$	$\checkmark^{10}$
8	$\check{\neg} q$	2; RC	$\{q \wedge \neg q\}$	
9	s	8, 5; RU	$\{q \wedge \neg q\}$	
10	$p \wedge \neg p$	1, 3; RU	$\emptyset$	

To derive $\sim q$ from $\neg q$ by $\mathbf{C}_2$, one needs $\neg(q \wedge \neg q) \wedge \neg(\neg(q \wedge \neg q) \wedge \neg\neg(q \wedge \neg q))$, which is abbreviated as $q^1 \wedge q^2$. Let me spell out the proof, using the abbreviations in order to keep it within the margins.

1	$\neg\neg p$	Prem	$\emptyset$	
2	$\neg q$	Prem	$\emptyset$	
3	$\neg p$	Prem	$\emptyset$	
4	$p \vee r$	Prem	$\emptyset$	
5	$q \vee s$	Prem	$\emptyset$	
6	p^1	RC	$\{p \wedge \neg p\}$	$\checkmark^{14}$
7	p^2	RC	$\{p^1 \wedge \neg p^1\}$	
8	$\sim p$	3, 6, 7; RU	$\{p \wedge \neg p, p^1 \wedge \neg p^1\}$	$\checkmark^{14}$
9	r	4, 8; RU	$\{p \wedge \neg p, p^1 \wedge \neg p^1\}$	$\checkmark^{14}$
10	q^1	RC	$\{q \wedge \neg q\}$	

[12] No set of consistency statements is sufficient because there are no existentially quantified consistency statements in the $\mathbf{C}_n$ logics. Note that the premise set is normal; no *Dab*-formula is derivable from it.

[13] The role played by minimal disjunctions of abnormalities becomes clear in the next section.

11	q^2	RC	$\{q^1 \wedge \neg q^1\}$
12	$\sim q$	2, 10, 11; RU	$\{q \wedge \neg q, q^1 \wedge \neg q^1\}$
13	s	5, 12; RU	$\{q \wedge \neg q, q^1 \wedge \neg q^1\}$
14	$p \wedge \neg p$	1, 3; RU	$\emptyset$

The justification of lines 6, 7, 10, and 11 requires no premise because in each case the formula is derived on the condition in view of a $\mathbf{C}_2$-theorem of the form $\neg A \vee A$.

This proof is more complicated than the original one, but leads to the same consequence set. It is not difficult to prove that this obtains whenever the premise set does not contain any consistency statements of any $\mathbf{C}_n$—as all of these are **CL**-theorems, it would be odd that they occurred in a premise set intended to have **CL** as its underlying logic. But even if consistency statements of some $\mathbf{C}_n$ occur in the premise set, these have no effect in $\mathbf{C}_{\overline{\omega}}$, in which there are consistency statements. So the general conclusion is that applying ${\mathbf{C}_{\overline{\omega}}}^r$ or ${\mathbf{C}_{\overline{\omega}}}^m$ is the wiser decision. These logics will all by themselves minimize contradictions, including existentially quantified ones.

Wait a minute. These logics define a minimally abnormal interpretation of the premise sets. So they have as final consequences all consistency statements the addition of which is justified by logical reasons.[14] As ${\mathbf{C}_{\overline{\omega}}}^r$ and ${\mathbf{C}_{\overline{\omega}}}^m$ are simpler than any other ${\mathbf{C}_n}^r$ or ${\mathbf{C}_n}^m$ and deliver the same consequence sets, they deserve to be preferred. But this does not mean that applying the stratagem is a complex matter full of hazards whereas the adaptive logics deliver the right consequences on the spot. Determining the adaptive consequence set is itself a complex matter, which will be discussed in Chapter 10. What the adaptive logics do, however, is define the right consequence set.

At this point I should turn to the Jaśkowski logics, but I shall not discuss adaptive logics that have Jaśkowski logics as their lower limit. To be in line with the present chapter, I should do so in terms of the axiomatizations in $\mathcal{L}_s$. As noted in the previous section, the axiomatization of **D2** is of recent vintage and I am not aware of similar axiomatizations of other Jaśkowski logics. Quite some work has been done on the adaptive logics in terms of the modal transformation by Joke Meheus in [Meh06] and by Marek Nasieniewski in [Nas02, Nas03, Nas04, Nas08]. All this is very enlightening, but does not fit into the present chapter. I also hope that Marek will translate his book in English; summarizing it would not do justice to it.

While **AN** is a fascinating paraconsistent logic, the adaptive logics $\mathbf{AN}^r$ and $\mathbf{AN}^m$ do not have much to offer. These logics are defined by the lower limit logic **AN**, the set of abnormalities $\Omega = \{\exists(A \wedge \neg A) \mid A \in \mathcal{F}_s^p\}$, and, respectively, Reliability and Minimal Abnormality. The only gain the adaptive logics offer over **AN** comes to a set of formulas that are weaker than the **AN**-consequences.[15] If, for example, $\Gamma \vdash_{\mathbf{AN}} p$, then one can derive $p \vee q$ in the adaptive proof on the condition $\{p \wedge \neg p\}$. So there is a gain. Moreover, deriving $p \vee q$ brings us closer to the **CL**-consequence set. However, the conditionally derivable formulas are useless for deriving further consequences. To see this,

[14] If $(p \wedge \neg p) \check{\vee} (q \wedge \neg q)$ is a minimal *Dab*-consequence of the premises, one may add $\check{\neg}(p \wedge \neg p)$ to the premise set, but there is no *logical* justification to choose $\check{\neg}(p \wedge \neg p)$ rather than $\check{\neg}(q \wedge \neg q)$—see footnote 11 on adaptive logics that guide the addition of consistency-statements that rely on extra-logical considerations.

[15] This obviously includes conjunctions of weaker formulas.

consider the case where $p, \neg(p \vee q) \vee r$ are members of the premise set. One might think that the conditional rule is useful here to derive r. Here is a $\mathbf{AN}^r$-proof.

1	p	Prem	$\emptyset$
2	$\neg(p \vee q) \vee r$	Prem	$\emptyset$
3	$p \vee q$	1; RC	$\{p \wedge \neg p\}$
4	r	2, 3; RU	$\{p \wedge \neg p\}$

However, r may also be derived on the empty condition.

5	$(\neg p \wedge \neg q) \vee r$	2; RU	$\emptyset$
6	$\neg p \vee r$	5; RU	$\emptyset$
7	r	1, 6; RU	$\emptyset$

Note that the transition from 2 to 6 is validated by the semantics because $(\neg p \vee r) \wedge (\neg q \vee r)$ is the PCNF of $\neg(p \vee q) \vee r$ and the semantics validates Simplification. The same reasoning applies to every case in which one would try to analyse a complex formula, like 2, by means of a conditional formula, like 3.

The adaptive logics $\mathbf{AN}^r$ and $\mathbf{AN}^m$ have another peculiar property. Note that $p \vee q \vdash_{\mathbf{AN}} ((p \vee q) \vee r) \check{\vee} ((p \vee q) \wedge \neg(p \vee q))$. As one would expect, it holds that $p \vee q \vdash_{\mathbf{AN}} ((p \vee q) \vee r) \check{\vee} (p \wedge \neg p) \check{\vee} (q \wedge \neg q)$. Less expected will be that both $p \vee q \vdash_{\mathbf{AN}} ((p \vee q) \vee r) \check{\vee} (p \wedge \neg p)$ and $p \vee q \vdash_{\mathbf{AN}} ((p \vee q) \vee r) \check{\vee} (q \wedge \neg q)$ also hold. So, in an $\mathbf{AN}^r$-proof, one has the choice to derive $(p \vee q) \vee r$ from $p \vee q$ on the condition $\{p \wedge \neg p\}$ or $\{q \wedge \neg q\}$. This strongly suggests that Reliability and Minimal Abnormality coincide, in other words that they both come to the Simple strategy.

Actually, Joke Meheus claims in [Meh00], that if $(p \wedge \neg p) \vee (q \wedge \neg q)$ is $\mathbf{AN}$-derivable from the premise set Γ, then so are $p \wedge \neg p$ and $q \wedge \neg q$. This is correct, but it does not concern the standard format—her paper was written before the standard format was formulated and her argument concerns the standard disjunction of abnormalities rather than a classical disjunction of abnormalities. Remember, however, the general convention that the premises belong to $\mathcal{W}_s$ and that the classical logical symbols have a merely technical use. Given this, the argument may be adjusted to the standard format. Suppose indeed that $(p \wedge \neg p) \check{\vee} (q \wedge \neg q)$ is $\mathbf{AN}$-derivable from $\Gamma \subseteq \mathcal{W}_s$. A little reflection readily shows that this is only possible if (i) $p \wedge \neg p$ or $q \wedge \neg q$ is $\mathbf{AN}$-derivable from Γ or (ii) $(p \wedge \neg p) \vee (q \wedge \neg q)$ is $\mathbf{AN}$-derivable from Γ. In case (ii), both $p \wedge \neg p$ and $q \wedge \neg q$ are $\mathbf{AN}$-derivable from Γ. So in neither case is $(p \wedge \neg p) \check{\vee} (q \wedge \neg q)$ a minimal *Dab*-consequence of Γ. So Reliability and Minimal Abnormality coincide and come to the Simple strategy. In other words $\mathbf{AN}^r$ and $\mathbf{AN}^m$ coincide with $\mathbf{AN}^s$.

7.4 As Normal As Possible

Inconsistency-adaptive logics have the aim to offer a maximally consistent interpretation of premise sets, or theories, that were intended as consistent but have turned out to be inconsistent. So when it was recently found possible to realize the aim in a more efficient way, this came as a shock.

Two other problems are solved at once. Inconsistency-adaptive logics are instruments: formal characterizations of defeasible reasoning forms. We want to have a manifold of them around to suit specific application purposes. While there is a lot of variation with respect to the lower limit logic and the strategy, every lower limit logic seems to determine a unique set of abnormalities. This holds for the logics introduced in Chapter 2 and also for the logics introduced in the present chapter, if flip-flops are disregarded.

The second problem concerns the comparison between different lower limit logics. Stronger paraconsistent logics have in general larger consequence sets than weaker ones, but also spread inconsistencies. While the former property makes more formulas derivable on the empty condition, the latter restricts the number of formulas that are finally derivable on a non-empty condition. In general, varying the lower limit logic often leads to incomparable adaptive consequence sets. The result presented in this section changes the picture drastically. By varying the set of abnormalities, adaptive logics with a very weak lower limit logic may be given an extremely rich and sensible consequence set.

The idea behind the enriched set of abnormalities is surprisingly simple. When complex **CLuN**^{m}-abnormalities are derivable, these may have different causes. Thus if $(p \vee q) \wedge \neg(p \vee q)$ is **CLuN**-derivable from the premises, the cause may be that $p \wedge \neg(p \vee q)$ is so derivable, or that $q \wedge \neg(p \vee q)$ is, or $(p \vee q) \wedge \neg(p \vee q)$ is whereas neither $p \wedge \neg(p \vee q)$ nor $q \wedge \neg(p \vee q)$ is. These three cases can be distinguished.

Consider the premise set $\Gamma_4 = \{\neg(p \vee q), q, p \vee r\}$ and let the underlying logic be **CLuN**^{m}. Here is an instructive **CLuN**^{m}-proof.

1	$\neg(p \vee q)$	Prem	$\emptyset$	
2	q	Prem	$\emptyset$	
3	$p \vee r$	Prem	$\emptyset$	
4	$\neg p$	1; RC	$\{(p \vee q) \wedge \neg(p \vee q)\}$	$\checkmark^7$
5	$\neg q$	1; RC	$\{(p \vee q) \wedge \neg(p \vee q)\}$	$\checkmark^7$
6	r	3, 4; RC	$\{(p \vee q) \wedge \neg(p \vee q), q \wedge \neg q\}$	$\checkmark^7$
7	$(p \vee q) \wedge \neg(p \vee q)$	1, 2; RU	$\emptyset$	

The formula $\neg p$ is derivable on the condition $\{(p \vee q) \wedge \neg(p \vee q)\}$ and hence r is derivable on the condition $\{(p \vee q) \wedge \neg(p \vee q), p \wedge \neg p\}$. By the presence of q and $\neg(p \vee q)$, however, $(p \vee q) \wedge \neg(p \vee q)$ is derivable from Γ_4 on the empty condition and so cannot be taken to be false. So neither $\neg p$ nor r are **CLuN**^{m}-derivable from Γ_4.

At first sight, this seems justified. Note, however, that the derivability of $(p \vee q) \wedge \neg(p \vee q)$ is caused by the derivability of $q \wedge \neg(p \vee q)$, not by the derivability of $p \wedge \neg(p \vee q)$. If it were possible to distinguish between those two, $\neg p$ and r would be final consequences while $\neg q$ is not.

Is it possible to turn this idea in a technically feasible procedure? It is. In the presence of $\neg(p \vee q)$, each of $p \vee q$, p, and q may cause the abnormality. The disjunction is derivable from either disjunct. Moreover, any **CLuN**-model verifying $p \vee q$ verifies p or q, but not necessarily both. This suggests that we consider $(p \vee q) \wedge \neg(p \vee q)$, $p \wedge \neg(p \vee q)$, and $q \wedge \neg(p \vee q)$ as separate abnormalities. The gain of doing so is clear: $\neg(p \vee q) \vdash_{\mathbf{CLuN}} \neg p \vee (p \wedge \neg(p \vee q))$, whence (if $p \wedge \neg(p \vee q)$ counts as an abnormality) $\neg p$ is derivable from $\neg(p \vee q)$ on the condition $\{p \wedge \neg(p \vee q)\}$, which is provably not a disjunct of any minimal *Dab*-consequence of Γ_4. On this understanding, the proof goes as follows.

1	$\neg(p \vee q)$	Prem	$\emptyset$	
2	q	Prem	$\emptyset$	
3	$p \vee r$	Prem	$\emptyset$	
4	$\neg p$	1; RC	$\{p \wedge \neg(p \vee q)\}$	
5	$\neg q$	1; RC	$\{q \wedge \neg(p \vee q)\}$	$\checkmark^{8}$
6	r	3, 4; RC	$\{p \wedge \neg(p \vee q), p \wedge \neg p\}$	
7	$(p \vee q) \wedge \neg(p \vee q)$	1, 2; RU	$\emptyset$	
8	$q \wedge \neg(p \vee q)$	1, 2; RU	$\emptyset$	

No hocus-pocus is going on here. For example, the stage of the proof may be extended with the following lines, but nothing changes to the marks of lines of the preceding stage.

9	$\neg p$	1; RC	$\{(p \vee q) \wedge \neg(p \vee q)\}$	$\checkmark^{7}$
10	$\neg q$	1; RC	$\{(p \vee q) \wedge \neg(p \vee q)\}$	$\checkmark^{7}$
11	r	3, 10; RC	$\{(p \vee q) \wedge \neg(p \vee q), p \wedge \neg p\}$	$\checkmark^{7}$

So r is a final consequence of Γ_4. Of course, this is merely an example. The matter needs to be elaborated.

Remember that $\mathcal{A}$ is the set of atoms (primitive formulas and their negations). Formulas that are not atoms are classified as $\mathfrak{a}$-formulas or $\mathfrak{b}$-formulas, varying on a theme from [Smu95]. To each of them, two other formulas are assigned according to the following table.

$\mathfrak{a}$	$\mathfrak{a}_1$	$\mathfrak{a}_2$		$\mathfrak{b}$	$\mathfrak{b}_1$	$\mathfrak{b}_2$
$A \wedge B$	A	B		$A \vee B$	A	B
$A \equiv B$	$A \supset B$	$B \supset A$		$A \supset B$	$\check{\neg} A$	B
$\check{\neg}(A \vee B)$	$\check{\neg} A$	$\check{\neg} B$		$\check{\neg}(A \wedge B)$	$\check{\neg} A$	$\check{\neg} B$
$\check{\neg}(A \supset B)$	A	$\check{\neg} B$		$\check{\neg}(A \equiv B)$	$\check{\neg}(A \supset B)$	$\check{\neg}(B \supset A)$
				$\neg A$	$\check{\neg} A$	$\check{\neg} A$

Next, a set $\text{sp}(A)$ of *specifying parts* is assigned to every open or closed formula A as follows:

1. Where $A \in \mathcal{A}$, $\text{sp}(A) = \{A\}$.
2. $\text{sp}(\mathfrak{a}) = \{\mathfrak{a}\} \cup \{A \wedge B \mid A \in \text{sp}(\mathfrak{a}_1); B \in \text{sp}(\mathfrak{a}_2)\}$.
3. $\text{sp}(\mathfrak{b}) = \{\mathfrak{b}\} \cup \text{sp}(\mathfrak{b}_1) \cup \text{sp}(\mathfrak{b}_2)$.
4. $\text{sp}(\forall \alpha A) = \{\forall \alpha B \mid B \in \text{sp}(A)\}$.
5. $\text{sp}(\exists \alpha A) = \{\exists \alpha B \mid B \in \text{sp}(A)\}$.
6. if $B, C \in \text{sp}(A)$, then $B \vee C \in \text{sp}(A)$.

The adaptive logic $\mathbf{CLuN}_1^m$ is defined by the following triple: (i) lower limit: **CLuN**, (ii) set of abnormalities: $\Omega^s = \{\exists(B \wedge \neg A) \mid A \in \mathcal{F}; B \in \text{sp}(A)\}$, and (iii) strategy: Minimal Abnormality.

No part of the previous construction is missing. The underlying idea is that $B \in \text{sp}(A)$ iff B is a truth-function of subformulas of A and $B \vdash_{\mathbf{CLuN}} A$. All those truth-functions are actually (possibly quantified) $\wedge$-$\check{\neg}$-compounds of subformulas of A—the other logical symbols may obviously occur within the subformulas themselves. Relying on this idea, we want to count as abnormalities not only formulas of the form $\exists(A \wedge \neg A)$, but also formulas of the form $B \wedge C$ for which $B \in \mathrm{sp}(A)$ and $C \in \mathrm{sp}(\neg A)$. Three comments are useful in this respect. First, if A is a conjunction of (one or more) atoms, possibly preceded by a sequence of quantifiers, then $\mathrm{sp}(A) = \{A\}$ in view of clauses 1 and 2. Next, the only specifying parts of the right conjunct of $A \wedge \neg A$ are formulas from ref in fn which $\check{\neg}A$ is **CLuN**-derivable.[16] However, in the presence of A, $\check{\neg}A$ (causes triviality and) leads to a **CLuN**-falsehood, and lower limit falsehoods may be filtered out of the set of abnormalities in view of Fact 5.9.8. So we may forget about specifying parts of the second conjunct of $A \wedge \neg A$. Finally, it is needless as well as useless to define specifying parts of formulas of the form $\check{\neg}\neg B$ that are themselves specifying parts of the first conjunct of $A \wedge \neg A$. A specifying part of that first conjunct can only have the form $\check{\neg}\neg B$ if A has the form $\neg\neg B$. However, $\neg\neg B$ is not **CLuN**-derivable from any truth-function of subformulas of B.[17] So $\check{\neg}\neg A$ does not have any specifying parts and hence should not occur as an $\mathfrak{a}$-formula or as a $\mathfrak{b}$-formula in the table.

The logics $\mathbf{CLuN}^m_1$ and $\mathbf{CLuN}^r_1$ offer a *refinement* in comparison to $\mathbf{CLuN}^m$ and $\mathbf{CLuN}^r$. Even if $(p \vee q) \wedge \neg(p \vee q)$ is true in some models of a premise set, either $p \wedge \neg(p \vee q)$ or $q \wedge \neg(p \vee q)$ may be false in some of those models and this enables us to rule out some further models as more abnormal than required by the premises.

We have seen that the logic $\mathbf{CLuN}^m_1$ is richer than $\mathbf{CLuN}^m$ with respect to Γ_4. However, the enrichment is not restricted to similar cases. Let me mention two further examples. Consider first $\Gamma_5 = \{p \vee q, \neg(p \vee q), p \vee r, q \vee s\}$. In view of the explicit contradiction between the first two premises, one might expect to obtain no gain in this case. Yet, there is one. Let me spell out the $\mathbf{CLuN}^m_1$-proof.

1	$p \vee q$	Prem	$\emptyset$	
2	$\neg(p \vee q)$	Prem	$\emptyset$	
3	$p \vee r$	Prem	$\emptyset$	
4	$q \vee s$	Prem	$\emptyset$	
5	$\neg p$	2; RC	$p \wedge \neg(p \vee q)$	$\checkmark^{12}$
6	r	3, 5; RU	$p \wedge \neg(p \vee q)$	$\checkmark^{12}$
7	$r \vee s$	6; RU	$p \wedge \neg(p \vee q)$	
8	$\neg q$	2; RC	$q \wedge \neg(p \vee q)$	$\checkmark^{12}$
9	s	4, 8; RU	$q \wedge \neg(p \vee q)$	$\checkmark^{12}$
10	$r \vee s$	9; RU	$q \wedge \neg(p \vee q)$	
11	$(p \vee q) \wedge \neg(p \vee q)$	1, 2; RU	$\emptyset$	
12	$(p \wedge \neg(p \vee q)) \vee (q \wedge \neg(p \vee q))$	11; RU	$\emptyset$	

The only minimal *Dab*-consequences of Γ_5 are 11 and 12. So $\Phi(\Gamma_5) = \{\{(p \vee q) \wedge \neg(p \vee q), p \wedge \neg(p \vee q)\}, \{(p \vee q) \wedge \neg(p \vee q), q \wedge \neg(p \vee q)\}\}$. The formula $r \vee s$ is derived at line 10 on a condition that does not overlap with the first member

[16] If this is not obvious at once, have another look at the **CLuN**-semantics. If the truth of $\neg A$ is not caused by the truth of $\check{\neg}A$, the falsehood of A that is, then it is caused directly by the assignment, in other words not by the truth of any other formula (it hangs from a skyhook—see [Bat03a]).

[17] As the semantics reveals, a truth-function of subformulas of B may **CLuN**-entail $\check{\neg}B$ (and hence $\neg B$), but cannot possibly **CLuN**-entail $\neg\neg B$. Put differently, with the exception of $\check{\neg}\neg B$ itself, no truth-function of subformulas of $\neg B$ can warrant that $\neg B$ if false.

of $\Phi(\Gamma_5)$ and is derived at line 7 on a condition that does not overlap with the second member of $\Phi(\Gamma_5)$. So, on the minimal Abnormality strategy, $r \vee s$ is a $\mathbf{CLuN}^m_1$-consequence of Γ_5 whereas it is not a $\mathbf{CLuN}^m$-consequence of this premise set.

The next example illustrates another way in which $\mathbf{CLuN}^m_1$ differs from $\mathbf{CLuN}^m$. Here is a $\mathbf{CLuN}^m_1$-proof from $\Gamma_6 = \{\neg\neg(p \wedge q), \neg p, \neg q \vee r\}$.

1	$\neg\neg(p \wedge q)$	Prem	$\emptyset$
2	$\neg p$	Prem	$\emptyset$
3	$\neg q \vee r$	Prem	$\emptyset$
4	q	1; RC	$\{\check{\neg} q \wedge \neg\neg(p \wedge q)\}$
5	r	3, 4; RC	$\{\check{\neg} q \wedge \neg\neg(p \wedge q), q \wedge \neg q\}$
6	$(\neg(p \wedge q) \wedge \neg\neg(p \wedge q)) \vee (p \wedge \neg p)$	1, 2; RU	$\emptyset$
7	$(\check{\neg} p \wedge \neg\neg(p \wedge q)) \vee (p \wedge \neg p)$	1, 2; RU	$\emptyset$

Note that 6 and 7 are the only *Dab*-formulas that are **CLuN**-derivable from Γ_6 and that 4 and 5 are unmarked. Γ_6 has two kinds of minimal abnormal **CLuN**-models. Those of the first kind verify only the abnormalities $\neg(p \wedge q) \wedge \neg\neg(p \wedge q)$ and $\check{\neg} p \wedge \neg\neg(p \wedge q)$. An example is the model in which $v(q) = v(r) = v(\neg\neg(p \wedge q)) = 1$ whereas $v(A) = 0$ for all other members of $\mathcal{W}_{\mathcal{O}}$. The minimal abnormal **CLuN**-models of the second kind verify only $p \wedge \neg p$. An example is the model in which $v(p) = v(\neg p) = v(q) = v(r) = 1$ whereas $v(A) = 0$ for all other members of $\mathcal{W}_{\mathcal{O}}$. All these models of Γ_6 verify q as well as r.

If one transforms the last $\mathbf{CLuN}^m_1$-proof into a $\mathbf{CLuN}^m$-proof from Γ_6, the condition of line 4 reads $\{\neg(p \wedge q) \wedge \neg\neg(p \wedge q)\}$ and the condition of line 5 reads $\{\neg(p \wedge q) \wedge \neg\neg(p \wedge q), q \wedge \neg q\}$—remember that $\check{\neg} q \wedge \neg\neg(p \wedge q)$ is not a $\mathbf{CLuN}^m$-abnormality. So both lines are marked in view of line 6. Obviously 7 is still derivable, but is not a *Dab*-formula with respect to $\mathbf{CLuN}^m$.

Some readers may wonder about the use of clause 6 from the specifying part definition. Consider a premise set containing $\neg(p \vee (q \vee r))$ as well as $p \vee r$. It is obviously possible to derive $p \vee (q \vee r)$ from $p \vee r$, but neither p nor $q \vee r$, nor q, nor r is a **CLuN**-consequence of it. Nevertheless, we want $p \vee r$ to be a specifying part of $p \vee (q \vee r)$, and one that is different from the other five.

We have seen three examples in which the $\mathbf{CLuN}^m$-consequence set of a premise set is a proper subset of the $\mathbf{CLuN}^m_1$-consequence set of the same premise set. Moreover, as was already noted, $\Omega \subset \Omega^1$. All this suggests that the adaptive logic $\mathbf{CLuN}^m_1$ is stronger than $\mathbf{CLuN}^m$, but this is not the case. To show this consider the premise set $\Gamma_7 = \{\neg(\neg s \vee (\neg p \wedge \neg r)), \neg(\neg p \vee \neg q), \neg(s \vee p)\}$.

It is provable that a formula of the form $A \wedge \neg A$ is only a member of a minimal *Dab*-consequence of Γ_7 (with respect to both Ω and Ω^1) if it is one of 1–9 below. All are members of Ω^1 and 1–3 are also members of Ω.

1	$(\neg s \vee (\neg p \wedge \neg r)) \wedge \neg(\neg s \vee (\neg p \wedge \neg r))$
2	$(\neg p \vee \neg q) \wedge \neg(\neg p \vee \neg q)$
3	$(s \vee p) \wedge \neg(s \vee p)$
4	$\neg s \wedge \neg(\neg s \vee (\neg p \wedge \neg r))$
5	$(\neg p \wedge \neg r) \wedge \neg(\neg s \vee (\neg p \wedge \neg r))$
6	$\neg p \wedge \neg(\neg p \vee \neg q)$
7	$\neg q \wedge \neg(\neg p \vee \neg q)$
8	$s \wedge \neg(s \vee p)$
9	$p \wedge \neg(s \vee p)$

p	p	p	p	p	p	p	p	$\check{\neg}p$	$\check{\neg}p$	$\check{\neg}p$	$\check{\neg}p$	$\check{\neg}p$	$\check{\neg}p$	$\check{\neg}p$	$\check{\neg}p$
q	q	q	q	$\check{\neg}q$	$\check{\neg}q$	$\check{\neg}q$	$\check{\neg}q$	q	q	q	q	$\check{\neg}q$	$\check{\neg}q$	$\check{\neg}q$	$\check{\neg}q$
r	r	$\check{\neg}r$	$\check{\neg}r$	r	r	$\check{\neg}r$	$\check{\neg}r$	r	r	$\check{\neg}r$	$\check{\neg}r$	r	r	$\check{\neg}r$	$\check{\neg}r$
s	$\check{\neg}s$	s	$\check{\neg}s$	s	$\check{\neg}s$	s	$\check{\neg}s$	s	$\check{\neg}s$	s	$\check{\neg}s$	s	$\check{\neg}s$	s	$\check{\neg}s$
	1		1		1		1		1	1	1		1	1	1
				2	2	2	2	2	2	2	2	2	2	2	2
3	3	3	3	3	3	3	3	3		3		3		3	
∗		∗							∗		∗		∗		∗
	4		4		4		4		4		4		4		4
										5	5			5	5
								6	6	6	6	6	6	6	6
				7	7	7	7					7	7	7	7
8		8		8		8		8		8		8		8	
9	9	9	9	9	9	9	9								
∗		∗							∗						

Table 7.1: **CLuN**-models of Γ_7

No **CLuN**-model of Γ_7 is minimal abnormal (with respect to either Ω or Ω^1) and verifies $A \wedge \neg A$ for any propositional letter A. So we may restrict our attention to models that verify the premises together with some of the relevant propositional letters and the classical negation of the others. I list the *types* of models of Γ_7 in Table 7.1. The value of unmentioned letters is arbitrary, provided that, for each unmentioned letter A, the model does not verify $A \wedge \neg A$.

In every column I list the abnormalities verified by the corresponding models. The abnormalities are referred to by numbers that correspond to the preceding list. There are two rows of stars. The stars of the upper row indicate the types of models that are minimally abnormal with respect to Ω; those of the lower row indicate which types of models are minimally abnormal with respect to both Ω and Ω^1. So the stars in the lower row indicate which of the models selected in view of Ω are also selected in view of Ω^1.

Note that the second, fourth, sixth, eighth, and ninth columns of the table summarize models that are minimally abnormal with respect to Ω^1-abnormalities, but none of them is minimally abnormal with respect to Ω-abnormalities. Thus, $p \vee \neg s$ is a $\mathbf{CLuN}^m$-consequence but not a $\mathbf{CLuN}_1^m$-consequence of Γ_7. Incidentally, the selection of models indicated by the lower row of stars cannot be expressed in terms of the typical $\mathbf{CLuN}_1^m$-abnormalities $\Omega^1 - \Omega$ alone.

In order to obtain the desired maximally consistent interpretation of Γ_7, we need a *combined* adaptive logic. Let us call this logic $\mathbf{CLuN}_c^m$ and let $Cn_{\mathbf{CLuN}_c^m}(\Gamma) = Cn_{\mathbf{CLuN}_1^m}(Cn_{\mathbf{CLuN}^m}(\Gamma))$, which defines the logic in view of what was said at the end of Section 6.2.2—note that $\Omega \subset \Omega^1$. Thus, in semantic terms, $\mathbf{CLuN}_c^m$ selects from all **CLuN**-models of Γ_7 those that are minimally abnormal with respect to Ω and next from the latter selects those that are minimally abnormal with respect to Ω^1.

It is easily seen from Table 7.1 that $\mathbf{CLuN}_c^m$ delivers a number of consequences of Γ_7 on top of those delivered by $\mathbf{CLuN}^m$: q, $p \vee r$, $s \vee r$, ... Given the way in which $\mathbf{CLuN}_c^m$ is defined, $Cn_{\mathbf{CLuN}^m}(\Gamma) \subseteq Cn_{\mathbf{CLuN}_c^m}(\Gamma)$ and for many premise sets the inclusion is proper. Given that $\Omega \subset \Omega^1$, $\mathbf{CLuN}_c^m$-proofs are actually just like $\mathbf{CLuN}_1^m$-proofs except that, at every stage, one first marks in view of Ω and next in view of Ω^1.

As was explained before, the strength of the lower limit logic seems to have an odd effect on the resulting inconsistency-adaptive logic. Some lower limit logics are weak, as is the case for **CLuN**, others are strong, as is the case for **CLuNs**. **CLuN** maximally isolates inconsistencies. The effect on the adaptive logic is that, in the presence of less inconsistencies, more formulas are derivable on a condition that does not cause the line to be marked. Thus $p, \neg p \vee r, \neg(p \vee q) \nvdash_{\mathbf{CLuN}} \neg p$, whence $p, \neg p \vee r, \neg(p \vee q) \vdash_{\mathbf{CLuN}^m} r$. **CLuNs** spreads inconsistencies, as the example illustrates: $p, \neg p \vee r, \neg(p \vee q) \vdash_{\mathbf{CLuNs}} \neg p$, whence $p, \neg p \vee r, \neg(p \vee q) \nvdash_{\mathbf{CLuNs}^m} r$. So, in this case, $\mathbf{CLuN}^m$ does better.

The weakness of the lower limit logic, however, has also disadvantages: less formulas are unconditionally derivable, which diminishes the inferential power of the logic. This is nicely illustrated by $\Gamma_4 = \{\neg(p \vee q), q, p \vee r\}$. As $\Gamma_4 \vdash_{\mathbf{CLuN}} (p \vee q) \wedge \neg(p \vee q)$, $\Gamma_4 \nvdash_{\mathbf{CLuN}^m} \neg p$ and hence $\Gamma_4 \nvdash_{\mathbf{CLuN}^m} r$. The matter is very different for **CLuNs**: $\Gamma_4 \vdash_{\mathbf{CLuNs}} \neg p$ and, as this does not cause an inconsistency, $\Gamma_4 \vdash_{\mathbf{CLuNs}^m} r$. So here the richer lower limit logic **CLuNs** causes $\mathbf{CLuNs}^m$ to do better than $\mathbf{CLuN}^m$.

It is instructive to compare the way in which $\mathbf{CLuN}^m$, $\mathbf{CLuN}^m_c$, and $\mathbf{CLuNs}^m$ handle different premise sets. I shall at once include $\mathbf{LP}^m$ in the comparison. $\mathbf{LP}^m$ is defined exactly as $\mathbf{CLuNs}^m$—simply replace **CLuNs** by **LP**. **LP**m from [Pri91] and [Pri06] is identical to $\mathbf{LP}^m$ at the propositional level, but differs drastically from it at the predicative level—a matter that should not concern us here.

For the sake of transparency, I shall present the comparison in the form of tables. $\mathbf{CLuNs}^m$ and $\mathbf{LP}^m$ occur in the same column when all formulas in that column are in both consequence sets. Their consequence sets are different, however, in view of the different meaning of the implication.

Let us first consider $\Gamma_8 = \{\neg\neg(p \wedge q), \neg p, \neg q \vee r, p \vee s\}$, which is an extension of Γ_6.

$\mathbf{CLuN}^m$	$\mathbf{CLuN}^m_c$	$\mathbf{CLuNs}^m/\mathbf{LP}^m$
		$p \wedge q$
		p
$\neg p$	$\neg p$	$\neg p$
	q	q
$\neg q \vee r$	r	r
s	s	

The comparison requires some study on the part of the reader. Note that I do not repeat all the premises. Like $\mathbf{CLuNs}^m$ and $\mathbf{LP}^m$, $\mathbf{CLuN}^m_c$ delivers q as well as r, which is a gain with respect to $\mathbf{CLuN}^m$. $\mathbf{CLuNs}^m$ and $\mathbf{LP}^m$ also deliver p, and hence $p \wedge q$. This, however, causes an inconsistency itself, because $\neg p$ is a premise. As a result, s is not derivable from the premise set according to $\mathbf{CLuNs}^m$ and $\mathbf{LP}^m$, whereas it is a $\mathbf{CLuN}^m_c$-consequence.

The situation is perfectly similar for $\Gamma_9 = \{\neg(p \vee q), p, \neg p \vee s, q \vee r\}$:

$\mathbf{CLuN}^m$	$\mathbf{CLuN}_c^m$	$\mathbf{CLuNs}^m/\mathbf{LP}^m$
p	p	p
		$\neg p$
	$\neg q$	$\neg q$
$q \vee r$	r	r
s	s	

Again, $\mathbf{CLuN}_c^m$ does better than $\mathbf{CLuN}^m$. Like $\mathbf{CLuNs}^m$ and $\mathbf{LP}^m$, it delivers r, but unlike those logics, it does not deliver $\neg p$, thus avoiding an inconsistency which prevents that r is a $\mathbf{CLuNs}^m$-consequence of the premise set.

Another interesting example is $\Gamma_5 = \{p \vee q, \neg(p \vee q), p \vee r, q \vee s\}$, which we have seen before.

$\mathbf{CLuN}^m$	$\mathbf{CLuN}_c^m$	$\mathbf{CLuNs}^m/\mathbf{LP}^m$
		$\neg p$
	$\neg p \vee \neg q$	$\neg q$
$p \vee q$	$p \vee q$	$p \vee q$
$p \vee r$	$p \vee r$	$p \vee r$
$q \vee s$	$q \vee s$	$q \vee s$
	$r \vee s$	

Here $\mathbf{CLuN}_c^m$ delivers $\neg p \vee \neg q$, which is a gain with respect to $\mathbf{CLuN}^m$, but a loss with respect to $\mathbf{CLuNs}^m$ and $\mathbf{LP}^m$ which deliver $\neg p$ and $\neg q$. Note that $\neg p$ and $\neg q$ do not cause an explicit inconsistency, but with $p \vee q$ they add up to the *Dab*-formula $(p \wedge \neg p) \check{\vee} (q \wedge \neg q)$. The absence of this *Dab*-formula from the $\mathbf{CLuN}_c^m$-consequences leads to a gain, viz. $r \vee s$.

Let me present two examples in which the table needs different columns for $\mathbf{CLuNs}^m$ and $\mathbf{LP}^m$. Here is the table for $\Gamma_{10} = \{p, \neg p \vee q, \neg(p \vee r), \neg\neg p \supset s\}$:

$\mathbf{CLuN}^m$	$\mathbf{CLuN}_c^m$	$\mathbf{CLuNs}^m$	$\mathbf{LP}^m$
p	p	p	p
		$\neg p$	$\neg p$
$\neg\neg p$	$\neg\neg p$	$\neg\neg p$	$\neg\neg p$
	$\neg r$	$\neg r$	$\neg r$
q	q		
s	s	s	

and here the table for $\Gamma_{11} = \{p, \neg p \vee q, \neg(p \vee r), \neg\neg p \supset s, \neg q \vee t, r \vee u\}$:

$\mathbf{CLuN}^m$	$\mathbf{CLuN}_c^m$	$\mathbf{CLuNs}^m$	$\mathbf{LP}^m$
p	p	p	p
		$\neg p$	$\neg p$
$\neg\neg p$	$\neg\neg p$	$\neg\neg p$	$\neg\neg p$
	$\neg r$	$\neg r$	$\neg r$
q	q		
s	s	s	
t	t		
	u	u	u

An expected advantage of $\mathbf{CLuN}_c^m$ over $\mathbf{CLuN}^m$ is that the former delivers a strictly stronger consequence set. More interesting is the comparison between $\mathbf{CLuN}_c^m$ on the one hand and $\mathbf{CLuNs}^m$ and $\mathbf{LP}^m$ on the other hand. In general,

$\mathbf{CLuN}_c^m$ does 'better' than $\mathbf{CLuNs}^m$ and $\mathbf{LP}^m$ in avoiding inconsistencies that derive from the analysis of negated complex formulas and, in doing so, it often makes more consequences derivable. Both examples illustrate this.

It is useful to comment on the difference between $\mathbf{CLuN}_c^m$ on the one hand and $\mathbf{CLuNs}^m$ and $\mathbf{LP}^m$ on the other hand. Let the premise set contain $\neg(A \vee B)$. According to $\mathbf{CLuNs}^m$ and $\mathbf{LP}^m$, $\neg A$ and $\neg B$ are unconditionally derivable from this. According to $\mathbf{CLuN}_c^m$, they are at best conditionally derivable. In the presence of both A and B, neither $\neg A$ nor $\neg B$ nor their disjunction is a $\mathbf{CLuN}_c^m$-consequence. In the presence of one of them only, the negation of the other is a $\mathbf{CLuN}_c^m$-consequence. If neither is present, but the disjunction $A \vee B$ is, then $\neg A \vee \neg B$ is a $\mathbf{CLuN}_c^m$-consequence. If not even the disjunction is present, then both $\neg A$ and $\neg B$ are $\mathbf{CLuN}_c^m$-consequences.

If the strategy is Reliability, the combined logic—call it $\mathbf{CLuN}_c^r$ and remember that it requires a logic $\mathbf{CLuN}_1^r$—still delivers more consequences than $\mathbf{CLuN}^r$, but the gain is less impressive than in the case of $\mathbf{CLuN}_c^m$. Some illustrations: if marking proceeds on the Reliability strategy, all lines that have a non-empty condition in the proof from Γ_5 are marked, whereas the two lines that have a non-empty condition in the proof from Γ_6 are unmarked.

The gain illustrated in this section for $\mathbf{CLuN}_c^m$ may be obtained for many other inconsistency-adaptive logics by defining a corresponding combined logic. Obviously, there will only be a gain if the 'first conjunct' of the abnormality has specifying parts that are not identical to this first conjunct. So there is no gain for $\mathbf{CLuNs}^m$ and $\mathbf{LP}^m$. A similar gain may be obtained for other adaptive logics as well. It is rather obvious in which way specifying parts should be defined and again there will be no gain if the different 'parts' of an abnormality have only themselves as specifying parts. So I leave the matter here.

7.5 Identifying Inconsistent Objects

In [Pri91], Graham Priest formulates an inconsistency-adaptive logic which he now calls **LP**m and which has **LP** as its lower limit. **LP**m is not an adaptive logic in standard format and, as already explained in Section 5.2, Strong Reassurance does not hold for it. Another feature, which was actually denied in [Pri91], is that a model with domain D may be less abnormal than a model with a domain of which D is a proper subset—see the correction in [Pri02]; the corrected version also appears in [Pri06]. A consequence of this is that inconsistent objects are identified whenever this does not lead to other inconsistencies. For example, from $\{Pa \wedge \neg Pa, Pb \wedge \neg Pb\}$ one derives $a = b$ because by identifying a and b, the premise set has a model in which there is only one inconsistent *object.*[18] From $\{Pa \wedge \neg Pa, Pb \wedge \neg Pb, Qa, \neg Qb\}$ one can neither derive $a = b$ nor $\neg a = b$. If a and b denote the same object, this object has property P as well as property $\neg P$ and it has property Q as well as property $\neg Q$; if a and b do not denote the same object, *two* objects have property P as well as $\neg P$ and no object is inconsistent with respect to Q. So the sets of 'abnormalities' are incomparable.[19]

[18] Here lies an essential difference with adaptive logics in standard format, which measure the abnormal part of a model in terms of the abnormal formulas that the model verifies, in the present case existentially closed inconsistencies.

[19] That an element or a tuple of elements of the domain is inconsistent is technically realized by assigning to every predicate π a positive extension $v^+(\pi)$ as well as a negative extension

While the absence of Strong Reassurance is obviously unacceptable, identifying inconsistent objects that are not known to be different in other respects may very well be attractive in certain circumstances. Independent of Graham Priest's **LP**m, who actually discovered this property of his logic after the first publication, Peter Verdée had the idea (unpublished note) that identifying names of inconsistent entities is attractive in certain circumstances. Moreover, he formulated the basic case for an adaptive logic in the standard format. His idea combined with the lower limit logic **CLuN** is that the formulas $Pa \wedge \neg Pa$ and $Pb \wedge \neg Pb$ count as abnormalities, but that the formula $(Pa \wedge \neg Pa) \wedge (Pb \wedge \neg Pb) \wedge \check{\neg}\, a = b$ counts as a different abnormality. On this convention, the only models of $\Gamma_{12} = \{Pa \wedge \neg Pa, Pb \wedge \neg Pb, Ra\}$ are those in which $a = b$ and hence Rb.

Why is this proposal attractive? Suppose you have met a man who has a very exceptional property, for example he interrupted his statements by barks. Suppose moreover a friend tells you that he has met someone who interrupted his statements by barks. You will probably reply that you have met the same person. Your conclusion relies on defeasible reasoning, but it seems sensible nevertheless, unless you friend's barker was a woman, or was French while yours was British, or was a dog while yours was a human. If inconsistencies are indeed exceptional, then we may justifiedly identify two entities that are inconsistent with respect to the same property if nothing else known about them makes identifying them even more exceptional.

The occurrence of classical negation in the new kind of abnormalities involves a subtle point. Consider the premise set $\Gamma_{13} = \{Pa \wedge \neg Pa, Pb \wedge \neg Pb, \neg\, a = b\}$. This premise set has **CLuN**-models in which $v(a) = v(b)$ as well as models in which $v(a)$ and $v(b)$ are different elements of the domain. First consider the case in which classical negation occurs in the new abnormalities. The minimally abnormal models of both kinds then verify the following abnormalities.[20]

$v(a) = v(b)$	$v(a) \neq v(b)$
$Pa \wedge \neg Pa$	$Pa \wedge \neg Pa$
$Pb \wedge \neg Pb$	$Pb \wedge \neg Pb$
$a = b \wedge \neg a = b$	$(Pa \wedge \neg Pa) \wedge (Pb \wedge \neg Pb) \wedge \check{\neg} a = b$

If the new abnormalities were formulated with the standard negation, the abnormalities verified by the least abnormal models of both kinds would be as follows.

$v(a) = v(b)$	$v(a) \neq v(b)$
$Pa \wedge \neg Pa$	$Pa \wedge \neg Pa$
$Pb \wedge \neg Pb$	$Pb \wedge \neg Pb$
$(Pa \wedge \neg Pa) \wedge (Pb \wedge \neg Pb) \wedge \neg a = b$	$(Pa \wedge \neg Pa) \wedge (Pb \wedge \neg Pb) \wedge \neg a = b$
$a = b \wedge \neg a = b$	

In this case no model in which $v(a) = v(b)$ would be minimally abnormal, just as in $\mathbf{CLuN}^m$.

So it is better to formulate the abnormalities with classical negation. Let me try to make this completely transparent. On the one hand, an abnormality is avoided by deriving $\check{\neg}\, a = b$ from $\neg\, a = b$. On the other hand, the abnormalities

$v^-(\pi)$. Every member of $v^+(\pi) \cap v^-(\pi)$ is inconsistent.

[20] I list only basic abnormalities, not derivable ones like $\exists(Px \wedge \neg Px)$.

of the new kind are intended to identify objects that are inconsistent with respect to the same properties. In the case of Γ_{13} these two conflict with each other and the most sensible decision is to consider models of both kinds to be minimally abnormal.

Obviously the idea needs elaboration. Some abnormalities of $\mathbf{CLuN}^m$ are existentially quantified, as $\exists x(Px \wedge \neg Px)$. If one wants to identify objects denoted by different individual constants, then one certainly also wants to identify objects not denoted by such constants. So one should count $\exists x \exists y((Px \wedge \neg Px) \wedge (Py \wedge \neg Py) \wedge \check{\neg}\, x = y)$ as an independent abnormality—independent of $\exists x(Px \wedge \neg Px)$ that is. If objects are named by individual constants, the matter is simple. Let $\Gamma_{14} = \{Pa \wedge \neg Pa, Pb \wedge \neg Pb, Pc \wedge \neg Pc, Ra\}$. If a **CLuN**-model M of Γ_{14} verifies $a = b$ as well as $a = c$, and hence also Rb and Rc, then $Ab(M) = \{Pa \wedge \neg Pa, Pb \wedge \neg Pb, Pc \wedge \neg Pc\}$. All other **CLuN**-models of Γ_{14} verify more abnormalities, for example $(Pa \wedge \neg Pa) \wedge (Pb \wedge \neg Pb) \wedge \check{\neg}\, a = b$ or $(Pa \wedge \neg Pa) \wedge (Pc \wedge \neg Pc) \wedge \check{\neg}\, a = c$ or $(Pb \wedge \neg Pb) \wedge (Pc \wedge \neg Pc) \wedge \check{\neg}\, b = c$, or several of these. However, if objects are not named by individual constants, the matter is more complicated. Consider the premise set $\Gamma_{15} = \{\exists x(Px \wedge \neg Px \wedge Qx \wedge Rx), \exists y(Py \wedge \neg Py \wedge \neg Qy \wedge Ry), \exists z(Pz \wedge \neg Pz \wedge Qz \wedge \neg Rz)\}$. We certainly want to allow for minimally abnormal **CLuN**-models of Γ_{15} in which three different objects are inconsistent with respect to property P—if we do not do so, we obtain objects that display inconsistencies with respect to Q or R. So, following the idea underlying Peter Verdée's proposal, **CLuN**-models of Γ_{15} in which more than three different objects are inconsistent with respect to P should not be minimally abnormal. But how to realize this?

One might think about introducing such abnormalities as $\exists x_1 \ldots \exists x_n((Px_1 \wedge \neg Px_1) \wedge \ldots \wedge (Px_n \wedge \neg Px_n) \wedge (\check{\neg}\, x_1 = x_2 \wedge \ldots \wedge \check{\neg}\, x_1 = x_n \wedge \check{\neg}\, x_2 = x_3 \wedge \ldots \wedge \check{\neg}\, x_{n-1} = x_n))$ for any $n \geq 1$. But this leads nowhere. Even on the new understanding, some premise sets have minimally abnormal models in which infinitely many objects are and are not P, but in which not all objects are inconsistent with respect to P. An example of such a premise set is $\Gamma_{16} = \{\exists x(Rx \wedge Q_1x \wedge Q_2x \wedge \ldots \wedge Q_nx) \mid n \in \{1, 2, \ldots\} \cup \{\exists x_1 \ldots \exists x_n((Px_1 \wedge \neg Px_1) \wedge \ldots \wedge (Px_n \wedge \neg Px_n) \wedge (\neg Q_1x_1 \wedge Q_2x_1 \wedge \ldots \wedge Q_nx_1) \wedge (Q_1x_2 \wedge \neg Q_2x_2 \wedge \ldots \wedge Q_nx_2) \wedge \ldots \wedge (Q_1x_n \wedge Q_2x_n \wedge \ldots \wedge \neg Q_nx_n)) \mid n \in \{1, 2, \ldots\}\}$. The intended minimally abnormal **CLuN**-models of Γ_{16} can clearly not be identified by means of abnormalities of the form $\exists x_1 \ldots \exists x_n((Px_1 \wedge \neg Px_1) \wedge \ldots \wedge (Px_n \wedge \neg Px_n) \wedge (\check{\neg}\, x_1 = x_2 \wedge \ldots \wedge \check{\neg}\, x_1 = x_n \wedge \check{\neg}\, x_2 = x_3 \wedge \ldots \wedge \check{\neg}\, x_{n-1} = x_n))$.

A viable approach consists in mentioning 'other' predicates in the abnormalities. Thus, if the premise set is Γ_{15}, all **CLuN**-models of the premise set will verify $\exists x \exists y((Px \wedge \neg Px \wedge Qx) \wedge (Py \wedge \neg Py \wedge Qy)) \wedge \check{\neg} x = y) \,\check{\vee}\, \exists x(Rx \wedge \neg Rx)$ as well as $\exists x \exists y((Px \wedge \neg Px \wedge Rx) \wedge (Py \wedge \neg Py \wedge Ry)) \wedge \check{\neg} x = y) \,\check{\vee}\, \exists x(Qx \wedge \neg Qx)$. In the case of Γ_{16}, no minimal abnormal model will verify $\exists x((Px \wedge \neg Px) \wedge Rx)$. Phrasing this correctly requires some care.

Let $\overline{\alpha}$ and $\overline{\beta}$ be sequences of members of $\mathcal{C} \cup \mathcal{V}$ which have the same number $n \geq 1$ of elements and which are such that, for all i, the ith elements of the sequences both belong to $\mathcal{C}$ or both belong to $\mathcal{V}$. Let $A(\overline{\alpha})$ and $B(\overline{\alpha})$ be formulas in which the members of $\overline{\alpha}$ occur free—taking for granted that all constants always occur free,[21] and let $A(\overline{\beta})$, respectively $B(\overline{\beta})$, be obtained by replacing in $A(\overline{\alpha})$, respectively $B(\overline{\alpha})$, the ith element of $\overline{\alpha}$ by the ith element of $\overline{\beta}$ for

[21] Alternatively, variables in $\overline{\alpha}$ occur free in $A(\overline{\alpha})$ and in $B(\overline{\alpha})$.

all i $(1 \geq i \geq n)$. Finally, where $\overline{\alpha}$ is $\alpha_1, \ldots, \alpha_n$ and $\overline{\beta}$ is $\beta_1, \ldots, \beta_n$, let $\overline{\alpha} \nparallel \overline{\beta}$ abbreviate $\check{\neg}(\alpha_1 = \beta_1 \wedge \ldots \wedge \alpha_n = \beta_n)$. The set of abnormalities, Ω, will be the union of the following sets:

$$\{\exists(A \wedge \neg A) \mid A \in \mathcal{F}_s\}$$
$$\{\exists(A(\overline{\alpha}) \wedge \neg A(\overline{\alpha}) \wedge B(\overline{\alpha})) \mid A(\overline{\alpha}) \in \mathcal{F}_s\}$$
$$\{\exists((A(\overline{\alpha}) \wedge \neg A(\overline{\alpha})) \wedge (A(\overline{\beta}) \wedge \neg A(\overline{\beta})) \wedge \overline{\alpha} \nparallel \overline{\beta}) \mid A(\overline{\alpha}), A(\overline{\beta}) \in \mathcal{F}_s\}$$
$$\{\exists((A(\overline{\alpha}) \wedge \neg A(\overline{\alpha}) \wedge B(\overline{\alpha})) \wedge (A(\overline{\beta}) \wedge \neg A(\overline{\beta}) \wedge B(\overline{\beta})) \wedge \overline{\alpha} \nparallel \overline{\beta}) \mid A(\overline{\alpha}), A(\overline{\beta}) \in \mathcal{F}_s\}$$

The first and third set may be removed as $B(\overline{\alpha})$ may be $A(\overline{\alpha})$ as well as $\neg A(\overline{\alpha})$. Note that the second set introduces another abnormality, actually another set of abnormalities, for every new sequence of objects that are inconsistent with respect to A and have some properties different from other objects inconsistent with respect to A. Some abnormalities of the set are existentially quantified and these do the job. Thus, in the case of Γ_{12}, **CLuN**-models verifying $\neg Rb$ are more abnormal than models verifying Rb.[22] Similarly, the third set introduces a new set of abnormalities whenever two (sequences of) objects that are inconsistent with respect to A are not identical. The fourth set introduces a new set of abnormalities whenever two (sequences of) objects that are inconsistent with respect to A and share B are not identical. There is obviously a lot of redundance in this set of abnormalities, but actually there is always redundance in all sets of abnormalities.

The adaptive logics to which this leads are $\mathbf{CLuN}^r_p$ and $\mathbf{CLuN}^m_p$, so named in honour of Peter Verdée, defined by the lower limit **CLuN**, the just defined set of abnormalities, and respectively Reliability and Minimal Abnormality. Note that the gain provided by these logics may be combined with the gain described in Section 7.4. Just replace $\mathbf{CLuN}^r$ or $\mathbf{CLuN}^m$ by, respectively, $\mathbf{CLuN}^r_p$ or $\mathbf{CLuN}^m_p$ in the definition of the combined logic. Obviously, the gain arrived at by the new set of abnormalities may also be obtained in the case of other lower limit logics. For example, one may replace the lower limit logic by **CLuNs** and adjust the set of abnormalities, replacing $\mathcal{F}_s$ by $\mathcal{F}^p_s$ in the definition of the four subsets.

Some further comments are in place. The first two concern inconsistency. The elaboration of Verdée's idea enables one to incorporate an attractive aspect of Priest's approach within the standard format. One still refers only to abnormal *formulas*. In other words, one depends only on the realm of our reasoning and not on the realm of the possible situations underlying our reasoning. Nevertheless, one is able to minimize the number of inconsistent *objects*. The second comment is that this approach is in standard format. So we have all the nice properties proven in Chapters 4 and 5, including Strong Reassurance. The third comment is that, as Peter Verdée indicated in his note, the idea may not only be applied to inconsistency-adaptive logics, but to all adaptive logics. The proposed extension of the set of abnormalities indeed applies generally. There are, however, cases where the extension does not seem to make sense. An example are the logics of inductive generalization discussed in Chapter 3. Where it makes sense, the extension is very welcome. Adaptive logics articulate *methods*

[22] All models of Γ_{12} verify $\exists x(Px \wedge \neg Px \wedge Rx)$ but models verifying $\neg Rb$ moreover verify $\exists x(Px \wedge \neg Px \wedge \neg Rx)$.

whence all multiplicity and refinement is most welcome. in Intro zetten