From Problem Solving to Argumentation: Pacioli's Appropriation of Abbacus Algebra

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Introduction

While Euclidean axiomatic geometry was a model for reasoning to certainty, algebra before Viète and Descartes did not enjoy this epistemic status. Algebra was considered an art, and although it was called the *ars magna*, it was still distant from an Aristotelian *scientia*. Algebra was introduced in Europe through the first Latin translations of al-Khwārizmī's *Algebra* by Robert of Chester (c. 1145), Gerard of Cremona (c. 1150) and Guglielmo de Lunis (c. 1215). The epistemic validity of algebraic problem solving depended on correctly performing the basic operations of algebra such as *al-jabr* and *al-muqābala*, translated in Latin as *restoration* and *opposition*. However, these operations were not explicitly defined. The meaning of the terms used for the operations is ambiguous within the work of a single author and between Arab authors and their Latin translations. These terms were used for different kinds of operations (Heeffer 2007a, Oaks and Alkhateeb 2007). Two centuries of algebraic practice within the abbacus tradition partially resolved the issue of ambiguity but no explicit description of the rules or a demonstration of their validity was provided. While the geometrical demonstrations for the canonical rules were reiterated from Arab authors, there is no attempt to argue for the validity of the analytical part of algebraic problem solving based on restoration and opposition. The absence of a description and / or explanation of rules is surprising, the more so as many abacus treatises do discuss and argue for the validity of other operations, for example the expansion of powers of irrational binomials. The only argumentation for the correctness of algebraic operations is a numerical test with the determined values of the problem. From 1494, with the publication of Pacioli's *Summa de arithmetica geometria proportioni et proportionalita*, this situation changes. During the sixteenth century, humanists such as Ramus, Peletier, Viète and Clavius participated in a systematic program to set up sixteenth-century algebra on solid Greek foundations. This was motivated by a need to cover up the Arabic (read barbaric) origins of algebra, rather than a sincere feeling of necessity for foundational work. The late discovered *Arithmetica* of Diophantus was taken as an opportunity by Viète to restore algebra "which was spoiled and defiled by the barbarians" to a fictitious pure form. To that purpose he devised a new vocabulary of Greek terms to obscure the Arabic roots of algebra "lest it should retain its filth and continue to stink in the old way".[2] The reality was that, with some exceptions, ancient Greek mathematics was more foreign to European mathematics than Indian and Arabic arithmetic and algebra were; the latter were well digested within the vernacular tradition (Høyrup 1996, Heeffer 2007b).

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[2] Viète 1591, translation by W. Smith in Klein 1968, 318.

Euclidean axiomatic geometry and logical syllogisms functioned as a model for reformulating the basic operations of algebra as 'common notions' (*notiones communes*). The Greek equivalent 'κοινναι έννοιαι' was derived from Proclus's commentary on the first book of Euclid where he quotes Aristotle in an attempt to demonstrate that "axiom and 'common notion' mean the same thing" (Proclus 1992, 152). He lists five common notions: 1) things which are equal to the same thing are equal to one another, 2) if equals be added to equals the wholes are equal, 3) if equals be subtracted from equals the remainders are equal, 4) the whole is greater than the part, and 5) things which coincide with one another are equal to one another. Cifoletti (2006) convincingly traces the path of Proclus's 'common notions' to the foundations of algebraic operations in *L'algebre* by Peletier (1554) and later Gosselin (1577) and Viète (1591). We may add that Michael Stifel in his *Arithmetica integra*, published a decade before Peletier, already demonstrated how the rules of algebra, which he calls *regulae Gebri*, can be derived from the common meanings (*communes sententiae*) as the common notions were labeled in the Euclid edition of Oronce Finé (1536).[3] During the seventeenth century, it was common practice to list the common notions in the introduction to algebra textbooks. William Leybourn (1660) added a translation of de Billy's *Abrégé des préceptes d'algèbre* with the list of common notions as the fourth part of his *Arithmetic*, first published in 1657. Sixteenth-century algebraic practice typically required experience and knowledge of many rules, which each had their own name such as the *regula alligationis* or *regula augmenti*.[4] The idea of a universal *mathesis* rendered knowledge of such rules superfluous. For John Wallis (1657, 85), algebra was basically not different from geometry or arithmetic. In the seventeenth century algebra starts from simple facts which can be formulated as axioms. All other knowledge about algebraic theorems can be derived from these axioms by deduction. Wallis introduced the term 'axioms' in relation to algebra in his early work, *Mathesis universalis*, included in his *Operum mathematicorum* (1657, 85). With specific reference to Euclid's *Elements*, he gives nine *Axiomata*, and also calls them *communes notationes*. Some years later, John Kersey (1673, Book IV, 179) expanded on Wallis' theory and formulated 29 axioms "or common notions, upon which the force of inferences or conclusions, about the equality, majority and minority of quantities compared to one another, doth chiefly depend". Although using many more axioms, he basically reformulates those from Wallis. The epistemic validity of seventeenth-century algebra not only depended on axioms or common notions. Also the use of theorems and lemmata as known from Euclidean geometry became a common practice in seventeenth-century algebra textbooks. However, it took about 150 years before this practice became established. The aspect of argumentative reasoning in algebra will be the focus of this paper. The idea that valid algebraic derivations can be generalized and formulated as theorems with a general validity was a completely new concept for the abbacus tradition. The idea is essential to the degree that its first appearance in the *Summa* by Pacioli makes it the first textbook on algebra.

[3] Stifel (1545) p. 229: "Si ab aequalibus aequalia auferas, quae remanent aequalia erunt. Et si ad aequalia addantur aequalia, quae super excrescunt aequalia erunt", and "Quae uni et eidem sunt aequalia, etiam inter se sunt aequalia". Compare with Finé (ed. 1544, pp-9-10).

[4] Widmann (1489) list more than twenty rules. Also Pacioli (1494) used many names, which was ridiculed by Cardano (1539, *Opera Omnia* IV, 79), who showed that you can turn any generalized derivation into a rule and give a name to it. He called the method *Regula de Modo* in his *Ars Magna*. See Heeffer (2007c) for the origin and evolution of proto-algebraic rules.

Before getting into detail on how this transition took place, I will first give an overview of the European algebraic tradition before Pacioli.

Abbacus algebra: a brief characterization

With some exceptions, algebraic practice was completely absent from the scholarly tradition or university curriculum before the mid-sixteenth century.[5] It took until the late seventeenth century before algebra became taught at universities. Instead, algebra flourished within the vernacular tradition of the abbacus schools in Italian cities during the fourteenth and fifteenth century. We call this the abbacus or abbaco tradition, spelled as in the *Liber abbaci* of Fibonacci (1202) to distinguish it from the abacus. The abbacists practised calculation with hindu-arabic numerals as opposed to calculation using material means such as the *tavola* or the abacus. The abbacus masters were hardly known before the first transcriptions of their manuscript treatises by Gino Arrighi during the 1960s and 1970s. It is only with Warren van Egmond's extensive catalogue of manuscripts that we have a fairly complete picture of this tradition (van Egmond 1980). Abbacus masters earned a living from teaching commercial arithmetic to sons of merchants and artisans, renting rooms and occasionally surveying assignments (Goldthwaithe 1972-3). For the sake of prestige and also out of genuine interest many of them wrote long treatises on arithmetic and algebra in which they solved hundreds of problems. Such manuscripts were often illustrated and presented as gifts to patrons and important merchants. Van Egmond's catalogue lists about 250 extant abbacus manuscripts kept in libraries all over the world, many dealing with algebra. The seemingly evident narrative that the tradition was initiated with Fibonacci's book is currently challenged by Jens Høyrup (2005). Although the first abbacus manuscript dealing with algebra dates from 1307, there is evidence that the tradition existed at Fibonacci's time. Furthermore it seems that it originated from the Provençe (south of France) and Catalan regions (north of Spain) (Høyrup 2006).

Abbacus algebra is all about problem solving. Most of the folios of these sometimes hefty collections deal with arithmetical and algebraic solutions to a large number of problems. In these treatises the introduction – if there is one – explains the rules of algebra, possibly with a geometrical demonstration. The early treatises are limited to the six rules of Arabic algebra, but later *maestri d'abbaco* extend the list to more types, resulting in the rather preposterous list of 198 equation types of Maestro Dardi (van Egmond 1983) accompanied by problems to illustrate each of them. Later treatises occasionally discuss addition and multiplication of polynomials as an introduction to algebra. But that is as far as it goes for the theory. The bulk of the text is pure problem solving. There is a surprising consistency in the structure, style and rhetoric of abbacus texts during the two centuries of their existence. Practically every text dealing with algebra follows the same rigid structure which can be divided into six parts:

1. *problem enunciation*: in a first section the problem text is provided and a question is posed. Most problems are set in a practical context.

[5] One such exception is the *Quadripartitum numerorum* of Jean de Murs (1343) (L'Huillier 1990), which provides evidence that algebra, as known from Fibonacci's *Liber abbaci*, was studied in scholarly circles in Paris. However, it is unlikely that it was ever taught within the *quadrivium*.

2. *choice of the rhetorical unknown*: every solution start with the sentence “pose that <some unknown quantity of the problem> equals <some quantity of> the *cosa*” (the rhetorical unknown). Often a clever choice of the unknown or a power of the unknown is an important step in the solution of the problem. Most abbacus texts deal with a single unknown, though there are some exceptions.[6] A straightforward translation of unknown quantities of the problem into symbolic form is a practice which is established only during the eighteen century.
3. *manipulation of polynomials*: using the unknown, the problem text is formulated in terms of coequal polynomials and manipulated in such a way that these are kept equal. The vernacular terms *ristorare* and later *ragguagliare* are used for both the restoration and opposition operations.
4. *reduction to a canonical form*: the purpose of manipulating the polynomials is to reduce them to a form in which a standard rule applies. This marks the end of the analytical part of the reasoning.
5. *applying a rule*: usually the rule is reformulated and literally applied. Typically it includes the normalization of the equation by dividing it by the coefficient of the square term even if this amounts to dividing by one.
6. *numerical test*: often, but not always, the validity of the solution is checked by a numerical test using the root of the equation. This test is always performed on the problem enunciation and not on the equation.

The lack of symbolism in abbacus algebra is compensated by the rigid rhetorical structure. Each problem is dealt with in the same way. Every rule is reformulated and applied as it were for the first time. Repetition, cadence and structure facilitate the understanding and memorization of the problem solving procedure. Only in very rare cases are problems and solutions generalized or is there a transfer of results from one problem to another. Almost all extant texts before the end of fifteenth century are characterised by this kind of abbacus problem solving. In the south of France, the situation changes with *Larismethique nouellement compose* of 1520 by Estienne de La Roche. In Germany we have the anonymous Vienna codex 5277, written between 1500 and 1518 (Kaunzner, 1972). Together with Pacioli’s *Summa*, published in Italy, these works on algebra share a dramatic departure from algebra as problem solving to an argumentative form of reasoning.

Plagiarism vs. appropriation

Ever since Giorgio Vasari’s (1550) encyclopedic biography of painters, sculptures and architects it was suspected that Pacioli based his published work on several manuscripts from the abbacus tradition. These claims have partially been substantiated in relation to the *Geometry*. Gino Loria was the first to show that the *Libellus* by Pacioli is a literal translation of *De corporibus regularibus* of Piero della Francesa. Margaret Daly Davis (1977) demonstrated that 27 of the problems on regular bodies in the geometry part of Pacioli’s *Summa* are reproduced almost literally from Pierro della Francesco’s *Trattato d’abaco*. Ettore Picutti (1989) cites the historian Girolamo Mancini (915) who discovered that treatise XI of distinction 9 of the *Summa*, entitled *De scripturis*, is a transcription of a

[6] See my forthcoming “The *Regula Quantitatis*: From the Second Unknown to the Symbolic Equation”.

manuscript by Giorgi Charini. Picutti himself has shown that "all the 'geometria' of the *Summa*, from the beginning to page 59^v (119 folios), is the transcription of the first 241 folios of the Codex Palatino 577".[7] He includes a reproduction of one part of f. 51^r of the geometry part and the corresponding text from the manuscript to prove his claim. In relation to the algebra contents Franci and Rigatelli (1985) further claim that a detailed study of the sources of the *Summa* would yield many surprises. Yet, for the part dealing with algebra, no hard evidence for plagiarism has been given. While studying the history of problems involving numbers in geometric progression (*GP*), we found that a complete section of the *Summa* is based on the *Trattato di Fioretti* of Maestro Antonio de Mazzinghi written before 1383.[8] However, on closer inspection we cannot use the qualification of plagiarism for Pacioli's use of problems and problem solving methods by Antonio. Instead, the chapter provides us with a rare insight in Pacioli's restructuring old texts, and as such, in the shift of rhetoric and use of argumentation in algebra books. Pacioli's appropriation of abbacus texts is an exemplary first step of the humanist project for providing solid foundations to the art of algebra.

There is a strong parallel between accusations of Pacioli's plagiarism and the algebra book of 1520 by de La Roche. Aristide Marre discovered that this printed work contained large fragments that were literally copied from Chuquet's manuscript (Marre 1880, introduction). Indeed, especially on the *Appendice*, which contains the solution to many problems, Marre (1881) writes repeatedly in footnotes "This part is reproduced word by word in the *Larismethique*" with due references. However, giving a transcription of the problem text only, Marre withholds that for many of the solutions to Chuquet's problems de la Roche uses different methods and an improved symbolism.[9] In general, the *Larismethique* is a much better structured text than the *Triparty*. de la Roche reorganizes Chuquet's manuscript according to the structure of Pacioli's *Summa*. He adds introductions explaining problem solving methods, as the one using *la regle de la quantite*. He even adopts Pacioli's classification in distinctions and chapters. Marre's conclusion on de la Roche is very harsh (Marre 1881, 28; translation mine):

> One can state, pure and simple that, [de la Roche] copied a mass of excerpts from the *Triparty*, that he omitted several important passages, especially on algebra, that he abridged and extended others for producing the *Arismetique*, a work much inferior to the *Triparty*.

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[7] Picutti (1989, 76): "tutta la «Geometria» della *Summa* dagli inizi a p. 59v (cioè 119 pagine in folio) è trascrizione delle prime 241 carte del codice Palatino 577 della Biblioteca Nazionale di Firenze, di autore ignoto (ma che anni fa abbiamo attribuito e continuiamo ad attribuire tuttora a maestro Benedetto da Firenze)." Simi and Rigatelli (1993, 463) wrongly cite Picutti that 'all the 'geometria' of the Summa, from the beginning on page 59v. (119 folios), is the transcription of the first 241 folios of the Codex Palatino 57". This quote has been repeated by other authors but lead to confusion because the page numbers do not match. I am grateful to Alan Sangster who pointed out the mistake and provided me a with copy of Picutti's article.

[8] Mazzinghi died in 1383. His original writings are lost, but 42 of his problems are reproduced in the Siena manuscript L. IV. 21 of c.1463. A transcription is given by Arrighi 1967. We also consulted the unpublished manuscript Magl. XI.120, *Regolo del'arzibra*, which adds several other problems on numbers in *GP*, but following the same methods.

[9] The improvements relate especially to the use of multiple unknowns. See my "The *Regula Quantitatis*" for more details.

Comparing the problem texts only, the denunciation of de la Roche would also apply to numerous others, including Chuquet's use of various problems from Fibonacci and Barthelemy of Romans (Spiesser 2003). Because of Marre's misrepresentation of the *Larismethique* as a grave case of plagiarism the importance of this work has been seriously underestimated.

Extracting general principles from algebraic practice

Pacioli discusses thirty problems on numbers in *GP* (from the 35 problems in distinction 6, treaty 6, article 14), before he treats algebra itself. Most of these problems correspond with problems from Maestro Antonio, often using the same values. More importantly, the original problem solving methods are reproduced literally by Pacioli, including one rare instance using two unknowns and one which Antonio calls "without *cosa*". Relevant for our discussion are two introductory sections preceding the problems. Pacioli gives some theoretical principles on three numbers in GP in the section called *De tribus quantitatibus continue proportionalium* (distinction 6, treaty 6, article 12, f. 88^v).[10] Another section on keys, lists theoretical principles on four numbers in GP under the heading *De clavibus seu evidentiis quantitatum continue proportionalium*, (distinction 6, treaty 6, article 11, f. 88^r). Pacioli does not explain where these principle are derived from. He only gives some numerical examples. However, a close comparison with the *Trattato di Fioretti* shows that several are extracted from Maestro Antonio's solution. Let us look at one example involving three numbers in GP with their sum given and an additional condition.

Pacioli	Maestro Antonio
Make three parts of 13 in continuous proportion so that the first multiplied with the sum of the other two, the second [multiplied] with the [sum of the] other two, the third [multiplied] with the [the sum of the] other two, and these multiplications added together makes 78.[11]	Make three parts of 19 in continuous proportion so that the first multiplied with [the sum of] the other two, the second part multiplied with the [sum of the] other two, the third part multiplied with the [the sum of the] other two, and these sums added together makes 228. Asked is what are the parts.[12]

In modern symbolism, using multiple unknowns, the general structure of the problem is as follows :

[10] I have used the 1523 edition but the numbering of pages and sections is practically identical with the original.

[11] Pacioli, f. 91^r: "Famme de 13 tre parti continue proportionali che multiplicata la prima in laltre dui, la seconda in laltre dui, la terça in laltre dui, e queste multiplicationi gionti asiemi facino 78".

[12] Arrighi 1967, p. 15: "Fa' di 19, 3 parti nella proportionalità chontinua che, multiplichato la prima chontro all'altre 2 e lla sechonda parte multiplichato all'altre 2 e lla terza parte multiplichante all'altre 2, e quelle 3 somme agunte insieme faccino 228. Adimandasi qualj sono le dette parti".

$$\frac{x}{y} = \frac{y}{z}$$
$$x + y + z = a$$
$$x(y+z) + y(x+z) + z(x+y) = b$$

The *Trattato di Fioretti* is the first historical source in which this type problem is being treated and Maestro Antonio poses the problem with values $a = 19$ and $b = 228$. Expanding the products and summing the terms gives

$$2xy + 2xz + 2yz = 228,$$

but as $y^2 = xz$ we can write this expression also as

$$2xy + 2y^2 + 2yz = 228, \text{ or } 2y(x + y + z) = 228.$$

Given that the sum of the three terms is 19, dividing 228 by 19 thus gives us the double of the middle part. Therefore the middle part is 6. Antonio then proceeds to find the other terms with the procedure of dividing a number into two extremes such that their product is equal to the square of the middle term. The problem thus reduces to dividing 19 – 6 = 13 into two parts so that 6 is the middle term. Thus the product of the two parts is the square of the middle term or 36. Given the product and the sum of two numbers their values can be found as the roots of the quadratic equation[13]

$$x^2 + 36 = 13x.$$

Maestro Antonio uses the rule for the two positive roots of the quadratic equation to find the two parts as 4 and 9.[14]

Pacioli solves the problem in essentially the same way. However, the rhetorical structure is quite different. He poses the problem with values $a = 13$ and $b = 78$. Maestro Antonio performs an algebraic derivation on a particular case. Instead, Pacioli justifies first part of the solution as an application of a more general principle, defined by a general key:[15] (Pacioli 1494, f. 91^r):

> This can be solved using the fourteenth key. Which says that you have to divide the sum of these multiplications, thus 78, by the double of 13. And this 13 is the sum of

[13] This is an old Babylonian problem solved by *igūm* and *igibūm* (Høyrup 2002, 55-6) and the likely source for the prototype problem of early Arabic algebra.

[14] The recognition of two positive roots for this type of equations was known from Arabic and even Babylonian algebra. However, during the abbacus tradition it gradually disappears. In Heeffer (2007a) we argue that this evolution is invoked by the specific rhetoric of abbacus problem solving.

[15] Pacioli 1494, f. 91^r: "Questa solverai p*er* la 14^a chiave. La q*uel*l*a* dice che stu p*ar*tirai la su*m*ma de ditte multiplicatio*n*i, cioè 78 per lo doppio de 13. E q*uel*l*a* 13 sira la summa de ditte q*uan*tita ne virra la 2^a parte. Donca parti 78 in 26 neve 3 p. la 2^a parte".

> these quantities, which will give you the second part. Thus divide 78 by 26 gives 3 for the second part.

The fourteenth key he is referring to, is formulated as follows, some pages earlier:[16] (Pacioli 1494, f. 89^{v}):

> On three quantities in continuous proportion, when multiplying each with the sum of the other two and adding these products together. Then divide this by double the sum of these three quantities and this always gives the second quantity.

This particular key is one of several variations on the algebraic derivation of Maestro Antonio, each presented as a general principle. In modern notation:

$$y = \frac{x(y+z)+y(x+z)+z(x+y)}{2(x+y+z)} \qquad \text{(Key 14)}$$

Having determined the value for the middle part, Pacioli continues to solve Antonio's problem in a different way, by means of algebra. The problem reduces to one of dividing 10 into two parts so that 3 is the middle term in *GP*. Using the *cosa* for the smaller term and $(10 - x)$ for the larger, the product of the two is 9, the square of the middle term. He arrives at the quadratic equation with 1 and 9 as its roots. Elsewhere Pacioli writes that this sub problem, finding the two extremes of three numbers in *GP* with the middle term given, can be solved either by algebra or following a theorem of Euclid.[17] Drawing on Euclid, he provides legitimation for the procedure which needs no further explanation or proof.

Cardano also frequently uses this procedure, in his *Practica arithmetice*. Instead of referring to Euclid he defines a general rule for the procedure in chap. 42, paragraph 116, a divides into two parts x_1 and x_2 with b as mean proportional as follows:[18]

$$x_{1,2} = \frac{a}{2} \pm \sqrt{\left(\frac{a}{2}\right)^2 - b^2}$$

To show one more example of the way Pacioli turns the solutions of Maestro Antonio into theorems of algebra let us look at a problem involving four numbers in *GP*. In modern symbolic form we can formulate the problem as:

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[16] Pacioli 1494, f. 89^{v}: "De 3 quantita c*on*tinue proportionali che multiplicata ciascuna in laltre doi e quelli multiplicationi gionti insiemi. E poi questo p*ar*tito nel doppio de la summa de ditte 3 qua*n*tita e sempre laverimento sera la 2^{a} quantita".

[17] Pacioli 1494, f. 89^{r}. Pacioli only refers to "the second of Euclid". On other occasions he uses "the fourth of the second of Euclid". This leads use to book 2, prop. 4.

[18] Si sint duo numeri utpote 24 et 10 et velis dividere 24 in duas partes in quarum medio cadat 10 in continua proportionalitate, quadra dimidium maioris quod est 12 sit 144. Detrahe quadratum minoris quod est 100 remanet 44, cuius $\Re$ addita ad 12 et diminuta faciet duos numeros inet quos 10 cadit in medio in continua proportionalitate, et erunt 12 p $\Re$ 44 et 10 et 12 m $\Re$ 44.

$$\frac{x}{y} = \frac{y}{z} = \frac{z}{u}$$
$$x + u = a$$
$$xyzu = b$$

Both Maestro Antonio and Pacioli use the values $a = 17\frac{1}{2}$ and $b = 2916$ (Antonio, problem 8, Arrighi 26-29; Pacioli 1494, problem 25, f 95^v). Antonio commences with the keen observation that the square root of the product of all terms is equal to the product of the first and the fourth. Such pertinent perception of the master is again an opportunity for Pacioli to turn this into a general theorem formulated as key 7: [19]

given that $\frac{x}{y} = \frac{y}{z} = \frac{z}{u}$ we can derive $xu = yz$

Antonio finds that the square root of 2916 is 54. Knowing the sum and the product of two numbers within a *GP*, he then reduces the problem to one of dividing 17 ½ into two parts so that their product equals 54, following a procedure discussed before. Hence he follows the standard abbacus method of deriving a form in which one of the standard rules can be applied. Pacioli instead refers to Euclid for finding the values of the two extremes and then follows Antonio's solution for determining the other parts.

In his *Practica Arithmeticae* of 1539, Cardano continues to build on what was initiated by Pacioli. In chapters 42 and 51 he lists many rules which are algebraic theorems. Some of them are directly taken from Pacioli, such as Pacioli's key 14.[20] Many others are derivations by Cardano himself as the following on four quantities in *GP*:[21]

$$\left(\frac{x+y+z+u}{x+u}\right) = \frac{x+z}{x+z-y} + \frac{y+u}{y+u-z}$$

Conclusion

We can be certain that Pacioli mined Antonio's treatise for general principles such as the one we have discussed, because they are used nowhere else than for solving the problems taken from Antonio. Pacioli has chosen to present some typical derivations as general rules which are later applied to solve problems in a clear and concise way. Even with the body of evidence against him, we should be careful in accusing Pacioli of plagiarism. At best, we observe here an appropriation of problems and methods. The restructuring of

[19] Pacioli 1494, f. 88v : "sempre tanto fa a multiplicare la prima nela quarta quanto a multiplicare la seconda nela terza". Pacioli adds that the square of the multiplications equals the product of all four.

[20] Cardano 1539, Cap. 42, rule 94: "Pendet haec ex dicendis in regula 3 cum fuerint tres quantitates continuae proportionales, quod ex ductu uniuscuiusque partis in alteram fiet, si divindatur per duplatum aggregatum omnium, exhibit secunda quantitatis".

[21] Cardano 1539, Cap. 51, rule 31: "Omnium quatuor quantitatum continuae proportionalium proportio totius aggregati ex omnibus quatuor ad aggregatum primae et quartae est veluti aggregati primae et tertiae ad aggregatum ipsum, dempta secundae, aut aggregatis secundae et quartae ad ipsummet aggregatum demta tertia".

material and the shift in rhetoric is in itself an important aspect in the development of sixteenth-century textbooks on algebra. Pacioli raised the testimonies of algebraic problem solving from the abbacus masters to the next level of scientific discourse, the textbook. When composing the *Summa*, Pacioli had almost twenty years of experience in teaching mathematics at universities all over Italy. His restructuring of abbacus problem solving methods is undoubtedly inspired by this teaching experience.[22] By reformulating algebraic derivations of abbacus masters as theorems of algebra, and using Euclid's theorems for algebraic quantities, Pacioli introduces a new style of argumentative reasoning which was absent from abbacus algebra. Pacioli's *Summa* and Cardano's *Practica Arithmeticae* had a decisive influence and the two works together shaped the structure of future treatises on algebra. Authors such as Michael Stifel, Pedro Nunez, Jacques Peletier, Buteo and also Viète continued the new argumentative rhetoric. Although Viète does not mention any authors except for Diophantus, from the examples he discusses in the *Zetetica* (1591) and *De æqvationvm recognitione et emendation* (1615), the influence of Cardano is clearly present.

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[22] Pacioli is often wrongly considered an abbacist (e.g. Biagioli, 1989). In fact, he enjoyed the social status of a well-paid university professor. Between 1477 and 1514, he taught mathematics at the universities of Perugia, Zadar (Croatia), Florence, Pisa, Naples and Rome (Taylor, 1942).

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