Appendix to Towards More Conflict-Tolerant Deontic Logics By Relaxing the Interdefinability Between Obligations And Permissions[*]

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APPENDIX

This Appendix contains in Part A some results concerning strengthenings of the transformation presented in Section 5 and in Part B all the meta-proofs for our results.

A Strengthenings of the Transformation

In this Section we demonstrate that many of the results presented in Section 6 for our transformation $\mathbf{L}\star$ are also valid for strengthenings of $\mathbf{L}\star$.

(AND-OP) expresses that if we have $\mathsf{P}(A \vee B)$ and there is no OP-conflict concerning the latter, then either $\mathsf{P}A$ or $\mathsf{P}B$ is the case.

$$\big(\mathsf{P}(A \vee B) \wedge \neg\mathsf{O}\neg(A \vee B)\big) \supset \big(\mathsf{P}A \vee \mathsf{P}B\big) \tag{AND-OP}$$

Even stronger is

$$\mathsf{P}(A \vee B) \supset \big(\mathsf{P}A \vee \mathsf{P}B\big) \tag{P-AND}$$

The corresponding frame conditions to these axioms are:

$$\text{For all } X, Y \subseteq W, \text{ if } X \cup Y \in \mathcal{P}_w \text{ and } W \setminus (X \cup Y) \notin \mathcal{O}_w, \text{ then } X \in \mathcal{P}_w \text{ or } Y \in \mathcal{P}_w \tag{FP-AND-OP}$$

$$\text{For all } X, Y \subseteq W, \text{ if } X \cup Y \in \mathcal{P}_w, \text{ then } X \in \mathcal{P}_w \text{ or } Y \in \mathcal{P}_w \tag{FP-P-AND}$$

The intuition behind the restricted inheritance principle (RPM) was to restrict inheritance to obligations that are not involved in conflicts. The same idea may be generalized for permissions in terms of the following rule:

$$\text{If } \vdash A \supset B, \text{ then } \big(\mathsf{P}A \wedge \neg\mathsf{O}\neg A\big) \supset \mathsf{P}B \tag{RM-OP}$$

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(RM-OP) states that if there is no OP-conflict concerning $\mathsf{P}A$ and A entails classically B, then inheritance is applicable to $\mathsf{P}A$ in order to derive $\mathsf{P}B$.

The intuition behind the restricted inheritance principle (RPM') was to restrict inheritance in such a way that it does not create additional conflicts. The same idea may be generalized for permissions in terms of the following rule:

$$\text{If } \vdash A \supset B, \text{ then } \big(\mathsf{P}A \wedge \neg\mathsf{O}\neg B\big) \supset \mathsf{P}B \qquad \text{(RM'-OP)}$$

The following result shows that there is no need to strengthen $\mathbf{L}\star$ be these principles.

Theorem A.1.

(i) If $\mathbf{L}\star$ *validates* (RPM)*, then it validates* (RM-OP)*.*

(ii) $\mathbf{L}\star$ *validates* (RM'-OP)*.*

Note that a strengthening of $\mathbf{L}\star$ by (RM-OP) is not necessary for the transformation of an A-CTDL since these systems already validate the stronger (P-RM).

Where Δ is a set of axioms and rules, we write $\mathbf{L} \oplus \Delta$ for the logic that is the result of adding the rules and axioms in Δ to the axiomatization of $\mathbf{L}$. Let in the following $\mathbf{K}\star \in \big\{\mathbf{L}\star \oplus \{\text{(P-AND)}\}, \mathbf{L}\star \oplus \{\text{(AND-OP)}\}\big\}$.

In the remainder of this section we demonstrate that the meta-theory for $\mathbf{K}\star$ is nearly analogous to the results presented for $\mathbf{L}\star$ in Section 6. Let us first compare the consequence relations characterized by $\mathbf{L}$ and by $\mathbf{K}\star$.

Theorem A.2. *Where* $\Gamma \subseteq \mathcal{W}_{\mathsf{O}}$ *and* $A \in \mathcal{W}_{\mathsf{O}}$, $\Gamma \vdash_{\mathbf{L}} A$ *iff* $\Gamma \vdash_{\mathbf{K}\star} A$.

Corollary A.1. $\mathbf{K}\star$ *is* $\mathbf{L}$*-conservative.*

Theorem A.3. *Where* Γ *is* $\mathbf{L}$*-consistent, if* $\Gamma \vdash_{\mathbf{L}} \mathsf{P}A$ *then* $\Gamma_{\mathsf{O}} \vdash_{\mathbf{K}\star} \mathsf{P}A$.

Let us now focus on the conflict-tolerance of $\mathbf{K}\star$.

Theorem A.4. *If* $\mathbf{L}$ *satisfies (†), then* $\mathbf{K}\star$ *satisfies (‡).*

Theorem A.5. *If* $\mathbf{L}$ *strengthened by (D) and the set of axioms and rules* Θ *characterizes the same consequence relation as* **SDL***, then* $\mathbf{K}\star$ *strengthened by* (DfP1) *and the axioms and rules in* Θ *characterizes the same consequence relation as* **SDL***.*

Remark A.1. Note that Theorem 5 does in general not hold for $\mathbf{L}\star \oplus \{\text{(P-AND)}\}$ where $\mathbf{L}$ is an I-CTDL or an H-CTDL. Take the following example: $\Gamma = \{\mathsf{O}(A \vee B), \mathsf{O}\neg A, \mathsf{O}\neg B, \neg\mathsf{O}A, \neg\mathsf{O}B\}$. By (D) we get $\mathsf{P}(A \vee B)$ from $\mathsf{O}(A \vee B)$. By (P-AND) we get $\mathsf{P}A \vee \mathsf{P}B$. Hence, $\Gamma \vdash_{\mathbf{L}\star\oplus\{\text{(P-AND)}\}} \big(\mathsf{O}\neg A \wedge \mathsf{P}A \wedge \neg\mathsf{O}A\big) \vee \big(\mathsf{O}\neg B \wedge \mathsf{P}B \wedge \neg\mathsf{O}B\big)$. A similar case can be made for $\mathbf{L}\star \oplus \{\text{(OP-AND)}\}$. This is left to the reader.

Let in the following $\gamma = \{\neg\mathsf{O}E, \neg\mathsf{O}\neg E\}$.

Theorem A.6. *Let* $\delta \in \{\mathsf{OO}, \mathsf{OP}\}$*,* $\beta \subseteq \gamma \cup \{\mathsf{O}C, \neg\mathsf{O}\neg C, \mathsf{P}D, \neg\mathsf{O}\neg D, \mathsf{P}F, \mathsf{P}\neg F\}$*,* $\tau \in \{\mathsf{O},\ \mathsf{P},\ \mathsf{OO}\neg,\ \mathsf{O}\neg\mathsf{P},\ \mathsf{O}\neg\mathsf{O}\neg\}$*. (i) If* $\mathbf{K}\star$ *is* δ*-EX-*OP*-*γ*-tolerant than it is also* δ*-EX-*τ*-*β*-tolerant. (ii) If* $\mathbf{K}\star$ *is* OO*-EX-*OP*-*γ*-tolerant and* OP*-EX-*OP*-*γ*-tolerant, then* $\mathbf{K}\star$ *is* CONFLICT-TOLERANT.

Theorem A.7. *If* $\mathbf{L}$ *is* OO*-EX-*OP*-*γ*-tolerant, then* $\mathbf{K}\star$ *is* CONFLICT-TOLERANT.

Of course, if $\mathbf{L}$ is OO-CONFLICT-TOLERANT, then it is also OO-EX-OP-γ-tolerant and hence $\mathbf{K}\star$ is CONFLICT-TOLERANT.

Theorem A.8. *Where* $\tau \in \{\mathsf{O}, \mathsf{P}, \mathsf{OO}\neg, \mathsf{O}\neg\mathsf{P}, \mathsf{OP}, \mathsf{O}\neg\mathsf{O}\neg\}$ *and* β *is a set of wffs, if* $\mathbf{K}\star$ *is* OO*-EX-*τ*-*β*-tolerant, then it is also* OP*-EX-*τ*-*β*-tolerant.*

Theorem A.9. *Let* β *be set of formulas of the form* $\mathsf{O}A$*,* $\neg\mathsf{O}A$ *and* $\mathsf{P}A$*, and* $\tau \in \{\mathsf{O}, \mathsf{OO}\neg, \mathsf{O}\neg\mathsf{P}, \mathsf{OP}, \mathsf{O}\neg\mathsf{O}\neg\}$*. If* $\mathbf{L}$ *is* OO*-EX-*τ*-*β*-tolerant, then*

(i) $\mathbf{K}\star$ *is* OO*-EX-*τ*-*β*-tolerant,*

(ii) $\mathbf{K}\star$ *is* OP*-EX-*τ*-*β*-tolerant,*

(iii) if $\tau = \mathsf{O}\neg\mathsf{O}\neg$*, then* $\mathbf{K}\star$ *is* OO*-EX-*P*-*β*-tolerant and* OP*-EX-*P*-*β*-tolerant.*

B Meta-Proofs

B.1 Meta-Theory for $\mathbf{L}\star$

The authors in [2] have presented a generic soundness and (strong) completeness result for the canonical form of neighborhood semantics for which models do not feature an actual world. It can easily be shown that our representation of neighborhood semantics that makes use of an actual world is equivalent. Where F is a neighborhood frame, F-models without an actual world, shortly F-models', are defined by a pair $\langle F, v\rangle$ where $v : \mathcal{S} \to \wp(W)$. Evidently, the only difference to the models we have presented is the lack of an actual world. Validity in a model at a world, $M, w \models A$, is defined as usual (see Section 5.3.1). For a given frame $F = \langle W, \mathcal{O}\rangle$ resp. $F = \langle W, \mathcal{O}, \mathcal{P}\rangle$ and $\Gamma \subseteq \mathcal{W}$, $\Gamma \Vdash'_F A$ iff for all F-models' M and for all $w \in W$, if $M, w \models B$ for all $B \in \Gamma$, then $M, w \models A$. Moreover, where $\mathcal{F}$ is a class of frames, $\Gamma \Vdash'_{\mathcal{F}} A$ iff $\Gamma \Vdash'_F A$ for all $\mathcal{F}$-frames F.

The following result shows that all the soundness and completeness results for neighborhood semantics presented in [2] carry forward to our neighborhood semantics with an actual world.

Theorem B.1. *Where* $\Gamma \subseteq \mathcal{W}$ *and* $\mathcal{F}$ *is a class of O- resp. OP-frames,* $\Gamma \Vdash'_{\mathcal{F}} A$ *iff* $\Gamma \Vdash_{\mathcal{F}} A$*.*

Proof. "$\Leftarrow$": Let $\Gamma \Vdash_{\mathcal{F}} A$ and $F = \langle W, \mathcal{O}\rangle \in \mathcal{F}$ resp. $F = \langle W, \mathcal{O}, \mathcal{P}\rangle \in \mathcal{F}$. Suppose there is an F-model' M' and a world $w \in W$ for which $M', w \not\models A$ and $M', w \models B$ for all $B \in \Gamma$. Note that $M = \langle F, v, w\rangle$ is an F-model of Γ for which $M \not\models A$,—a contradiction.

“$\Rightarrow$”: Let $\Gamma \Vdash'_{\mathcal{F}} A$. Suppose for some frame $F = \langle W, \mathcal{O} \rangle \in \mathcal{F}$ resp. $F = \langle W, \mathcal{O}, \mathcal{P} \rangle \in \mathcal{F}$ there is an F-model $M = \langle F, v, @ \rangle$ of Γ for which $M \not\models A$. Let $M' = \langle F, v \rangle$. Note that $M', @ \models B$ for all $B \in \Gamma$ and $M', @ \not\models A$,—a contradiction. $\square$

Strong completeness and soundness for the systems defined in Sections 5, 7 and 8 are immediate consequences of the strong completeness and soundness results presented in [2] and Theorem B.1.

Note that the following fact holds by definition:

Fact B.1. *Where $F = \langle W, \mathcal{O} \rangle$ is an O-frame resp. $F = \langle W, \mathcal{O}, \mathcal{P} \rangle$ is an OP-frame, for any model $M = \langle F, v, @ \rangle$, (i) $|A \wedge B|_M = |A|_M \cap |B|_M$, (i) $|A \vee B|_M = |A|_M \cup |B|_M$, (iii) $|\neg A|_M = W \setminus |A|_M$, (iv) $|\top|_M = W$, and (v) $|\bot|_M = \emptyset$.*

We will make use of it without further notice.

Lemma B.1. *Let $F = \langle W, \mathcal{O} \rangle$ be an O-frame, $F_\star = \langle W, \mathcal{O}', \mathcal{P} \rangle$ be an OP-frame that satisfies (FP-DfP2) and for which $\mathcal{O}'_@ = \mathcal{O}_@$ where $@ \in W$. Where $M = \langle F, v, @ \rangle$ and $M_\star = \langle F_\star, v, @ \rangle$, we have: (i) For all $A \in \mathcal{W}_{\mathsf{O}}$, $M \models A$ iff $M_\star \models A$, and (ii) if $M \models \mathsf{P}A$ then $M_\star \models \mathsf{P}A$.*

Proof. Ad (i): We prove the claim by an induction over the length of A. Let $A \in \mathcal{S}$: $M \models A$ iff $M, @ \models A$ iff $@ \in v(A)$ iff $M', @ \models A$ iff $M' \models A$. Let A be a wff without occurrences of O's and P's. Let first $A = B \wedge C$. $M \models A$ iff $M \models B \wedge C$ iff $M, @ \models B \wedge C$ iff $M, @ \models B, C$ iff $M \models B, C$ iff (by induction hypothesis) $M' \models B, C$ iff $M', @ \models B, C$ iff $M', @ \models B \wedge C$ iff $M' \models B \wedge C$. The proof is similar for the other Boolean combinations. We so far have shown that $|A|_M = |A|_{M'}$ for all wffs A without occurrences of modal operators. We will now shown that (i) holds for all $A \in \mathcal{W}_{\mathsf{O}}$. Let first $A = \mathsf{O}B$. $M \models A$ iff $M, @ \models A$ iff $M, @ \models \mathsf{O}B$ iff $|B|_M \in \mathcal{O}_@$ iff $|B|_{M'} \in \mathcal{O}_@$ iff $|B|_{M'} \in \mathcal{O}'_@$ iff $M', @ \models \mathsf{O}B$ iff $M', @ \models A$ iff $M' \models A$. Let now $A = B \wedge C$. $M \models A$ iff $M \models B \wedge C$ iff $M, @ \models B \wedge C$ iff $M, @ \models B, C$ iff $M \models B, C$ iff (by induction hypothesis) $M' \models B, C$ iff $M', @ \models B, C$ iff $M', @ \models B \wedge C$ iff $M' \models B \wedge C$ iff $M' \models A$. The argument is analogous for the other Boolean combinations.

Ad (ii): Let $M \models \mathsf{P}A$. Then by (M-DfP), $M \models \neg\mathsf{O}\neg A$. Hence, $|\neg A|_M \notin \mathcal{O}_@$. Hence, as shown in (i), $|\neg A|_{M'} \notin \mathcal{O}'_@$. Thus, by (FP-DfP2), $|A|_{M'} \in \mathcal{P}_@$. Hence, $M', @ \models \mathsf{P}A$ and thus, $M' \models \mathsf{P}A$. $\square$

Proof of Fact 2. (i) and (ii) follow immediately by Lemma B.1. Ad (iii): In the proof of Lemma B.1 this has been shown for all formulas without occurrences of modal operators. Let $A = \mathsf{O}B$. $M, w \models A$ iff $M, w \models \mathsf{O}B$ iff $|B|_M \in \mathcal{O}_w$ iff $|B|_{M_\star} \in \mathcal{O}_w$ iff $M_\star, w \models \mathsf{O}B$. The rest we show by induction. Let $A = B \wedge C$ where $B, C \in \mathcal{W}_{\mathsf{O}}$. $M, w \models A$ iff $M, w \models B \wedge C$ iff $M, w \models B, C$ iff $M_\star, w \models B, C$ iff $M_\star, w \models B \wedge C$ iff $M_\star, w \models A$. The proof is analogous for the other classical connectives. $\square$

Proof of Fact 1. We show that for each $\mathcal{F}_O$-model M there is an $\mathcal{F}_{OP}$-model $M_\star$ such that

$$\text{for all } A \in \mathcal{W}, M \models A \text{ iff } M_\star \models A, \tag{1}$$

and vice versa. Let $M = \langle W, \mathcal{O}, v, @\rangle$ be an $\mathcal{F}_O$-model. We define the $\mathcal{F}_{OP}$-model $M_\star = \langle W, \mathcal{O}, \mathcal{P}, v, @\rangle$ where $\mathcal{P}_w = \{X \subseteq \mathcal{W} \mid W \setminus X \notin \mathcal{O}_w\}$. Note that (FP-DfP1) and (FP-DfP2) hold by definition. We show that (1) holds by an induction over the length of A. By Fact 2 we know that (1) holds for all $A \in \mathcal{W}_O$. Let $A = \mathsf{P}B$. $M_\star \models A$ iff $M_\star \models \mathsf{P}B$ iff $M_\star, @ \models \mathsf{P}B$ iff $|B|_{M_\star} \in \mathcal{P}_@$ iff (by (FP-DfP1)) $W \setminus |B|_{M_\star} \notin \mathcal{O}_@$ iff $|\neg B|_{M_\star} \notin \mathcal{O}_@$ iff (by Fact 2) $|\neg B|_M \notin \mathcal{O}_@$ iff $M, @ \models \neg\mathsf{O}\neg B$ iff $M, @ \models \mathsf{P}B$ iff $M \models \mathsf{P}B$. Suppose $A = B \wedge C$. $M \models A$ iff $M, @ \models A$ iff $M, @ \models B \wedge C$ iff $M, @ \models B, C$ iff $M \models B, C$ iff (by induction hypothesis) $M_\star \models B, C$ iff $M_\star \models B \wedge C$ iff $M_\star \models A$. The other Boolean combinations are handled in an analogous way.

Let $M_\star = \langle W, \mathcal{O}, \mathcal{P}, v, @\rangle$ be an $\mathcal{F}_{OP}$-model for which $M_\star \models A$. Moreover, let $M = \langle W, \mathcal{O}, v, @\rangle$. By Fact 2 we know that (1) holds for all $A \in \mathcal{W}_O$. Let $A = \mathsf{P}B$. $M_\star \models A$ iff $M_\star \models \mathsf{P}B$ iff $M_\star, @ \models \mathsf{P}B$ iff $|B|_{M_\star} \in \mathcal{P}_@$ iff (by (FP-DfP1)) $W \setminus |B|_{M_\star} \notin \mathcal{O}_@$ iff $|\neg B|_{M_\star} \notin \mathcal{O}_@$ iff (by Fact 2) $|\neg B|_M \notin \mathcal{O}_@$ iff $M, @ \models \neg\mathsf{O}\neg B$ iff $M, @ \models \mathsf{P}B$ iff $M \models \mathsf{P}B$. Let $A = \neg B$. $M_\star \models A$ iff $M_\star, @ \models \neg B$ iff $M_\star, @ \not\models B$ iff $M_\star \not\models B$ iff (by induction hypothesis) $M \not\models B$ iff $M, @ \not\models B$ iff $M, @ \models \neg B$ iff $M \models \neg B$. The other Boolean combinations are handled in an analogous way. $\square$

Lemma B.2. *Let* $\Gamma \subseteq \mathcal{W}_\mathsf{O}$ *and* M *be an* $\mathbf{L}$*-model of* Γ*. There is an* $\mathbf{L}\star$*-model* $M_\star$ *of* Γ *for which (i)* $M_\star \models \mathsf{O}A \wedge \mathsf{P}\neg A$ *iff* $M \models \mathsf{O}A \wedge \mathsf{O}\neg A$*, (ii) for all* $A \in \mathcal{W}_\mathsf{O}$*,* $M \models A$ *iff* $M_\star \models A$*, and (iii) if* $M \models \mathsf{P}A$*, then* $M_\star \models \mathsf{P}A$*.*

Proof. Let $M = \langle F, v, @\rangle$ and $F = \langle W, \mathcal{O}\rangle$. Let $F_\star = \langle W, \mathcal{O}, \mathcal{P}\rangle$ where $\mathcal{P}$ is constructed on the basis of $\mathcal{O}$ along the following lines. For all $w \in W$ proceed as follows:

Step 1: For all $X \in \mathcal{O}_w$ let $X \in \mathcal{P}^1_w$, hence $\mathcal{P}^1_w =_{\text{df}} \{X \subseteq W \mid X \in \mathcal{O}_w\}$.

Step 2: For all $X \notin \mathcal{O}_w$ let $W \setminus X \in \mathcal{P}^2_w$, hence $\mathcal{P}^2_w =_{\text{df}} \{X \subseteq W \mid W \setminus X \notin \mathcal{O}_w\}$.

In the case that $\mathbf{L}$ is an A-CTDL we add another step:

Step 3: For every $Y \in \mathcal{P}^1_w \cup \mathcal{P}^2_w$ add all supersets $X \supset Y$ to $\mathcal{P}^3_w$, hence $\mathcal{P}^3_w =_{\text{df}} \{X \subseteq W \mid X \supset Y \text{ where } Y \in \mathcal{P}^1_w \cup \mathcal{P}^2_w\}$.

If $\mathbf{L}$ is an A-CDTL let $\mathcal{P}_w = \mathcal{P}^1_w \cup \mathcal{P}^2_w \cup \mathcal{P}^3_w$, otherwise let $\mathcal{P}_w = \mathcal{P}^1_w \cup \mathcal{P}^2_w$. Next we demonstrate that all the frame conditions for $\mathbf{L}\star$ are satisfied for $F_\star$: (FP-D) and (FP-DfP2) are satisfied due to construction steps 1 and 2. In the case that $\mathbf{L}$ is an A-CTDL note that (FP-P-RM) is satisfied due to construction step 3. Evidently all the $\mathbf{L}$ frame conditions are satisfied since they only concern the set $\mathcal{O}$. Thus, we have shown that all the $\mathbf{L}\star$ frame conditions are valid for $F_\star$. Let $M_\star = \langle F_\star, v, @\rangle$.

(ii) and (iii) follow by Fact 2. This also shows that $M_\star$ is a model of Γ.

Ad (i): Let $M_\star \models \mathsf{O}A \wedge \mathsf{P}\neg A$. Thus, where $X = |A|_{M_\star}$, $X \in \mathcal{O}_@$ and $W \setminus X \in \mathcal{P}_@$. Thus, if $W \setminus X$ was added in step one, then also $W \setminus X \in \mathcal{O}_@$ and thus $M_\star \models \mathsf{O}A \wedge \mathsf{O}\neg A$. Note that $W \setminus X$ was not added in step 2, since then $X \notin \mathcal{O}_@$,—a contradiction. Now suppose that $W \setminus X$ was added in step 3 (in the case that **L** is an A-CTDL). Then there is a $Y \subset W \setminus X$ such that Y was added in step 1 or step 2. In the former case $Y \in \mathcal{O}_@$ and thus, due to (F-RM), $W \setminus X \in \mathcal{O}_@$. Then, $M_\star \models \mathsf{O}A \wedge \mathsf{O}\neg A$. In the latter case $W \setminus Y \notin \mathcal{O}_@$. Note that $X \subset W \setminus Y$ and thus, due to (F-RM), $W \setminus Y \in \mathcal{O}_@$,—a contradiction. Altogether we have shown that $M_\star \models \mathsf{O}A \wedge \mathsf{O}\neg A$. Hence, by (ii), $M \models \mathsf{O}A \wedge \mathsf{O}\neg A$.

Let now $M \models \mathsf{O}A \wedge \mathsf{O}\neg A$. By (ii), $M_\star \models \mathsf{O}A \wedge \mathsf{O}\neg A$. Thus $|\neg A|_{M_\star} \in \mathcal{O}_@$. By step 1, $|\neg A|_{M_\star} \in \mathcal{P}_@$. Hence, $M_\star \models \mathsf{P}\neg A$ and hence $M_\star \models \mathsf{O}A \wedge \mathsf{P}\neg A$. $\square$

Theorem B.2. *Where* Γ *is* **L**-*consistent and* $A \in \mathcal{W}_{\mathsf{O}}$, $\Gamma \vdash_{\mathbf{L}} A$ *iff* $\Gamma_{\mathsf{O}} \vdash_{\mathbf{L}\star} A$.

Proof. Let Γ be **L**-consistent and $A \in \mathcal{W}_{\mathsf{O}}$. Suppose $\Gamma \vdash_{\mathbf{L}} A$. Thus, by Fact 3, $\Gamma_{\mathsf{O}} \vdash_{\mathbf{L}} A$. Then $M \models A$ for all **L**-models of Γ_{O}. By Lemma B.2(ii) there is an **L**$\star$-model of Γ_{O}. Suppose there is an **L**$\star$-model $M' = \langle W, \mathcal{O}, \mathcal{P}, v, @\rangle$ of Γ_{O} such that $M' \not\models A$. Let $M'' = \langle W, \mathcal{O}, v, @\rangle$. Then $M'' \not\models A$ and M'' is an **L**-model of Γ_{O} due to Lemma B.1 (i). This is a contradiction. Thus, $M' \models A$ for every **L**$\star$-model M' of Γ_{O}. Hence, $\Gamma_{\mathsf{O}} \Vdash_{\mathbf{L}\star} A$ and thus $\Gamma_{\mathsf{O}} \vdash_{\mathbf{L}\star} A$.

Suppose $\Gamma_{\mathsf{O}} \vdash_{\mathbf{L}\star} A$. Then for all **L**$\star$-models M' of Γ_{O}, $M' \models A$. Take any **L**-model $M = \langle W, \mathcal{O}, v, @\rangle$ of Γ. By Fact 3, M is also a model of Γ_{O}. Suppose $M \not\models A$ and hence $M \models \neg A$. By Lemma B.2(ii) there is an **L**$\star$-model $M_\star$ of Γ_{O} for which $M_\star \models \neg A$,—a contradiction. Thus, $M \models A$ for all **L**-models M of Γ_{O}. Thus, $\Gamma_{\mathsf{O}} \Vdash_{\mathbf{L}} A$ and hence $\Gamma_{\mathsf{O}} \vdash_{\mathbf{L}} A$. By Fact 3, $\Gamma \vdash_{\mathbf{L}} A$. $\square$

Proof of Theorem 1. If Γ is **L**-consistent, this follows by Theorem B.2. Let Γ be **L**-inconsistent. Suppose there is an **L**$\star$ model $M\star$ of Γ. By Lemma B.1 there is a **L** model of Γ. This is a contradiction. $\square$

Proof of Theorem 2. If $\Gamma \vdash_{\mathbf{L}} \mathsf{P}A$ then, due to (DfP1), $\Gamma \vdash_{\mathbf{L}} \neg\mathsf{O}\neg A$. Hence, due to the **L**-conservativeness of **L**$\star$, $\Gamma_{\mathsf{O}} \vdash_{\mathbf{L}\star} \neg\mathsf{O}\neg A$. Due to (DfP2), $\Gamma_{\mathsf{O}} \vdash_{\mathbf{L}\star} \mathsf{P}A$. $\square$

Proof of Theorem 3. This is due to the fact that by definition **L**$\star$ enriched by (DfP1) characterizes the same consequence relation as **L** enriched by (D). Since the latter characterizes the same consequence relation as **SDL**, so does the former. Note that all instances of (P-RM) in the case that **L** is an A-CTDL and all instances of (P-RE) are theorems of **L**. $\square$

Proof of Theorem 4. Note that **L**$\star$ enriched by (DfP1) by definition characterizes the same consequence relation as **L** enriched by (D) since all instances of (P-RM) in the case that **L** is an A-CTDL and all instances of (P-RE) are theorems of **L**. Hence, **L**$\star$ enriched by (DfP1) and the rules and axioms in Θ characterizes the same consequence relation as **L** enriched by (D) and Θ. $\square$

Proof of Theorem 5. Suppose $\Gamma_{\mathsf{O}} \vdash_{\mathbf{L}\star} \bigvee_I (\mathsf{O}A_i \wedge \mathsf{P}\neg A_i \wedge \neg\mathsf{O}\neg A_i)$ for some index set I. Then for every **L**$\star$-model $M_\star$ of Γ_{O} there is an A_i such that $M_\star \models \mathsf{O}A_i \wedge \mathsf{P}\neg A_i \wedge \neg\mathsf{O}\neg A_i$. Since Γ is **L**-consistent also Γ_{O} is **L**-consistent. Thus, there is an **L**-model M of Γ_{O}. By Lemma B.2, there is an **L**$\star$-model $M'_\star$ of Γ_{O} for which $M'_\star \models \mathsf{O}A_i \wedge \mathsf{P}\neg A_i$ iff $M \models \mathsf{O}A_i \wedge \mathsf{O}\neg A_i$ iff $M'_\star \models \mathsf{O}A_i \wedge \mathsf{O}\neg A_i$ for all A_i where $i \in I$,—a contradiction. $\square$

Proof of Theorem 6. This follows by Lemma B.2 and the fact that due to (M-DfP) every **L**-model of Γ is also an **L**-model of Γ_{O}. $\square$

Proof of Theorem 8. Let $\tau = \mathsf{O}$. Suppose $\mathbf{L}\star$ is OO-EX-τ-β-tolerant. Hence, for some A and B, there is an $\mathbf{L}\star$-model M of $\beta \cup \{\mathsf{O}A, \mathsf{O}\neg A\}$ such that $M \models \neg\mathsf{O}B$. Due to (FP-D), $M \models \mathsf{O}A \wedge \mathsf{P}\neg A$. Thus, $\beta \cup \{\mathsf{O}A, \mathsf{P}\neg A\} \nvdash_{\mathbf{L}\star} \mathsf{O}B$. Hence, $\mathbf{L}\star$ is also OP-EX-τ-β-tolerant. The other cases are analogous. $\square$

Corollary B.1. $\mathbf{L}\star$ *is* CONFLICT-TOLERANT *iff it is* OO-CONFLICT-TOLERANT.

Proof. Follows by Theorem 8. $\square$

Lemma B.3. *Let* $\mathbf{K} \in \{\mathbf{L}, \mathbf{L}\star\}$*,* $\delta \in \{\mathsf{OO}, \mathsf{OP}\}$*,* $\beta' \subseteq \beta$ *where* β *is a set of wffs, and* $\tau \in \{\mathsf{O}, \mathsf{P}, \mathsf{OO}\neg, \mathsf{O}\neg\mathsf{P}, \mathsf{OP}, \mathsf{O}\neg\mathsf{O}\neg\}$*. If* $\mathbf{K}$ *is* δ*-EX-*τ*-*β*-tolerant, then it is also* δ*-EX-*τ*-*β'*-tolerant.*

Proof. Let $\tau = \mathsf{O}$ and $\delta = \mathsf{OO}$. Suppose $\mathbf{K}$ is δ-EX-τ-β-tolerant and let $\beta' \subseteq \beta$. Thus, for some A and B there is an $\mathbf{K}$ model M of $\beta \cup \{\mathsf{O}A, \mathsf{O}\neg A\}$ for which $M \models \neg\mathsf{O}B$. Obviously M is also a model of $\beta' \cup \{\mathsf{O}A, \mathsf{O}\neg A\}$. The other cases are analogous. $\square$

Lemma B.4. *Let* $\mathbf{K} \in \{\mathbf{L}, \mathbf{L}\star\}$*,* $\delta \in \{\mathsf{OO}, \mathsf{OP}\}$*, and* β *is a set of wffs. If* $\mathbf{K}$ *is* δ*-EX-*OP*-*β*-tolerant, then it is* δ*-EX-*O*-*β*-tolerant and* δ*-EX-*P*-*β*-tolerant.*

Proof. By the presupposition there is for some A and B a $\mathbf{K}$-model M of $\{\mathsf{O}A, \mathsf{O}\neg A\} \cup \beta$ in the case that $\delta = \mathrm{OO}$ resp. $\{\mathsf{O}A \wedge \mathsf{P}\neg A\} \cup \beta$ in the case that $\delta = \mathrm{OP}$ such that $M \not\models \mathsf{O}B \vee \mathsf{P}B$. Hence, $M \models \neg\mathsf{O}B, \neg\mathsf{P}B$. $\square$

Let in the following $\gamma = \{\neg\mathsf{O}E, \neg\mathsf{O}\neg E\}$ (see Section 6).

Theorem B.3. *Let* $\delta \in \{\mathsf{OO}, \mathsf{OP}\}$*,* $\beta \subseteq \gamma \cup \{\mathsf{O}C, \neg\mathsf{O}\neg C, \mathsf{P}D, \neg\mathsf{O}\neg D, \mathsf{P}F, \mathsf{P}\neg F\}$*,* $\tau \in \{\mathsf{O}, \mathsf{P}, \mathsf{OO}\neg, \mathsf{O}\neg\mathsf{P}, \mathsf{O}\neg\mathsf{O}\neg\}$*, and* $\mathbf{K} \in \{\mathbf{L}, \mathbf{L}\star\}$*. (i) If* $\mathbf{K}$ *is* δ*-EX-*OP*-*γ*-tolerant then it is also* δ*-EX-*τ*-*β*-tolerant. (ii) If* $\mathbf{K}$ *is* OO*-EX-*OP*-*γ*-tolerant and* OP*-EX-*OP*-*γ*-tolerant, then* $\mathbf{K}$ *is* CONFLICT-TOLERANT.

Proof. Ad (i): Suppose $\mathbf{K}$ is OO-EX-OP-γ-tolerant. Thus, for some A and B there is a $\mathbf{K}$-model M of $\{\mathsf{O}A, \mathsf{O}\neg A, \neg\mathsf{O}E, \neg\mathsf{O}\neg E\}$ such that $M \models \neg\mathsf{O}B, \neg\mathsf{P}B$. Note that since $M \models \mathsf{P}\neg B, \mathsf{O}\neg B, \mathsf{P}E, \mathsf{P}\neg E$ due to (FP-DfP2), M is also a model of $\beta'' = \{\mathsf{O}A,\ \mathsf{O}\neg A,\ \mathsf{O}C,\ \neg\mathsf{O}\neg C,\ \mathsf{P}D,\ \neg\mathsf{O}\neg D,\ \neg\mathsf{O}E,\ \neg\mathsf{O}\neg E,\ \mathsf{P}F,\ \mathsf{P}\neg F\}$ where $C = \neg B$, $D = \neg B$ and $F = E$. Hence, $\mathbf{K}$ is OO-EX-OP-β''-tolerant.

By Lemma B.4 it is also OO-EX-O-β'' and OO-EX-P-β''-tolerant. As $M \models \neg\mathsf{O}E \wedge \neg\mathsf{O}\neg E$, also $M \models \neg(\mathsf{O}E \vee \mathsf{O}\neg E)$. Thus, $\mathbf{K}$ is OO-EX-$\mathsf{OO}\neg$-β''-tolerant. Since $M \models \mathsf{O}C \wedge \neg\mathsf{O}\neg C$, also $M \models \neg(\mathsf{O}\neg C \vee \neg\mathsf{O}\neg\neg C)$. Thus, $\mathbf{K}$ is OO-EX-$\mathsf{O}\neg\mathsf{O}\neg$-$\beta''$-tolerant. Since $M \models \mathsf{P}E \wedge \neg\mathsf{O}E$, also $M \models \neg(\mathsf{O}E \vee \neg\mathsf{P}E)$. Hence, $\mathbf{K}$ is OO-EX-$\mathsf{O}\neg\mathsf{P}$-β''-tolerant.

The rest follows by Lemma B.3. The case for $\delta = \mathsf{OP}$ is analogous.
Ad (ii): Follows immediately by (i). $\square$

Theorem B.4. *Let β be a set of formulas of the form $\mathsf{O}A$, $\neg\mathsf{O}A$ and $\mathsf{P}A$, and $\tau \in \{\mathsf{O}, \mathsf{OO}\neg, \mathsf{O}\neg\mathsf{P}, \mathsf{OP}, \mathsf{O}\neg\mathsf{O}\neg\}$. If $\mathbf{L}$ is OO-EX-τ-β-tolerant, then $\mathbf{L}\star$ is OO-EX-τ-β-tolerant and OP-EX-τ-β-tolerant.*

Proof. Let $\tau = \mathsf{O}$. Since $\mathbf{L}$ is OO-EX-O-β-tolerant, there is, for some A and B, an $\mathbf{L}$-model M of $\{\mathsf{O}A \wedge \mathsf{O}\neg A\} \cup \beta$ such that $M \models \neg\mathsf{O}B$. By Lemma B.2, there is an $\mathbf{L}\star$-model $M_\star$ of $\{\mathsf{O}A \wedge \mathsf{O}\neg A\} \cup \beta$ for which $M_\star \models \neg\mathsf{O}B$. Hence, $\mathbf{L}\star$ is OO-EX-O-β-tolerant. Thus, by Theorem 8, $\mathbf{L}\star$ is also OP-EX-O-β-tolerant. The proof is analogous for $\tau \in \{\mathsf{OO}\neg, \mathsf{O}\neg\mathsf{O}\neg\}$.

Let $\tau = \mathsf{O}\neg\mathsf{P}$. Since $\mathbf{L}$ is OO-EX-$\mathsf{O}\neg\mathsf{P}$-β-tolerant, there is, for some A and B, an $\mathbf{L}$-model M of $\{\mathsf{O}A \wedge \mathsf{O}\neg A\} \cup \beta$ such that $M \models \neg\mathsf{O}B \wedge \mathsf{P}B$. By Lemma B.2, there is an $\mathbf{L}\star$-model $M_\star$ of $\{\mathsf{O}A \wedge \mathsf{O}\neg A\} \cup \beta$ for which $M_\star \models \neg\mathsf{O}B \wedge \mathsf{P}B$. Hence, $\mathbf{L}\star$ is OO-EX-$\mathsf{O}\neg\mathsf{P}$-β-tolerant. Thus, by Theorem 8, $\mathbf{L}\star$ is also OP-EX-$\mathsf{O}\neg\mathsf{P}$-β-tolerant.

Let $\tau = \mathsf{OP}$. Since $\mathbf{L}$ is OO-EX-OP-β-tolerant, there is, for some A and B, an $\mathbf{L}$-model M of $\{\mathsf{O}A \wedge \mathsf{O}\neg A\} \cup \beta$ such that $M \models \neg\mathsf{O}B \wedge \neg\mathsf{P}B$. Hence, due to (M-DfP), $M \models \mathsf{O}\neg B$. By Lemma B.2, there is a $\mathbf{L}\star$-model $M_\star$ of $\{\mathsf{O}A \wedge \mathsf{O}\neg A\} \cup \beta$ for which $M_\star \models \neg\mathsf{O}B, \mathsf{O}\neg B$ and $M_\star \models \mathsf{O}\neg B \wedge \mathsf{P}B$ iff $M \models \mathsf{O}\neg B \wedge \mathsf{O}B$. Since $M \not\models \mathsf{O}B$, $M_\star \not\models \mathsf{P}B$. Hence $M_\star \models \neg\mathsf{P}B$. Hence, $\mathbf{L}\star$ is OO-EX-OP-β-tolerant. Thus, by Theorem 8, $\mathbf{L}\star$ is also OP-EX-OP-β-tolerant. ☐

Proof of Theorem 7. If $\mathbf{L}$ is OO-EX-OP-γ-tolerant, then $\mathbf{L}\star$ is OO-EX-OP-γ-tolerant and OP-EX-OP-γ-tolerant by Theorem B.4. Moreover, by Theorem B.3 (ii), $\mathbf{L}\star$ is CONFLICT-TOLERANT. ☐

Theorem B.5. *Let β be a set of formulas of the form $\mathsf{O}A$, $\neg\mathsf{O}A$ and $\mathsf{P}A$, and let $\tau \in \{\mathsf{O}, \mathsf{P}, \mathsf{OP}, \mathsf{O}\neg\mathsf{O}\neg\}$. If $\mathbf{L}$ is OO-EX-$\mathsf{O}\neg\mathsf{O}\neg$-$\beta$-tolerant, then $\mathbf{L}\star$ is OO-EX-τ-β-tolerant and OP-EX-τ-β-tolerant.*

Proof. By supposition there is, for some A and B, an $\mathbf{L}$-model M of $\{\mathsf{O}A \wedge \mathsf{O}\neg A\} \cup \beta \cup \{\mathsf{O}\neg B \wedge \neg\mathsf{O}B\}$. By Lemma B.2, there is an $\mathbf{L}\star$-model $M_\star$ of $\{\mathsf{O}A \wedge \mathsf{O}\neg A\} \cup \beta \cup \{\mathsf{O}\neg B \wedge \neg\mathsf{O}B\}$ for which $M_\star \models \mathsf{O}\neg B \wedge \mathsf{P}B$ iff $M \models \mathsf{O}\neg B \wedge \mathsf{O}B$ and hence $M_\star \not\models \mathsf{P}B$. Hence, $\mathbf{L}\star$ is OO-EX-P-β-tolerant and OO-EX-O-β-tolerant. Note that $M_\star \models \neg(\mathsf{O}B \vee \mathsf{P}B)$. Hence, $\mathbf{L}\star$ is OO-EX-OP-β-tolerant. Moreover, $M_\star \models \neg(\mathsf{O}B \vee \neg\mathsf{O}\neg B)$. Hence, $\mathbf{L}\star$ is OO-EX-$\mathsf{O}\neg\mathsf{O}\neg$-$\beta$-tolerant. The OP-EX-$\mathsf{O}\neg\mathsf{O}\neg$-$\beta$-tolerance follows immediately by Theorem 8. ☐

Proof of Theorem 9. Follows by Theorem B.4 and Theorem B.5. ☐

B.2 Meta-Theory for Some Strengthenings of $\mathbf{L}\star$

Proof of Theorem A.1. Ad (i): By (RPM) and (RE), $(\mathsf{O}\neg B \wedge \neg\mathsf{O}B) \supset \mathsf{O}\neg A$. Hence, $\neg\mathsf{O}\neg B \vee \mathsf{O}B \vee \mathsf{O}\neg A$. Hence,

$$\neg\mathsf{O}\neg A \supset (\neg\mathsf{O}\neg B \vee \mathsf{O}B) \tag{2}$$

Note that by (DfP2)

$$\neg\mathsf{O}\neg A \equiv (\mathsf{P}A \wedge \neg\mathsf{O}\neg A) \tag{3}$$

and by (DfP2) and (D),

$$(\neg\mathsf{O}\neg B \vee \mathsf{O}B) \supset \mathsf{P}B \tag{4}$$

Hence, by (2), (3) and (4), $(\mathsf{P}A \wedge \neg\mathsf{O}\neg A) \supset \mathsf{P}B$.

Ad (ii): This is a direct consequence from (DfP2). $\square$

Lemma B.5. *Where $M = \langle W, \mathcal{O}, v, @\rangle$ is an $\mathbf{L}$-model and $M_\star = \langle W, \mathcal{O}', \mathcal{P}, v, @\rangle$ is a $\mathbf{K}\star$-model for which $\mathcal{O}'_@ = \mathcal{O}_@$, we have: (i) For all $A \in \mathcal{W}_\mathsf{O}$, $M \models A$ iff $M_\star \models A$, (ii) if $M \models \mathsf{P}A$ then $M_\star \models \mathsf{P}A$. (iii) $|A|_M = |A|_{M_\star}$ for all A without occurrences of modal operators.*

Proof. The proof of (i) and (ii) is analogous to the proof of Lemma B.1. Ad (iii): Let $A \in \mathcal{S}$. $M, w \models A$ iff $w \in v(A)$ iff $M_\star, w \models A$. We show the rest by an induction on the length of A. Let $A = B \wedge C$. $M, w \models A$ iff $M, w \models B \wedge C$ iff $M, w \models B, C$ iff $M_\star, w \models B, C$ iff $M_\star, w \models B \wedge C$ iff $M_\star, w \models A$. The proof is analogous for the other classical operators. $\square$

Lemma B.6. *Where $M = \langle F, v, @\rangle$ is an $\mathbf{L}$-model, there is a $\mathbf{K}\star$-model $M' = \langle F', v, @\rangle$ such that (i) F' satisfies* (FP-P-AND) *and* (FP-AND-OP)*; (ii) for all $A \in \mathcal{W}_\mathsf{O}$, $M \models A$ iff $M' \models A$; and (iii) if $M \models \mathsf{P}A$ then $M' \models \mathsf{P}A$.*

Proof. Where $F = \langle W, \mathcal{O}\rangle$ let $F' = \langle W, \mathcal{O}, \mathcal{P}\rangle$ and for all $w \in W$, $\mathcal{P}_w = \wp(W)$. (FP-DfP2) holds since for all $X \subseteq W$ for which $W \setminus X \notin \mathcal{O}_w$, $X \in \mathcal{P}_w$. Also (FP-D) holds, since for all $X \subseteq W$ for which $X \in \mathcal{O}_w$, $X \in \mathcal{P}_w$. (FP-P-AND) holds due to the fact that $\mathcal{P}_w$ is closed under subsets. The rest follows by Lemma B.5. $\square$

Theorem B.6. *Where Γ is $\mathbf{L}$-consistent and $A \in \mathcal{W}_\mathsf{O}$, $\Gamma \vdash_\mathbf{L} A$ iff $\Gamma_\mathsf{O} \vdash_{\mathbf{K}\star} A$.*

Proof. The proof is analogous to the proof of Theorem B.2, where all references to Lemma B.2(ii) are replaced by references to Lemma B.6(ii), all references to Lemma B.1 are replaced by references to Lemma B.5, and all references to $\mathbf{L}\star$ are replaced by references to $\mathbf{K}\star$. $\square$

Proof of Theorem A.2. If Γ is $\mathbf{L}$-consistent, this follows by Theorem B.6. Let Γ be $\mathbf{L}$-inconsistent. Suppose there is an $\mathbf{K}\star$ model $M\star$ of Γ. By Lemma B.5 there is a $\mathbf{L}$ model of Γ. This is a contradiction. $\square$

Proof of Theorem A.3. If $\Gamma \vdash_\mathbf{L} \mathsf{P}A$ then, due to (DfP1), $\Gamma \vdash_\mathbf{L} \neg\mathsf{O}\neg A$. Hence, due to the $\mathbf{L}$-conservativeness of $\mathbf{K}\star$, $\Gamma_\mathsf{O} \vdash_{\mathbf{K}\star} \neg\mathsf{O}\neg A$. Due to (DfP2), $\Gamma_\mathsf{O} \vdash_{\mathbf{K}\star} \mathsf{P}A$. $\square$

Proof of Theorem A.4. Note that by Theorem 3 $\mathbf{L}\star \oplus \{(\text{DfP1})\}$ characterizes the same consequence relation as **SDL**. Since **SDL** features all instances of (P-AND) and (AND-OP) as theorems, also $\mathbf{K}\star \oplus \{(\text{DfP1})\}$ characterizes the same consequence relation as **SDL**. $\square$

Proof of Theorem A.5. By Theorem 4, $\mathbf{L}\star\oplus\{(\text{DfP1})\}\cup\Theta$ characterizes the same consequence relation as **SDL**. Since **SDL** features all instances of (P-AND) and (AND-OP) as theorems, also $\mathbf{K}\star\oplus\{(\text{DfP1})\}$ characterizes the same consequence relation as **SDL**. $\square$

Proof of Theorem A.8. The proof is identical to the proof of Theorem 8 where all references to $\mathbf{L}\star$ are replaced by references to $\mathbf{K}\star$. $\square$

Corollary B.2. $\mathbf{L}\star$ *is* CONFLICT-TOLERANT *iff it is* OO-CONFLICT-TOLERANT.

Proof. Follows by Theorem A.8. $\square$

Lemma B.7. *Let* $\delta \in \{\mathsf{OO}, \mathsf{OP}\}$*,* $\beta' \subseteq \beta$ *where* β *is a set of wffs, and* $\tau \in \{\mathsf{O}, \mathsf{P}, \mathsf{OO}\neg, \mathsf{O}\neg\mathsf{P}, \mathsf{OP}, \mathsf{O}\neg\mathsf{O}\neg\}$*. If* $\mathbf{K}\star$ *is* δ*-EX-*τ*-*β*-tolerant, then it is also* δ*-EX-*τ*-*β'*-tolerant.*

Proof. The proof is identical to the proof of Lemma B.3 where all references to **K** are replaced by references to $\mathbf{K}\star$. $\square$

Lemma B.8. *Let* $\delta \in \{\mathsf{OO}, \mathsf{OP}\}$*, and* β *is a set of wffs. If* $\mathbf{K}\star$ *is* δ*-EX-*OP*-*β*-tolerant, then it is* δ*-EX-*O*-*β*-tolerant and* δ*-EX-*P*-*β*-tolerant.*

Proof. The proof is analogous to the proof of Lemma B.4 where all references to **K** are replaced by references to $\mathbf{K}\star$. $\square$

Proof of Theorem A.6. The proof is analogous to the proof of Theorem B.3 where all references to Lemma B.4 are replaced by references to Lemma B.8, all references to Lemma B.3 are replaced by references to Lemma B.7 and all references to **K** are replaced by references to $\mathbf{K}\star$. $\square$

Lemma B.9. *Let* $\Gamma \subseteq \mathcal{W}_{\mathsf{O}}$ *and* M *be an* **L***-model of* $\Gamma' = \Gamma \cup \{\mathsf{O}A, \mathsf{O}\neg A, \neg\mathsf{O}B, \neg\mathsf{P}B\}$*. There is a* $\mathbf{K}\star$*-model* $M_\star$ *of* Γ' *for which (i) for all* $C \in \mathcal{W}_{\mathsf{O}}$*,* $M \models C$ *iff* $M_\star \models C$*, and (ii) if* $M \models \mathsf{P}D$ *then* $M_\star \models \mathsf{P}D$*.*

Proof. Let $M = \langle F, v\rangle$ where $F = \langle W, \mathcal{O}\rangle$. Let $F_\star = \langle W, \mathcal{O}', \mathcal{P}\rangle$ such that $\mathcal{O}'_w = \mathcal{O}_@$ for all $w \in W$. We construct $\mathcal{P}$ on the basis of $\mathcal{O}'$ similar as in the proof of Lemma B.2, but with the addition of some steps. In the case that **L** is an A-CTDL we add a new step 2.1 to steps 1 and 2 and alter former step 3 accordingly:

Step 2.1 For all $X \in \mathcal{P}^1_w \cup \mathcal{P}^2_w$ add all $\{x\} \subseteq X \setminus |B|_M$ to $\mathcal{P}^{2.1}_w$. Hence $\mathcal{P}^{2.1}_w =_{\text{df}} \big\{\{x\} \subseteq X \setminus |B|_M : X \in \mathcal{P}^1_w \cup \mathcal{P}^2_w\big\}$.

Step 3.1 For every $Y \in \mathcal{P}^1_w \cup \mathcal{P}^2_w \cup \mathcal{P}^{2.1}_w$ add all supersets $X \supset Y$ to $\mathcal{P}^{3.1}_w$. Hence $\mathcal{P}^{3.1}_w =_{\text{df}} \big\{X \subseteq W : X \supset Y, Y \in \mathcal{P}^1_w \cup \mathcal{P}^2_w \cup \mathcal{P}^{2.1}_w\big\}$.

Let $\mathcal{P}_w = \mathcal{P}^1_w \cup \mathcal{P}^2_w \cup \mathcal{P}^{2.1}_w \cup \mathcal{P}^{3.1}_w$.

First notice that there is no $Y \in \mathcal{P}_w$ such that $Y \subseteq |B|_M$. Suppose for some $X \subseteq |B|_M$, $X \in \mathcal{O}'_w$. Then $X \in \mathcal{O}_@$. Due to (F-RM), $|B| \in \mathcal{O}_@$ and hence $M \models \mathsf{O}B$,—a contradiction. Hence, no $X \subseteq |B|_M$ is added to $\mathcal{P}_w$ in step 1.

Suppose some $X \subseteq |B|_M$ has been added in step 2. Then $W \setminus X \notin \mathcal{W}'_w$ and hence $W \setminus X \notin \mathcal{W}_@$. Note that $M \models \mathsf{O}\neg B$ due to $M \models \neg\mathsf{P}B$ and (M-DfP). Note that $W \setminus |B|_M = |\neg B|_M \subseteq W \setminus X$ and due to (F-RM), $W \setminus X \in \mathcal{W}_@$,—a contradiction. Hence, no $X \subseteq |B|_M$ is added to $\mathcal{P}_w$ in step 2. This, by definition, generalizes to steps 2.1 and 3.1. Now we show that $F_\star$ satisfies (FP-P-AND) and hence (FP-AND-OP). In order to show this let $Y = Y_1 \cup Y_2 \in \mathcal{P}_w$. The case $Y = Y_1 = Y_2 = \emptyset$ is trivial. Let $Y \neq \emptyset$. If Y was added in step 3.1 then by definition there is a $Y' \subseteq Y$ that was added in the previous steps. If Y was added in steps 1, 2 or 2.1 then let $Y' = Y$. We have shown that $Y' \not\subseteq |B|_M$. Hence, $Y' \setminus |B|_M \neq \emptyset$. By definition, there is a $x \in Y' \setminus |B|_M$. Wlog. let $x \in Y_1$. Since $\{x\} \in \mathcal{P}_w$ (by step 2.1), due to step 3.1 also $Y_1 \in \mathcal{P}_w$. Note that (FP-DfP2) is satisfied due the construction step 2. (FP-D) is satisfied due to the construction step 1. Step 3.1 takes care of (FP-P-RM). Thus, $F_\star$ is a $\mathbf{K}\star$-frame.

In the case that $\mathbf{L}$ is an I-CTDL or an H-CTDL and $\mathbf{K}\star \in \{\mathbf{L}\star\oplus(\text{AND-OP}), \mathbf{L}\star\oplus(\text{P-AND})\}$. We add the following step to steps 1 and 2:

Step 3': For all $X \in \mathcal{P}^1_w \cup \mathcal{P}^2_w$ add all Y to $\mathcal{P}^{3'}_w$ for which $Y \subset X$ and $Y \neq |B|_M$. Hence, $\mathcal{P}^{3'}_w =_{\text{df}} \{Y \subset X : Y \neq |B|_M, X \in \mathcal{P}^1_w \cup \mathcal{P}^2_w\}$.

Let $\mathcal{P}_w = \mathcal{P}^1_w \cup \mathcal{P}^2_w \cup \mathcal{P}^{3'}_w$.

Note that $|B|_M \notin \mathcal{P}_w$. It was not added in step one since otherwise $|B|_w \in \mathcal{O}_@$ and hence $M \models \mathsf{O}B$,—a contradiction. It was not added in step 2 since otherwise $W \setminus |B|_M = |\neg B|_M \notin \mathcal{O}_@$ and hence $M \models \neg\mathsf{O}\neg B$,—a contradiction. By definition also $|B|_M \notin \mathcal{P}^{3'}_w$. Thus, $|B|_M \notin \mathcal{P}_w$.

Note that (FP-DfP2) is satisfied due the construction step 2. (FP-D) is satisfied due to the construction step 1. Moreover, (FP-P-AND) and hence (FP-AND-OP) are valid due to construction step 3'. Where $Y = Y_1 \cup Y_2 \in \mathcal{P}_w$ we have to consider two cases: (1) $Y \setminus |B|_M \neq \emptyset$ and (2) $Y \subset |B|_M$. In the case (2) (FP-P-AND) holds by construction for Y. In the case (1) either $Y_1 \setminus |B|_M \supset \emptyset$ or $Y_2 \setminus |B|_M \supset \emptyset$. Thus, either $Y_1 \neq |B|_M$ or $Y_2 \neq |B|_M$. Thus, by construction step 3', either $Y_1 \in \mathcal{P}^{3'}_w$ or $Y_2 \in \mathcal{P}^{3'}_w$. Thus, $F_\star$ is a $\mathbf{K}\star$-frame.

Define $M_\star = \langle F_\star, v, @\rangle$. Both, (i) and (ii) hold due to Lemma B.5. Since $|B|_M \notin \mathcal{P}_@$ and since by Lemma B.5 $|B|_M = |B|_{M_\star}$, $M \models \neg\mathsf{P}B$. Hence, $M_\star$ models Γ'. ☐

Theorem B.7. *Let β be a set of formulas of the form $\mathsf{O}A$, $\neg\mathsf{O}A$ and $\mathsf{P}A$, and $\tau \in \{\mathsf{O}, \mathsf{O}\mathsf{O}\neg, \mathsf{O}\neg\mathsf{P}, \mathsf{O}\mathsf{P}, \mathsf{O}\neg\mathsf{O}\neg\}$. If $\mathbf{L}$ is* OO*-EX-τ-β-tolerant, then $\mathbf{K}\star$ is* OO*-EX-τ-β-tolerant and* OP*-EX-τ-β-tolerant.*

Proof. Let $\tau = \mathsf{O}$. Since $\mathbf{L}$ is OO-EX-O-β-tolerant, there is, for some A and B, an $\mathbf{L}$-model M of $\{\mathsf{O}A \wedge \mathsf{O}\neg A\} \cup \beta$ such that $M \models \neg\mathsf{O}B$. By Lemma B.9, there is a $\mathbf{K}\star$-model $M_\star$ of $\{\mathsf{O}A \wedge \mathsf{O}\neg A\} \cup \beta$ for which $M_\star \models \neg\mathsf{O}B$. Hence, $\mathbf{K}\star$ is OO-EX-O-β-tolerant. Thus, by Theorem A.8, $\mathbf{K}\star$ is also OP-EX-O-β-tolerant. The proof is analogous for $\tau \in \{\mathsf{O}\mathsf{O}\neg, \mathsf{O}\neg\mathsf{O}\neg\}$.

Let $\tau = \mathsf{O}\neg\mathsf{P}$. Since $\mathbf{L}$ is OO-EX-O¬P-β-tolerant, there is, for some A and B, an $\mathbf{L}$-model M of $\{\mathsf{O}A \wedge \mathsf{O}\neg A\} \cup \beta$ such that $M \models \neg\mathsf{O}B \wedge \mathsf{P}B$. By Lemma

B.9, there is a $\mathbf{K}\star$-model $M_\star$ of $\{\mathsf{O}A \wedge \mathsf{O}\neg A\} \cup \beta$ for which $M_\star \models \neg\mathsf{O}B \wedge \mathsf{P}B$. Hence, $\mathbf{K}\star$ is OO-EX-$\mathsf{O}\neg\mathsf{P}$-β-tolerant. Thus, by Theorem A.8, $\mathbf{K}\star$ is also OP-EX-$\mathsf{O}\neg\mathsf{P}$-β-tolerant.

Let $\tau = \mathsf{OP}$. Since $\mathbf{L}$ is OO-EX-OP-β-tolerant, there is, for some A and B, an $\mathbf{L}$-model M of $\{\mathsf{O}A \wedge \mathsf{O}\neg A\} \cup \beta$ such that $M \models \neg\mathsf{O}B \wedge \neg\mathsf{P}B$. Hence, due to (M-DfP), $M \models \mathsf{O}\neg B$. By Lemma B.9, there is a $\mathbf{K}\star$-model $M_\star$ of $\{\mathsf{O}A \wedge \mathsf{O}\neg A\} \cup \beta$ for which $M_\star \models \neg\mathsf{O}B, \neg\mathsf{P}B$. Hence, $\mathbf{K}\star$ is OO-EX-OP-β-tolerant. Thus, by Theorem A.8, $\mathbf{K}\star$ is also OP-EX-OP-β-tolerant. ☐

Proof of Theorem A.7. If $\mathbf{L}$ is OO-EX-OP-γ-tolerant, then $\mathbf{K}\star$ is OO-EX-OP-γ-tolerant and OP-EX-OP-γ-tolerant by Theorem B.7. Moreover, by Theorem A.6 (ii), $\mathbf{K}\star$ is CONFLICT-TOLERANT. ☐

Theorem B.8. *Let* $\tau \in \{\mathsf{O}, \mathsf{P}, \mathsf{OP}, \mathsf{O}\neg\mathsf{O}\neg\}$. *If* $\mathbf{L}$ *is* OO-*EX*-$\mathsf{O}\neg\mathsf{O}\neg$-$\beta$-*tolerant, then* $\mathbf{K}\star$ *is* OO-*EX*-τ-β-*tolerant and* OP-*EX*-τ-β-*tolerant.*

Proof. By supposition there is, for some A and B, an $\mathbf{L}$-model M of $\{\mathsf{O}A, \mathsf{O}\neg A\} \cup \beta \cup \{\mathsf{O}\neg B, \neg\mathsf{O}B\}$ and hence of $\{\mathsf{O}A, \mathsf{O}\neg A\} \cup \beta \cup \{\neg\mathsf{P}B, \neg\mathsf{O}B\}$. By Lemma B.9, there is a $\mathbf{K}\star$-model $M_\star$ of $\{\mathsf{O}A, \mathsf{O}\neg A\} \cup \beta \cup \{\neg\mathsf{P}B, \neg\mathsf{O}B\}$. Hence, $\mathbf{K}\star$ is OO-EX-P-β-tolerant, OO-EX-O-β-tolerant, and OO-EX-OP-β-tolerant. The OP-EX-O-β-tolerance, the OP-EX-P-β-tolerance and the OP-EX-OP-β-tolerance follows immediately by Theorem A.8.

Note that due to $M_\star \models \neg\mathsf{P}B$ and (FP-DfP2), $M_\star \models \mathsf{O}\neg B$. Hence, $M_\star \models \neg(\mathsf{O}B \vee \neg\mathsf{O}\neg B)$. Thus, $\mathbf{K}\star$ is OO-EX-$\mathsf{O}\neg\mathsf{O}\neg$-$\beta$-tolerant. The OP-EX-$\mathsf{O}\neg\mathsf{O}\neg$-$\beta$-tolerance follows immediately by Theorem A.8. ☐

Proof of Theorem A.9. Follows by Theorem B.7 and Theorem B.8. ☐

B.3 Meta-Theory for F and DPM

Proof of Theorem 10. We define an $\mathbf{F}$-model M on an O-frame $F = \langle W, \mathcal{O}\rangle$ such that M validates a counter-instance to the explosion principle OO-EX-OP-$\{\neg\mathsf{O}E, \neg\mathsf{O}\neg E\}$. Let $W = \wp(\mathcal{S})$ and p_1, p_2 be different propositional letters. We define $W_a = \{w \in W \mid p_1 \notin w, p_2 \notin w\}$, $W_b = \{w \in W \mid p_1 \notin w, p_2 \in w\}$, $W_c = \{w \in W \mid p_1 \in w, p_2 \notin w\}$ and $W_d = \{w \in W \mid p_1 \in w, p_2 \in w\}$. Let $\mathcal{O}_w = \big\{X \subseteq W \mid X \supseteq W_a \cup W_b\big\} \cup \big\{X \subseteq W \mid X \supseteq W_c \cup W_d\big\}$ for every $w \in W$. Let $M = \langle F, v, @\rangle$ where $v : \mathcal{S} \to \wp(W)$, $S \mapsto \{w \in W \mid S \in w\}$ and $@$ is an arbitrary world in W. It is an easy exercise for the reader to demonstrate that the $\mathbf{F}$ frame conditions hold for F. Note that $M \models \mathsf{O}A, \mathsf{O}\neg A, \neg(\mathsf{O}B \vee \mathsf{P}B), \neg(\mathsf{O}E \vee \mathsf{O}\neg E)$, where $A = p_1$, $B = \neg(p_1 \vee p_2)$, and $E = p_2$. The rest follows from Theorem B.3. ☐

Proof of Theorem 11. This has been shown in [1] (for **DPM.1**) and in [3] (for **DPM.2′**). ☐

Proof of Theorem 12. Let p_1 and p_2 be distinct propositional letters.

Where $W = \wp(\mathcal{S})$ and $v : \mathcal{S} \to \wp(W)$, $S \mapsto \{w \in W \mid S \in w\}$, we define a frame $F = \langle W, \mathcal{O}\rangle$ where for all $w \in W$, $\mathcal{O}_w = \{W' \subseteq W \mid W' \supseteq v(p_1)\} \cup \{\emptyset\}$.

Note that F is a **DPM.1**-frame. Let $M = \langle F, v, @\rangle$ where @ is any world in W. Evidently, $M \models \mathsf{O}A, \mathsf{O}\neg A, \neg(\mathsf{O}B \vee \mathsf{P}B), \neg(\mathsf{O}E \vee \mathsf{O}\neg E)$ where $A = \top$, $B = \neg p_1$, $E = p_2$.

For **DPM.2** and **DPM.2′** we construct a frame $F_2 = \langle W, \mathcal{O}^2\rangle$ as follows. We define $W_a = \{w \in W \mid p_1 \notin w, p_2 \notin w\}$, $W_b = \{w \in W \mid p_1 \notin w, p_2 \in w\}$, $W_c = \{w \in W \mid p_1 \in w, p_2 \notin w\}$ and $W_d = \{w \in W \mid p_1 \in w, p_2 \in w\}$. For all $w \in W$ let $\mathcal{O}^2_w = \{W\} \cup \{W_a \cup W_b\} \cup \{W_c \cup W_d\}$. Note that F_2 is a **DPM.2**-frame and a **DPM.2′**-frame as can easily be shown and is left to the reader. Let $M_2 = \langle F_2, v, @\rangle$. Note that $M_2 \models \mathsf{O}A, \mathsf{O}\neg A, \neg(\mathsf{O}B \vee \mathsf{P}B), \neg(\mathsf{O}E \vee \mathsf{O}\neg E)$, where $A = p_1$, $B = \bot$, and $E = p_2$.

The rest follows from Theorem B.3. □

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