On the Nature and Origin of Algebraic Symbolism

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THE MYTH OF SYNCOPATED ALGEBRA

Ever since Nesselmann's study on "Greek algebra" (1842), historical accounts on algebra draw a distinction in rhetorical, syncopated and symbolic algebra. This tripartite distinction has become such a common-place depiction of the history of algebraic symbolism that modern-day authors even fail to mention their source (e.g., Boyer 1968, 201; Flegg and Hay 1985; Struik 1987). The repeated use of Nesselmann's distinction in three *Entwickelungstufen* on the stairs to perfection is odd because it should be considered a highly normative view which cannot be sustained within our current assessment of the history of algebra. Its use in present-day text books can only be explained by an embarrassing absence of any alternative models. There are several problems with Nesselmann's approach.

A PROBLEM OF CHRONOLOGY

Firstly, if seen as steps within a historical development, as is most certainly the view by many who have used the distinction, it suffers from some serious chronological problems. Nesselmann (1942, 302) places Iamblichus, Arabic algebra, Italian abbacus algebra and Regiomontanus under rhetorical algebra ("Die erste und niedrigste Stufe") and thus covers the period from 250 tot 1470. A solution to the quadratic problem of al-Kwārizmī is provided as an illustration. The second phase, called syncopated algebra, spans from Diophantus's *Arithmetica* to European algebra until the middle of the seventeenth century, and as such includes Viète, Descartes and van Schooten. Nesselmann discusses problem III.7 of the *Arithmetica* as an example of syncopated algebra. The third phase is purely symbolic and constitutes modern algebra with the symbolism we still use today. Nesselmann repeats the example of al-Kwārizmī in modern symbolic notation to illustrate the third phase, thereby making the point that it is not the procedure or contextual elements but the use of symbols that distinguishes the three phases.

Though little is known for certain about Diophantus, most scholars situate the *Arithmetica* in the third century which is about the same period as Iamblichus (c. 245-325). So, syncopated algebra overlaps with rhetorical algebra for most of its history.

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This raises serious objections and questions such as "Did these two systems influence each other?". Obviously, historians as Tropfke (1933, II, 14) and Gandz (1936, 271) were struck my this chronological anomaly but formulated an explanation. They claim that Arabic algebra does not rely on Diophantus's syncopated algebra but descends instead from Egyptian and Babylonian problem-solving methods which were purely rhetorical. However, these arguments are now superseded by the discovery of the Arabic translations of the *Arithmetica* (Sesiano 1982). Diophantus was known and discussed in the Arab world ever since Qustā ibn Lūqā (c. 860). So if the syncopated algebra of Diophantus was known by the Arabs why did it not affect their rhetorical algebra?

THE ROLE OF SCRIBES

The earliest extant Greek manuscript, once in the hands of Planudes and used by Tannery, is Codex Matritensis 4678 (ff. 58-135) of the thirteenth century. The extant Arabic translation studied by Sesiano was completed in 1198. So no copies of the *Arithmetica* before the twelfth century are extant. The ten centuries separating the original text from the earliest extant Greek copy is a huge distance. Two important revolutionary changes took place around the ninth century: the transition of papyrus to paper and the replacement of the Greek uncial or majuscule script by a new minuscule one. Especially the transition to the new script was a drastic one. From about 850 every scribe copying a manuscript would almost certainly adopt the minuscule script (Wilson 1983, 1996: 66-7). Transcribing an old text into the new text was a laborious and difficult task. Certainly not an undertaking to be repeated when a copy in the new script was already somewhere available. It is therefore very likely that all extant manuscript copies are derived from one Byzantine archetype copy in Greek minuscule. Although contractions where also used in uncial texts, the new minuscule much facilitated the use of ligatures. This practice of combining letters, when performed with some consequence, saved considerable time and therefore money. Imagine the time savings by consistently replacing *άριθμος*, which appears many times for every problem, with ς in the whole of the *Arithmetica*. The role of professional scribes should therefore not be underestimated. Although we find some occurrences of shorthand notations in papyri, the paleographic evidence we now have on a consistent use of ligatures and abbreviations for mathematical words points to a process initiated by mediaeval scribes much more than to an invention by classic Greek authors. Whatever syncopated nature we can attribute to the *Arithmetica* it is mostly an unintended achievement of the scribes.[2] The complete lack of any syncopation in the Arabic translation further supports this thesis. The name for the unknown and the powers of the unknown and even numbers are written by words in Arabic translation. The lack of well-established Hindu-Arabic numerals seems to indicate that the Arabic translation was faithful to a Greek majuscule archetype. Sesiano (1882: 75) argues that the Arabic version relies on the commentary by Hypathia while the Greek versions relate to the original text with some early additions and interpolations.

In so far the *Arithmetica* deserves the special status of syncopated algebra, it is very unlikely that the practice of using ligatures in Greek texts is a practice that developed from the ninth century and not of Diophantus during the third century. This overthrows much of the chronology as proposed by Nesselmann.

[2] This view is also expressed in relation to Archimedes's works (Netz and Noel, 2007).

SYMBOLS OR LIGATURES?

A third problem concerns the interpretation of the qualifications ‘rhetorical’ and ‘syncopated’. Many authors of the twentieth century attribute a highly symbolic nature to the *Arithmetica* (e.g. Kline 1972, I: 139-40). Let us take Cajori (1928, I, 1993: 71-4) as the most quoted reference on the history of mathematical notations. Typical for Cajori’s approach is the methodological mistake of starting from modern mathematical concepts and operations and looking for corresponding historical ones. He finds in Diophantus no symbol for multiplication and addition is expressed by juxtaposition. For subtraction the symbol is an inverted ψ. As an example he writes the polynomial

x^3+13x^2+5x+2 as $K^{\Upsilon}\overline{\alpha}\Delta^{\Upsilon}\overline{\iota\chi}\varsigma\overline{\varepsilon}\mathring{\mathrm{M}}\overline{\beta}$

where K^{Υ}, Δ^{Υ}, ς are the third, second and first power of the unknown and $\mathring{\mathrm{M}}$ represents the units. Higher order powers of the unknown are used by Diophantus as additive combination of the first to third powers.

Cajori makes no distinction between symbols, notations or abbreviations. In fact, his contribution to the history of mathematics is titled *A History of Mathematical Notations*. In order to investigate the specific nature of mathematical symbolism one has to make the distinction somewhere between symbolic and non-symbolic mathematics. This was, after all, the purpose of Nesselmann’s distinction. We take the position together with Heath (1885), Ver Eecke (1926) and Jacob Klein, that the letter abbreviations in the *Arithmetica* should be understood purely as ligatures (Klein 1936; 1968: 146):

> We must not forget that *all* the signs which Diophantus uses are merely word abbreviations. This is true, in particular for the sign of “lacking”, ↑, and for the sign of the unknown number, ς, which (as Heath has convincingly shown) represents nothing but a ligature for αρ (*ἀριθμος*).

Even Nesselmann acknowledges that the ‘symbols’ in the *Arithmetica* are just word abbreviations (“sie bedient sich für gewisse oft wiederkehrende Begriffe und Operationen constanter Abbreviaturen statt der vollen Worte”). In his excellent French translation of Diophantus, Ver Eecke consequently omits all abbreviations and provides a fully rhetorical rendering of the text as in “Partager un carré proposé en deux carrés” (II.8), which makes it probably the most faithful interpretation of the original text.[3]

This objection marks our most important critique on the threefold distinction: symbols are not just abbreviations or practical short-hand notations. Algebraic symbolism is a sort of representation which allows abstractions and new kinds of operations. This symbolic way of thinking can use words, ligatures or symbols, as we will argue further.

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[3] This problem led Fermat to add the marginal note in his copy of Bachet’s translation “Cubum autem in duos cubos, aut quadratoquadratum in duos quadratoquadratos, et generaliter nullam in infinitum ultra quadratum potestatem in duos eiusdem nominis fas est dividere”. If Fermat had used the ‘syncopated’ algebra of Diophantus he might have had some marginal space left to add his “marvelous proof” for this theorem.

The distinction between words, word abbreviations and symbols is in some way irrelevant with regards to the symbolic nature of algebra.

COUNTER EXAMPLES

A final problem for Nesselmann's tripartite distinction is that now, almost two centuries later, we have a much better understanding of the history symbolic algebra. Nesselmann relied mostly on the Jesuit historian Cossali (1797-9) for a historical account of Italian algebra before the sixteenth century. Except for Rafello Canacci, Cossali does not discuss much the algebra as it was practiced within the abbacus tradition of the fourteenth and fifteenth century. Guillaume Libri, who had collected many manuscripts from this tradition, describes and published several transcriptions in his *Histoire des sciences mathématiques en Italie* published in 1838. Strangely, the well-informed Nesselmann does not seem to know the book and thus remains ignorant of the continuous practice of algebra in Italy since Fibonacci and the first Latin translations of al-Kwārizmī. It is only since the past decades that we have a more complete picture on abbacus algebra thanks to the work of Gino Arrighi, Warren van Egmond, and the *Centro studi della matematica medioevale* of Sienna. In our understanding, symbolic algebra is an invention of the sixteenth century which was prepared by algebraic practice of the abbacus tradition. At least abbacus algebra has to be called syncopated in the interpretation of Nesselmann. Many of abbacus manuscripts use abbreviations and ligatures for *cosa*, the unknown (as c, co. or ρ), censo or cienso, the second power of the unknown (ce. or ç), cubo, the third power (cu.) and beyond. Also plus, minus and the square root are often abbreviated as in p, m and R. From the fifteenth century we also find manuscripts that explicitly refer to a method of solving problems that is different from the regular rhetorical method. In an anonymous manuscript of c. 1437,[4] the author solves several standard problems in two ways. One he calls symbolical (*figuratuiamente*) and the other rhetorical (*per scrittura*). He explains:

> I showed this symbolically as you can understand from the above, not to make things harder but rather for you to understand it better. I intend to give it to you by means of writing as you will see soon.[5]

This is the first occasion in the history of algebra where an author makes an explicit reference to two different kinds of problem solving, which we would now call symbolical and non-symbolical. This manuscript or related copies may have influenced the German cossists. Regiomontanus, who maintained close contacts with practitioners of algebra in Italy, adopts the same symbolic way of solving problems. In his correspondence with Johannes Bianchini of 1463 we find problems very similar to the abbacus text: divide 10

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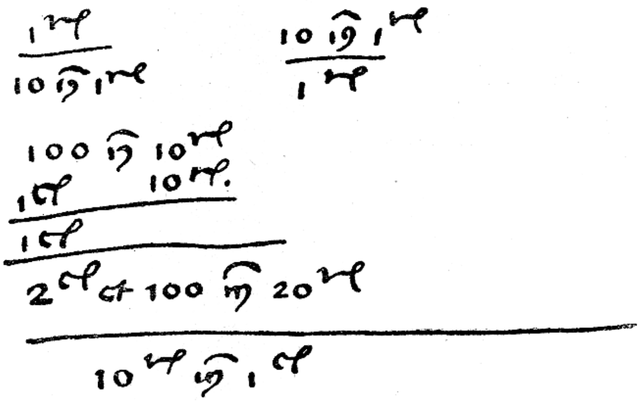
[4] Florence, Biblioteca Nazionale, Magl. Cl. XI. 119. A critical edition and translation of this manuscript is in preparation: A. Heeffer, "The algebra problems of *Ragioni apartenente all'arismetricha* with an English translation".

[5] f. 59r: "Ora io telo mostrata figuratuiamente come puoi comprendere di sopra bene che e lla ti sia malagievole ma per che tulla intenda meglio. Io intende di dartela a intendere per scrittura come apresso vedrai".

into two parts so that one divided by the other together with the other divided by the first equals 25.[6] In modern symbolic notation the problem can be formulated as follows:

$$\frac{x}{x-10}+\frac{10-x}{x}=25$$

Regiomontanus solves the problem in the same manner of abacus algebra but adopts only the symbolical version (as shown in Figure 1). He uses symbols for cosa and censo which we typically find in German cossist algebra from 1460 for a period of about 160 years.



$$\frac{x}{10-x},\frac{10-x}{x}$$

$$100-10x$$

$$\frac{x^2-10x}{x^2}$$

$$\frac{2x^2+100-20x}{10x-x^2}$$

FIGURE 1: THE SOLUTION OF AN ARABIC DIVISION PROBLEM BY REGIOMONTANUS (C 1460, NÜRNBERG CENT. V 56C, F. 23)

Literal transcription in modern symbolism.

While we see in later abbacus algebra and Regiomontanus the roots of symbolic algebra, Nesselmann places both within the stage of rhetorical algebra. According to Nesselmann's own definition these two instances of algebraic practice should at least be called syncopated.

CONCLUSION

We have argued that the interpretation of rhetorical, syncopated and symbolic algebra as three historical phases in the development of algebra cannot be sustained. The special

[6] The correspondence is kept in Nürnberg, City Library, Cent. V, 56c, ff. 11r-83v, The transcription is by Curtze 1902, 232-234: "Divisi 10 in duos, quorum maiorem per minorem divis, item minorem per maiorem. Numeros quotiens coniunxi, et fuit summa 25 : quero, que sint partes". The corresponding problem in Magl. Cl. XI. 119 is on f. 61v but uses a sum of 50 instead of 25.

status given to syncopated algebra seems to be an invention to provide the *Arithmetica* of Diophantus with a privileged status. Diophantus has always been difficult to place within the history of algebra. The humanist project of reviving ancient Greek science and mathematics played a crucial role in the creation of an identity for the European intellectual tradition. Beginning with Regiomontanus's 1464 lecture at Padua, humanist writers distanced themselves from "barbaric" influences and created the myth that all mathematics, including algebra descended from the ancient Greeks. Later writers such as Ramus, Peletier, Viète and Clavius participated in a systematic program to set up sixteenth-century mathematics on Greek foundations. The late discovered *Arithmetica* of Diophantus was taken as opportunity by Viète to restore algebra to a fictitious pure form.[7] The special status of syncopated algebra should be understood within this context. A symbolic interpretation of Diophantus's *Arithmetica* as a work of algebra by Bombelli, Stevin and Viète was made possible only through the developments before its rediscovery. Diophantus became important for algebra because symbolic algebra was already established by 1560.

Ironically, Renaissance humanist may be wrong about the Greek origin of the *Arithmetica* after all. Diophantus lived in Alexandria and there is no evidence that he was Greek. Hankel posed the provocative thesis that he was Arab.[8] If the *Arithmetica* was not written in Greek no one would have attributed it to the Greek tradition. Others conjectured he was a Hellenized Babylonian (Burton 1995). Precisely because the *Arithmetica* does not connect well with the Greek tradition of arithmetic and logistic provides impetus to a non-Greek origin.

As the tripartite distinction has obscured the true history of the development of symbolic algebra, we propose an alternative one.

AN ALTERNATIVE DISTINCTION

The term 'symbolic algebra' was introduced by the Cambridge wrangler George Peacock. in *A Treatise on Algebra* (1830). Peacock makes the distinction between arithmetical and symbolical algebra devoting a volume to each. Both kinds of algebra use symbols but in arithmetical algebra "we consider symbols as representing numbers, and the operations to which they are submitted as included in the same definitions" (1845, iv). In arithmetical algebra he allows only operations that are closed within this algebra thus avoiding negative and imaginary numbers. A quadratic equation is therefore no part of arithmetical algebra. Symbolical algebra is then considered to be a generalization of arithmetical algebra lifting the restrictions posed on operators. Though his book initiated work on the logical foundations of algebra, the restrictions set on arithmetical algebra are completely arbitrary and do not contribute to a historical assessment of symbolic algebra.

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[7] For a discussion on the creation of this new identity see Høyrup (1996) and Heeffer (2007).

[8] Hankel (1874: 157): "Wäre eine Conjectur erlaubt, ich würde sagen, er war kein Grieche; vielleicht stammte er von den Barbaren, welche später Europa bevölkerten; wären seine Schriften nicht in griechischer Sprache geschrieben, Niemand würde auf den Gedanken kommen, dass sie aus griechischer Cultur entsprossen wären".

In 1881 Léon Rodet questioned the threefold distinction by Nessselmann and proposed instead to draw the line between symbolic algebra and one dealing with abbreviations and numerical data (Rodet 1881, 69-70). Klein (1936; 1968, 146) took this as a departure for his seminal work on the emergence of symbolic algebra. Still, the focus on the use of symbols as a prerequisite leads to a limited view of symbolic algebra. As we will argue, a symbolic approach to algebra is perfectly possible without symbols. Moreover, symbols are usually introduced in a later stage towards symbolic algebra. Essentially there is only a distinction between symbolic and non-symbolic algebra, but to account for historical periods with symbolic practice without the use of symbols we propose a threefold distinction as follows:

1. Non-symbolic algebra: this is an algorithmic type of algebra dealing with numerical values only or with a non-symbolic model. Typical examples are Greek geometrical algebra or the Chinese method for solving linear problems with multiple unknowns (*Fāng chéng* 方程)

2. Proto-symbolic algebra: algebra which uses words or abbreviations for the unknown but is not symbolic in character. This would include Diophantus, Arabic algebra, early Abbacus algebra and early German cossic algebra.

3. Symbolic algebra: algebra using a symbolic model, which allows for manipulations on the level of symbols only. Established around 1560 and prepared by later abbacus and cossic algebra, Michael Stifel, Girolamo Cardano and the French algebraic tradition.

We now proceed to clarify the specificity of the symbolic mode of algebraic practice.

THE CONSTRUCTIVE FUNCTION OF SYMBOLISM

What is so specific about symbolic reasoning? What makes symbolism so powerful that it has completely conquered mathematical and scientific discourse since the seventeenth century? Many philosophers, from Descartes and Leibniz to Charles Sanders Peirce and Ernst Cassirer, have written extensively about the role of symbolism in mathematical problem solving. We will only touch upon some points of this long tradition. Our focus will be on the role of symbolism in the formation of new concepts in mathematics.

SYMBOLS VS. NOTATIONS

Part of the explanation of the emergence of symbolic algebra lies in the differentiation of the functions of symbols and notations. Both have a representative function but the role attributed to symbols surpasses its direct representational function. Notations have grown out of shorthand writing or abbreviations of words. As such, they directly represent the operations and concepts behind the abbreviation. Symbols add an extra to the function of notations, a distinction which has mostly been neglected in the history of mathematics.[9] Let us look at the function of some very essential symbols when they were first introduced.

[9] Montucla in his *Histoire des mathématiques* (1799, I, 587), sees no difference at all between words used for operations and symbols: "Notre algèbre ne diffère en aucune manière de ce qu'on

THE FIRST OCCURRENCE OF THE PLUS SIGN

Tropfke (1933, II, 14-8) describes how the addition operation was introduced. The + sign was first used in a printed arithmetic book by Johannes Widmann (1489, f. I vi^r).[10]

One could ask why the notation first appeared in Germany and not within the Italian abbacus tradition. We believe there is a valid explanation for this. The abbacus tradition after Fibonacci was fully vernacular. Later abbacus treatises use

$\bar{p}$ and $\bar{m}$

for the Italian words *plus* and *meno*. The German cossic tradition was originally Latin-based. Widmann found the sign in the manuscript collections he consulted for his arithmetic, particularly in the Dresden C80, (Wappler 1887, 13).

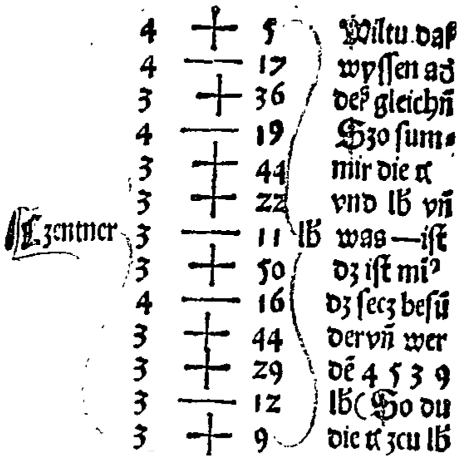

FIGURE 2: THE FIRST USE OF THE + AND – SIGN IN PRINT FROM WIDMANN 1989.

This manuscript uses the + notation as a ligature in a section written c. 1486. The plus sign is the shorthand for the Latin word *et*. The form of the crossed lines is evidently derived from the letter t in *et*. The use of + as ligature was not only used as a mathematical operation but also in the meaning of the word 'and' (e.g. Wappler 1887, 15). It first occurs within an algebraic context, as shown in Figure 3:

$x^3 + 2x^2$

FIGURE 3: THE FIRST APPEARANCE OF THE + SIGN IN THE DRESDEN CODEX C80, F. 350^{v}, WRITTEN AROUND 1486 (FROM TROPFKE 1933, II, 17).

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vient de voir. Il y a seulement ceci de plus, que les modernes affectant de mettre tout en signes, en ont imaginé pour désigner l'addition, la soustraction des grandeurs et leurs égalité. Les premiers algébristes du siezème siècle les indiquèrent par les lettres initiales de plus, moins, égal. Nous le faisons aujourd'hui par les signes +, -, =".

[10] This page of Widmann's book has lead to wild speculations on the origin of the signs (see Cajori 1928, I, 232-4). The meaning here is only one of aggregation but not relevant for our further discussion.

Because a ligature is essentially a shorthand notation, we would expect signs, which are based on ligatures, to be mathematical notations rather than symbols. This leads us to a key question: is the plus sign here a notation or a symbol? The answer will depend on the context. If printed in an early fifteenth century arithmetic book, the + sign in '3 + 5 makes 8', would be interpreted as a shorthand for 'and', meaning the addition of five to three. Here, 'plus' describes an operation, a mental or even physical action. There is some temporal element present in the description '3 + 5 makes 8'. First you have three; after adding five, you find out that you have eight. The + sign in this context is thus a direct representation of the action of adding things together. Therefore, we would not consider it as a symbol in the case of an arithmetic book. Interestingly, Widmann (1489), who uses the signs in an arithmetic book, does not so in the introductory chapters on the basic operations of addition and subtraction. Instead, he first employs the signs after 166 pages on mercantile problems. Which brings us to the context in which signs and symbols occur.

CONTEXTUAL ELEMENTS IN THE USE OF SIGNS

The context in which the + sign first appeared, illustrated in Figure 3, is very different from that of Widmann's arithmetic. Two important elements of this first occurrence are also present in the first use of operations on equations by Cardano (1539).[11]

Firstly, the + sign is introduced within the context of algebra, and not in arithmetic. It is part of a binomial expression with two cossic terms, x^3 and $2x^2$. One interpretation is to just see it as the addition of these two terms, similar to the arithmetical example above. However, the context of this important manuscript where operations on polynomials are introduced, asks for a more adequate interpretation.[12] The plus sign here has the additional function of creating a structure on which can be operated. For the abbacus masters and the early cossists binomials are more like primitive structures. In the late fourteenth century, algebra treatises typically have a section dealing with the addition, subtraction and multiplication of binomials. These binomials can be algebraic, as in the example from the illustration, but often also irrational. For example, Maestro Gilio, in the Siena L.IX.28 (Franci 1983, 7) has his algebra preceded by a "trattato dele radice" demonstrating how to add, subtract and multiply irrational binomials:

> Se avessi a multipricare 7 e R di 8 per 7 meno R di 8 fa così: multiprica 7 via 7 fa 49, et della multipricagione della R di 8 in meno R di 8 si ne perviene meno 8, tralo di 49 resta 41, e della multipricagione di 7 in più R di 8 e di 7 meno R di 8 si nne perviene ragiunti insieme nulla, adomque multipricando 7 per R di 8 in 7 meno R di 8 si nne viene 41.

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[11] As argued in Heeffer (2007a), Cardano (1539) was the first to show an explicit operation on an equation and the first to subtract two equations (Cardano 1545). Both instances are shown as marginal notations in the first printed editions.

[12] Grammateus (1518/21) also first uses the signs in his chapter on algebra, see Figure 5 and the discussion below.

The rule corresponds with the general formula:

$$\left(a+\sqrt{b}\right)\left(a-\sqrt{b}\right)=a^2-b$$

In abbacus treatises a binomial is considered a primitive element, a mathematical entity such as a number or a proportion. The plus sign is a constructive operator in the formation of a algebraic binomial. While the plus sign is derived from a ligature, its use for the representation of a binomial within an algebraic context points at a symbolic function.

A second contextual element of in the introduction of algebraic symbolism, is the function of clarification and illustration. Elsewhere we have shown how the first addition of two equations by Cardano appeared as an illustration in the *Ars Magna*.[13] Also in the Dresden C80, the expression $x^3 + 2x^2$ functions as an illustration between the text lines. The use of + in this illustration emphasizes the aggregate function of the sign. The binomial is 'constructed' by placing the sign in between the two terms. The temporal aspect of adding numbers together is absent here. The structure of the binomial is depending on the + as a connector. These contextual elements bring us to an interpretation of the + sign as a symbol rather than as a short hand notation.

PARADOXES OF SYMBOLISM?

During the seventeenth century we find discussions concerning conceptual difficulties with some basic features of symbolic reasoning which we now take for granted. Mersenne (1625, 522-1) talks about "a strange paradox" when discussing the rules of signs:

> Or plusierss s'étonnent comment il est possible que – multiplié par – , c'est-à-dire moins par moins fasse +, et que P multiplié par M, ou M par P fasse M, ce qui semble estre contre toute sorte de raison. Sur quoi vous pouvez voir Clavius au 6 chap. de son Algebre; neantmoins i'en ay veu qui nient cette proposition, sur laquelle ie ne m'arresterai pas davantage.

While the discussion of such "paradoxes" may seem idle to Mersenne, they increasingly appear during the seventeenth century. Antoine Arnauld, who wrote an important philosophical work know as *The Logic of Port-Royal* (Arnauld, 1662), also published a *Geometry* (Arnauld, 1667). In the book he includes an example of symbolic rules that he considers to be against our basic intuitions on magnitudes and proportions. His reasoning goes as follows. Suppose we have two numbers, a larger and a smaller one. The proportion of the larger to the smaller one should evidently be larger than the proportion of the smaller to the larger one. But if we use 1 as the larger number and – 1 as the smaller one this would lead to

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[13] See Heeffer 2007a). The illustration, an essential contribution to symbolic algebra, is omitted in the English edition by Witmer (1968).

$$\frac{1}{-1} > \frac{-1}{1}$$

which is against the rules of algebra. Witnessing the multiple instances in which this discussion turns up during the seventeenth century, the clash between symbolic reasoning and classic proportion theory, taught within the *quadrivium*, was experienced as problematic. Also Leibniz found it important enough to write an article about (Leibniz, 1712, 167). He acknowledges the problem as a genuine one, but states that the division should be performed as a symbolic calculation, the same way as we do with imaginary numbers. Indeed, when blindly applying the rules of signs there is no problem at all. When dividing a positive number by a negative one, the result is negative, and dividing a negative number by a positive one, the result is also negative. Therefore

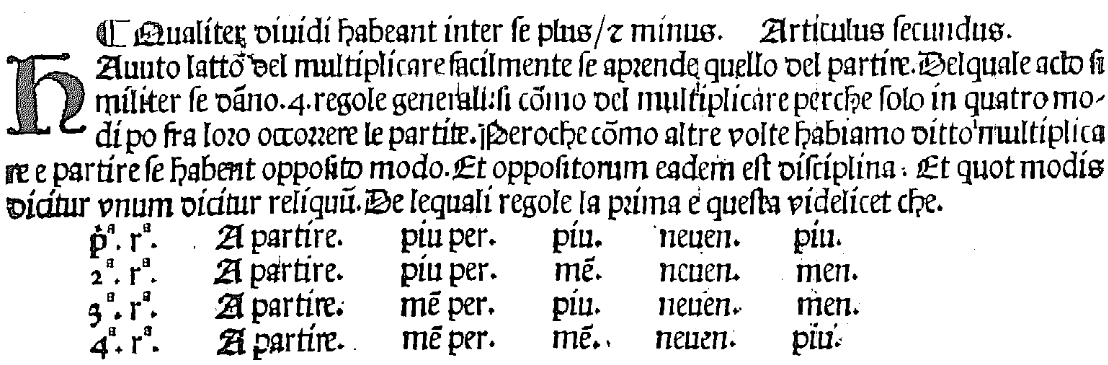
$$\frac{1}{-1} = \frac{-1}{1}$$

The discussion was not closed by Leibniz. Several eighteenth-century authors return to the question. E.g. Rolle (1690, 14-22), Newton (1707, 3), Maclaurin (1748, 6-7) and d'Alembert (1751-81).

SYMBOLIC REASONING WITHOUT SYMBOLS

Interestingly, the application of these rules posed no problems in the abbacus tradition before 1500. In the Summa, Pacioli lists the rules of signs for the arithmetical operations for addition, subtraction, multiplication and division. Dividing a positive by a negative produces a negative. Dividing a negative by a positive leads to a negative ("A partire piu per meno neven meno. A partire meno per piu neven meno") (Pacioli 1494, f. 113r), see

Figure 4.



¶ Qualiter diuidi habeant inter se plus/ꝛ minus. Articulus secundus.

Hauuto latto del multiplicare facilmente se aprende quello del partire. Del quale acto si militer se dano .4. regole generali: si como del multiplicare perche solo in quatro modi po fra loro occorrere le partite. Peroche como altre volte habiamo ditto multiplicare e partire se habent opposito modo. Et oppositorum eadem est disciplina: Et quot modis dicitur vnum dicitur reliquũ. De lequali regole la prima e questa videlicet che.

pª. rª.	A partire.	piu per.	piu.	neuen.	piu.
2ª. rª.	A partire.	piu per.	mẽ.	neuen.	men.
3ª. rª.	A partire.	mẽ per.	piu.	neuen.	men.
4ª. rª.	A partire.	mẽ per.	mẽ.	neuen.	piu.

FIGURE 4: PACIOLI'S RULES OF SIGNS FOR DIVISION

These rules were known implicitly and have been applied within the abbacus tradition, for example in the multiplication of irrational binomials in Fibonacci (1202; Boncompagni 1857, 370; Sigler 2003, 510):

$$\left(4-\sqrt{2}\right)\left(5-\sqrt{8}\right) = 22 - 4\sqrt{8} - 5\sqrt{2}$$

However, an explicit treatment was impeded by the immature status of negative quantities. As far as we know, Pacioli was the first to list these formal rules for the basic operations of arithmetic.[14] Importantly, Pacioli introduced these rules in distinction 8, as a preparation to his treatment of algebra. In contrast with the discussion of the basic operations of arithmetic, the rules of signs have a more formal and general character. Except for an illustrating example with numbers, the formulation of the rules does not refer to any sort of quantities, integers, irrational binomials or cossic numbers. The rules only refer to 'the negative' and 'the positive'. Despite the absence of any symbolism, we consider this an early instance of symbolic reasoning. Except for the ligatures $\overline{p}$ and $\overline{m}$, no symbols are used for plus and minus. Still, its use in the "formalism" of these rules makes *piu* and *meno* qualify as symbols.

TOWARDS OPERATIONAL SYMBOLISM

After Pacioli, the rules of signs appear more frequently in algebra textbooks. Exemplary is the anonymous Vienna codex 5277, written between 1500 and 1518 (Kaunzner, 1972). Here the rules of signs are introduced in relation to operations on polynomials and use the + and – sign introduced some decades before in the Dresden C80. For multiplication we find (f. 6r; Kaunzner 1972, 132):

$$\text{Si multiplicas} \begin{cases} +per+ \\ -per- \\ +per-et \quad econtra \end{cases} \text{fit} \begin{cases} + \\ + \\ - \end{cases}$$

Here, the rule appears to be symbolical, because we recognize our current symbols, but it is conceptually identical with that of Pacioli. Where we have previously denoted a constructive function to the plus sign in the Dresden C80, we can here discern an additional operative function. Not only can + and – be used to construct binomials, the signs now come into relation with the terms of the polynomials in which they appear. The example added in the Vienna 5277 show how to multiply two binomials:[15]

$$(6x+8\varphi)(5x-7\varphi)$$

The rule describes the following:

> Cumque in unitate + φ repetitur et in altera – φ, ducta x per + φ, exoritur + x. Si augetur x per – φ (ut praecedens edocuit regula), edocitur – x. Sed ex – per + vel + per – semper – perficiecitur, sicut sequens docebit exemplum.

The minus sign which was introduced for the construction of binomials is here used for the first time to denote a negative term! The text describes the multiplication of the

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[14] We checked about thirty transcriptions of abacus manuscripts published by Gino Arrighi and the Center for the Study of Medieaval Mathematics of Siena. Also Tropfke (1933, II, 124-8; 1980) lists no sources prior to Pacioli (1494).

[15] Codex Vindobonensis 5277, f. 6v; Kaunzner 1972, 132. We replaced the cossic sign of the unknown by *x*. The sign φ is used for units and has to be interpreted as x^0. The habit of using φ or Ø in German algebra textbooks is abandoned by the end of the sixteenth century.

positive term x with the negative $-x^0$. Where previous uses of negative values were highly problematic, we now witness how the use of symbolism facilitates the acceptance of negative terms. The modern interpretation of subtraction as the addition of a negative term now becomes realized. This is exemplified where the author introduces the first rule of algebra (on linear equations) with the following *cautela* (f. 13v; Kaunzner 1972, 139):

> Si radix in latera continet + φ, tunc is numerus. Quo radix subabundat, ex numero, cui radix aequatur, subtrahur. Si vero $-\ \varphi\ x$ continuerit, tunc addatur.

Where the original *al-jabr* operation from early Arabic algebra cannot be interpreted as the addition of a term to both sides of an equation to eliminate a negative term, such interpretation now becomes justified.[16] This rules describes that to solve the linear equation, for example,

$$ax + b = d - cx$$

you proceed by adding cx to both parts and subtracting b from both parts. By means of a symbolism for representing negative terms, a basic operation on equations now becomes commonplace.

The story does not end here. The Vienna codex is innovating in yet another aspect. The problem of a man making three business trips is one of the examples illustrating the first rule.[17] At each trip he doubles his income but spends 4 florenos. He ends up with nothing. The problem asks for the capital he started with. The author solves the problem by constructing an equation as follows. Take x for the original capital. After the first trip he has $2x - 4$. After the second he arrives at $4x - 8 - 4$, and after the third he end up with $8x - 16 - 8 - 4$. Using the new symbol for negative terms, the manuscript reads:[18]

$$8\ x\ -16\ \varphi\ -8\ \varphi - 4\ \varphi \text{ hoc totum est aequale } 0\ x^0$$

Though lacking a symbolic expression for the equation, the author puts the constructed polynomial equal to zero. This is highly uncommon for the beginning of the sixteenth century. This Vienna codex is an example of a sudden leap in the evolution towards

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[16] See Heeffer 2007d, and Oaks and Alkhateeb (2007) for such an interpretation of *al-jabr*.

[17] Better known as the monkey and coconut problem. See Heeffer "How Algebra Spoiled Renaissance Recreational Problems" for a more extensive discussion.

[18] Vienna 5277, f. 14r; Kaunzner 1972, 139: "Est quidam mercator, qui emit aliquot talenta piperis, et iterum vendit, et lucratur tantum, quantum summa valebat capitalis, et exponit 4 fl*orenos*. Cum residuo consimiliter tantum consequitur lucri, quantum restabat, et 4fl expendit. Itidem tertio modo facit, et 4fl exponit, et demum nihil, vel lucri, vel summae capitalis, remansit. Quaeritur iam de summa pecunia originali. Sit 1 x, et lucratur 1 x, ergo erunt 2 x. Ex his aufer 4 φ vel fl, restat $2x - 4\ \varphi$. Deinceps, cum eo lucratur totidem, quantum restabat, fiunt per consequens $4\ x - 8\ \varphi$. Ex quibus demendi sunt 4 fl, Stabit residuum $4\ x - 8\ \varphi - 4\ \varphi$. Tertio iterum tantum hicratur, quantum restabat, fiunt itaque $4\ x - 16\ \varphi - 8\ \varphi$. Ex his ultimo auferantur 4fl, et relinquuntur $8\ x\ - 16\ \varphi\ - 8\ \varphi - 4\ \varphi$ hoc totum est aequale 0 φ. Secundum cautelam addendi sunt 16 φ 8 φ 4 φ ad 0 φ. Summa, scilicet 28 φ, dividatur per 8 x, Quia aequivalent. Et quotiens, scilicet 3 ½, dicit florenorum in primo habitorum numerum".

algebraic symbolism. While it only adopts the + and – sign from a previous manuscript, the new symbols advance several conceptual steps: a symbolic expression for the rules of signs, the elimination of negative terms in an equation and the equation of a polynomial to zero.

THE SPREAD OF OPERATIVE SYMBOLISM

We have evidence that the Vienna codex was consulted by both Heinrich Schreyber (Grammateus) and Christoff Rudolff. Grammateus (1518/21, f. Gvi^{r}) uses the + and – signs and mentions the rules of signs while introducing operations on polynomials as shown in Figure 5.

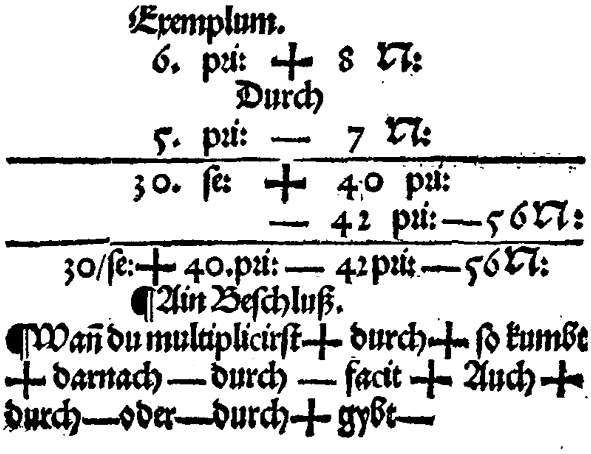
Exemplum.
6. pri: + 8 N:
Durch
5. pri: — 7 N:
30. se: + 40 pri:
— 42 pri: — 56 N:
30/se: + 40.pri: — 42pri: — 56 N:
¶Ain Beschluß.
¶Wañ du multiplicirst + durch + so kumbt + darnach — durch — facit + Auch + durch — oder — durch + gybt —

FIGURE 5: THE RULES OF SIGNS FOR MULTIPLICATION BY GRAMMATEUS

The *cautela* for the first rule from the Vienna codex is reproduced literally in a German translation by Grammateus.[19] That Rudolff in his *Coss* collected most of his material from Vienna manuscripts was known and discussed already in the sixteenth century.[20] With Pacioli (1494), Grammateus (1518/21) and Rudolff (1525) the symbolic approach to operate on positive and negative terms was spread all over Europe. This heralded the use of operative symbolism in algebra.

HOW NEW CONCEPTS ARE CREATED BY OPERATIVE SYMBOLISM

Let us return to the apparent paradox of Arnauld. Leibniz argued that symbolic reasoning resolves the paradox. We have shown that such kind of reasoning was common practice in the abbacus tradition of the late fifteenth century. Pacioli would respond to the discussion that

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[19] Grammateus, 1518/21, f. Jiiii^{r}: "Wan*n* do stet in ainer position der zwayer die sich mit ainander vorgleichen das zaich + so subtrahir sein zal von sienem gleichen in der andern position wird aber funde*n* – so addire die selbig zal zu der in der andern position".

[20] See the introduction by Stifel in the 1553 edition. Also discussed in my "The Rhetoric of Problems in Algebra Textbooks from Pacioli to Euler".

$$\frac{1}{-1} \text{ equals } -\frac{1}{1} \text{ and } \frac{-1}{1} \text{ also equals } -\frac{1}{1}\text{, therefore } \frac{1}{-1} = \frac{-1}{1}.$$

Although we do not find the symbols for division, negative numbers and equations in Pacioli (1494) or his predecessors, the common application of these operations provides evidence of a symbolic mode of reasoning. We will now discuss how symbolism has had a decisive role in the formation of three important new mathematical concepts of the sixteenth century. The first one relates to the discussion on negative quantities.

NEGATIVE NUMBERS

The Vienna 5277 manuscript was the first to apply the – sign to an algebraic term. We have pointed out the symbolic function of the sign. While the minus sign has been abstracted from the subtraction operation, it now incorporates the extra function of negation. By placing a – sign before an algebraic term, the term becomes negated. Two centuries later, d'Alembert will define 'negative' in the *Encyclopédie* as "the affection of term by the sign –" (Diderot and d'Alembert 1780, XXII, 289). d'Alembert rebukes "those who pretend that the ratio between 1 and – 1 is different from the ratio – 1 and 1". They are wrong in two respects, he claims. Firstly, because in algebra the division by negative numbers is common practice, and secondly because the value of the product of 1 and 1 is the same as the value of the product of – 1 and – 1.

His characterization 'affecting' is interesting. He makes the distinction between negation used for an isolated quantity (or term) and its use in the sense of $a - b$. A negative value must be understood in the first meaning, not in the second. He opposes the view of a negative quantity as a quantity less than zero, "as most mathematicians do".

We believe this is indeed the meaning attributed to a negative quantity by the early cossists. The first appearance of the negative symbol has the intention of affecting an algebraic term. The Vienna 5277 manuscript uses the minus sign to create negative quantities. Studies as by Sesiano (1985) and Gericke (1996) discussing several instances of so-called negative values from Fibonacci to the sixteenth century can be criticized for their all too casual interpretation of negatives. The concept of a negative quantity as a value smaller than zero was an unacceptable, and even ridiculous idea before the seventeenth century. However, the symbolic affection of an algebraic term did lead to the concept of a negative number. Negative numbers have become possible with the introduction of the minus sign. Where the – sign originally had the function of a constructive operator for binomials, in the early sixteenth century it became an operative symbol for the negation of algebraic terms. Once negative quantities had become established by this symbolic construction, the elimination of negative terms from an equation by addition became a common operation.

DEFINING IMAGINARY NUMBERS BY OPERATIONAL SYMBOLISM

Bombelli was the first to define imaginary numbers by the eight combinatorial operations that are possible with the products of the negative and positive roots of plus and minus one. Note the correspondence with Pacioli's rules of sign when Bombelli lists the following operations in Figure 6 (Bombelli 1572, f. 169^{r}). These operations defined imaginary numbers within the symbolic model. The interpretation of their arithmetical equivalence still remained a mystery. It took two more centuries to arrive at a geometrical interpretation of complex numbers. This story is well covered by Barry Mazur (2003). For a mathematician, Mazur pays surprisingly much attention to the conceptual evolutions which have lead to imaginary numbers. He makes an interesting observation with regard to possible forms of notation. In discussing dal Ferro's formula for one case of the cubic equation he remarks (Mazur 2003, 124-5):

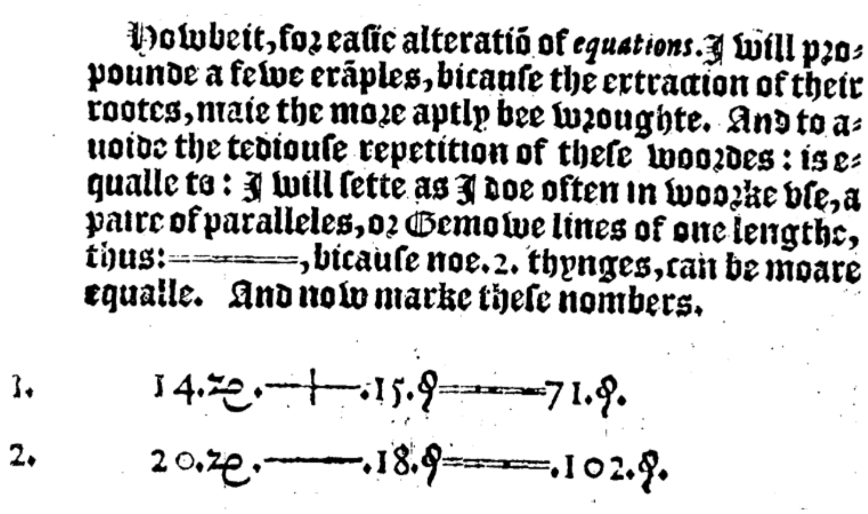
> In discussing the "easy" case in which the indicator $d = \frac{c^2}{4} - \frac{b^3}{27}$ is positive, I said that the manner in which Dal Ferro's expression is written tells us how to compute it (extract, as indicated, the roots and make the arithmetic operations requested by the formula). The expression doesn't provide a specific method for the extraction of those roots, but once we have such a method, the expression is itself interpretable as a possible algorithm for the production of a real number. It is often the case that our expressions for specific numbers suggest algorithms, or partial algorithms, for their computation. To take a random example, the number $2^{21} - 1$ happens to equal 7 x (300,000 - 407), and this number written in decimal notation is 2097151. Each way of writing this number hints at a specific strategy for its calculation (e.g., if you express the number as 221 – 1, the form of this expression bids you do what it tells you to do to calculate the number: raise 2 to the twenty-first power and then subtract 1 from the result).

Mazur here describes in different terms the same mechanism we have proposed to explain the function of symbolism. The "symbolic expression is itself interpretable as a possible algorithm for the production" of instances of the concept it represents. By using different symbolic expressions for a same number, we represent different algorithms or strategies for its computation. In other words, the possible combinatory operations on the object become embedded with the representation. The symbolism performs the task of representing these possible operations. We will now look at what this concretely means for the equation sign of a symbolic equation.

Più uia più di meno, fà più di meno.	$(+1).(+i) = +i$
Meno uia più di meno, fà meno di meno.	$(+1).(-i) = -i$
Più uia meno di meno, fà meno di meno.	$(+i).(+i) = -1$
Meno uia meno di meno, fà più di meno.	$(-i).(+i) = +1$
Più di meno uia più di meno, fà meno.	$(-1).(+i) = -i$
Più di meno uia men di meno, fà più.	$(-1).(+i) = -i$
Meno di meno uia più di meno, fà più.	$(-1).(-i) = +i$
Meno di meno uia men di meno fà meno.	$(-i).(-i) = -1$

FIGURE 6: A DEFINITION OF IMAGINARY NUMBERS BY THEIR POSSIBLE OPERATIONS (BOMBELLI 1572)

The equality sign evidently refers to the arithmetical equivalence of two expressions left and right from the sign. For example, the expression 3 + 5 = 8 denotes the arithmetical equivalence of the sum of three and five with eight, as well as of eight with its partitioning into the numbers three and five. However, if we look at the historical moment at which the equality sign was introduced, we arrive at a very different picture. The equality sign as we now use it, was introduced in a book on algebra by Robert Recorde (see Figure 7). He chose the sign of two parallel lines 'because no two things can be more equal'. This often quoted citation ignores the more important motivation for introducing the sign. Firstly, the equation sign was not introduced, either in his lengthy introduction, discussing the basic operations of arithmetic and extraction of roots, or in the dialogue on operations on polynomials or the rule of proportion. Instead, he introduced the sign in the chapter on the resolution of algebraic equations "For easie alteration of equations … And to avoid the tediouse repetition of these woordes: is equalle to: I will sette as I doe often in woorke use, a paire of parralle … lines of one lengthe, thus : ==, bicause noe 2, thynges, can be moare equalle", (Recorde 1557, fol. FFi^{v}).


Howbeit, for easie alteratiō of equations. I will propounde a fewe exāples, bicause the extraction of their rootes, maie the more aptly bee wroughte. And to auoide the tediouse repetition of these woordes: is equalle to: I will sette as I doe often in woorke vse, a paire of paralleles, or Gemowe lines of one lengthe, thus: =====, bicause noe. 2. thynges, can be moare equalle. And now marke these nombers.

1. 14.ze. + .15.ʒ. ===== 71.ʒ.
2. 20.ze. — .18.ʒ. ===== .102.ʒ.

FIGURE 7: THE FIRST USE OF THE EQUATION SIGN IN PRINT (FROM RECORDE 1557)

The use of the sign is thus specifically motivated by the alteration, or manipulation of equations. From this quote we can read the specific representational function that makes the equality sign the prime symbol of the concept of an equation. In addition to its direct reference to arithmetical equivalence, the equality symbol represents the combinatorial operations which are possible on an equation. These operations include adding or subtracting homogeneous terms to both sides of the equation, dividing or multiplying an equation by a constant or unknown (introduced by Cardano) and adding or subtracting two equations (introduced by Cardano and perfected by Peletier and Buteo). The equality sign symbolizes the algebraic equation. We have argued elsewhere that the concept of an equation fully emerged around 1560 (Heeffer 2007a). We also stated that symbols are introduced as a result of symbolic thinking. The introduction of the equality symbol provides historical evidence for the introduction of a symbol representing a newly emerged mathematical concept.

The introduction of the equation symbol completes the basic stage of development towards symbolic algebra, as initiated in Germany by the end of the fifteenth century. The time of the introduction, 1557, coincides perfectly with our conceptual analyses of algebra textbooks of the sixteenth century.

As the minus sign facilitated the acceptance of negative numbers, so did the equation sign contribute to the further development of algebra towards the study of the structure of equations. That the equation sign, as introduced by Recorde, was not universally accepted for another century, is irrelevant for our argumentation. Other signs or even words functioned as the equation symbol in the same way as the two parallel lines had done. Thomas Harriot, in his manuscripts, placed two short strokes between the parallel lines resembling 'II' and introduced the < and > signs as they are used today (Stedall 2003, 8). This was later abandoned in the printed edition and through its further use by Oughtred's *Clavis mathematicae*, the equation sign became generally accepted in England.

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