THE ORIGIN OF THE PROBLEMS IN EULER'S *ALGEBRA*

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1 Christoff Rudolff's *Coss* as a source

Leonard Euler's *Vollständige Anleitung zur Algebra* was published in two volumes by the Academy of Sciences in St-Peterburg in 1770 [2]. With the exception of Euclid's *Elements* it is the most printed book on mathematics [11, xxxiii]. It was translated into Russian (1768-9), Dutch (1773), French (1774), Latin (1790), English (1797, 1822) and Greek (1800). One popular German edition from Reclam Verlag sold no less than 108,000 copies between 1883 and 1943 [5]. Euler wrote his *Algebra* originally in German. Based on internal evidence, Fellmann dates the manuscript at 1765/1766 [3, 108], when he returned from Berlin to St-Petersburg, some years before he went completely blind.

In his selection of problems in the *Algebra*, Euler shows himself familiar with the typical recreational and practical problems of Renaissance and sixteenth-century algebra books. An extensive historical database with algebraic problems [4], immediately reveals Euler's use of the Stifel's edition of Rudolff's *Coss* for his repository of problems. This work, published in 1525 in Strassburg [6], was the first German book entirely devoted to algebra. Stifel used many problems from Rudolff in his *Arithmetica Integra* of 1544 and found the work too important not to publish his own annotated edition [9]. The table below shows an example of the many textual evidences of Euler's use of the book.

The first volume of Euler's *Algebra* on determinate equations contains 59 numbered problems. Two thirds of these can be directly matched with the problems from Rudolff. Some are literal reproductions, as the example given. Other were given new values or were slightly reformulated. The second part on indeterminate equations also has 59 problems and although the correlation here is manifest less, many problems still originate from Rudolff.

While the first sections include some illustrative examples, all the problems appear in the second part on equations, exactly as in Rudolff's book. The third chapter dealing with linear equations in one unknown has 21 problems. They clearly show how Euler successively selected suitable examples from Rudolff's book. The problems are put in practically the same order as Rudolff's.[1] They

[1] Euler's problem 8, 9, correspond with Rudolff's 16 and 9, 10 and 11 with 9, problems 12 to 21 with Rudolff's 24, 26, 6, 50, 53, 59, 68, 97, 98 and 110 respectively.

Rudolff, *Coss*, 1553	**Euler, *Opera Omnia*, I, I**
Drei haben ein hauss kaufft fur 100 fl. Begert der erst vom andern $\frac{1}{2}$ seyns gelts, so hette er das hauss alleyn zu bezalen. Der ander begat vom dritten $\frac{1}{3}$ seynes gelts das er das hauss alleyn könte bezalen. Der dritt begert vom ersten $\frac{1}{4}$ seyns gelts das er mochte das hauss alleyn bezalen. Wie vil hat yeder gelt gehabt? (f. 216^r, problem 123)	Drey haben ein Haus gekauft für 100 Rthl. der erste begehrt vom andern $\frac{1}{2}$ seines Gelds so könnte er das Haus allein bezahlen; der andere begehrt vom dritten $\frac{1}{3}$ seines Geldes, so könnte er das Haus allein bezahlen. Der dritte begehrt vom ersten $\frac{1}{4}$ seines Gelds so möchte er das Haus allein bezahlen. Wie viel hat jeder Geld gehabt? (p. 235)

include well-known problems from recreational mathematics, as discussed in [10] and [8]: the legacy problems, two cups and a cover, alligation, division and overtaking problems. The fourth chapter deals with linear problems in more than one unknown, including the mule and ass problem, doubling each other's money and men who buy a horse.[2] The fifth chapter is on the pure quadratic equation with five problems all taken from Rudolff.[3] The sixth has ten problems on the mixed quadratic equation, of which nine are taken from Rudolff.[4] Chapter eight, on the extraction of roots of binomials, has five problems, none from Rudolff. Finally, the chapter of the pure cubic has five problems, two from Rudolff and on the complete cubic there are six problems, of which four are from Stifel's addition. Cardano's solution to the cubic equation was published in 1545, between the two editions of the *Coss*. While Euler also treats logarithms and complex numbers, no problems on this subject are included.

Having determined the source for Euler's problems, the question remains why he went back almost 250 years. The motive could be sentimental. In the Russian Euler archives at St-Petersburg a manuscript is preserved containing a short autobiography dictated by Euler to his son Johann Albrecht on the first of December, 1767 [3, 11]. He states that his father Paulus taught him the basics of mathematics using the Stifel edition of Christoff Rudolff's *Coss*. The young Euler practiced mathematics for several years using this book, studying over four hundred algebra problems.[5] When he decided to write an elementary textbook, Euler conceived his *Algebra* as a self study book, much as he used Rudolff's *Coss*, the educational value of which Euler amply recognized.

[2] Euler's problem 3 to 7 correspond with Rudolff's 132, 112, 122, 123 and 128 respectively.

[3] Euler's first three are from the fifth rule, problems 2, 4 and 11. The fourth is problem 240 from the first rule and the last is problem 20 from the second rule.

[4] The first is the seventh problem from the seventh rule. The next are from Rudolff's fifth rule, problems 2, 4, 11, 17, 18, 19, 26 and 33 respectively.

[5] Fellmann 1995, 11: "vi ich bey Zeiten von meinem Vater den ersten Unterricht erhielt; und weil derselbe einer von den Discipeln des weltberühmten Jacobi Bernoulli gewesen, so tractete er mir sogleich die erste Gründe der Mathematic beizubringen, und bediente sich zu diesem End des Christophs Rudolphs Coss mit Micheals Stiefels Anmerckungen, worinnen ich mich einige Jahr mit allem Fleisch übte".

2 An example

Arithmetic books before the 16^{th} century use a great many recipes to solve a wide variety of problems. With the emergence of symbolic algebra in the second half of the 16^{th} century, many of these recipes became superfluous and the corresponding problems lost their appeal. Several types of problems disappeared from arithmetic and algebra books for the next two centuries. The algebra textbooks of the eighteenth century abandoned the constructive role of problems in producing algebraic theorems. Problems were used only to illustrate theory and practice the formulation of problems into the algebraic language. The new rhetoric of problems in algebra textbooks explains why Euler found in Rudolff's *Coss* a suitable repository of examples.

A typical example of this type of problems is a legacy problem, which emerged during the late Middle Ages and is found in Fibonacci's *Liber Abbaci* [7, 399]. It is a riddle about a dying man who distributes gold pieces to an unknown number of children, each receiving the same amount. With i children, each child gets ai plus $\frac{1}{n}^{th}$ of the rest. The question is how many children there are and what the original sum is. The solution in early textbooks depends on a rule of thumb which expresses the value of the legacy as $a\,(n-1)^2$. Rudolff provides the first algebraic solution in our database [6, 252^r]. The problem has $n = 10$ and $a = 1$. Using the unknown for the legacy, he expresses the share of the first person as $\frac{x-1}{10} + 1$ and calculates the share of the second as $\frac{9x-29}{100} + 2$. By equating these two, he arrives at a solution of 81. Also Cardano treats the problem but first giving $\frac{1}{7}^{th}$ of the remainder and then 100 [1, FFii^r]. He gives a solution by rule of thumb but adds "potest etiam fieri per algebra" (it can also be done by algebra) and provides an algebraic solution to the problem. He equates the share of the first son with that of the second

$$\frac{1}{7}x + 100 = \frac{1}{7}\left(\frac{6}{7}x - 100\right) + 200$$

and easily solves the equation to $x = 4200$. Several later works follow this reasoning. In contrast, Euler's solution is refreshingly simple and elegant. He uses the example in which $n = 10$ and $a = 100$. He notices that the differences between subsequent parts are the same and can be expressed as

$$100 - \frac{x + 100}{10}$$

with x as the share of each. As all children get an equal part, these differences must be zero, therefore $1000 - x - 100 = 0$ or $x = 900$.

After Euler, many of the textbooks on elementary algebra of the 19^{th} century include this and other problems from Rudolff as excercises. In this way, Euler's *Algebra* functioned as a gateway for the revival of Renaissance recreational problems.

References

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