Prime Implicates and Relevant Belief Revision

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Abstract
This article discusses Parikh’s axiom of relevance in belief revision, and recalls some results from Kourousias and Makinson (2007, *J. Symbolic Logic*, 72, 994–1002) in this context. The crucial distinction is emphasized between the uniqueness of the finest splitting of $K$ and the fact that $K$ has several normal forms associated with that finest splitting. The main new result of this article is a new proof for the theorem that the set of prime implicates of $K$ is a normal form for the finest splitting of $K$. It is explained how this proof avoids a mistake in an earlier proof from Wu and Zhang (2010, *Knowledge-Based Syst.*, 23, 70–76). As a corollary, relevance can be re-defined without reference to the finest splitting, using the notion of path-relevance from Makinson (2009, *J. Appl. Logic*, 7, 377–387). Finally, a weak yet sufficient condition for irrelevance is presented.

Keywords: Belief revision, relevance, splitting, prime implicates.

1 Motivation and outline of this article
This article discusses the relation of Parikh’s axiom of relevance in belief revision to the set of prime implicates of a belief set $K$. Its aim is threefold:

(i) to stress the importance of the distinction between the uniqueness of the finest splitting of a belief set, and the fact that this belief set may have several different canonical forms associated with that finest splitting (see Section 4)

(ii) to show that the set of prime implicates is one specific such normal form, correcting an earlier proof from [17] (see Section 5)

(iii) to prepare the study of dynamic and subclassical relevance (see Section 6)

2 Preliminaries
I will only consider the propositional fragment of classical logic (CL). I use small letters $a, b, c, \ldots$, $x, y, z, \ldots$ to denote formulas and capitals $A, B, \ldots, X, Y, \ldots$ to denote sets of formulas. The consequence relation associated with CL is denoted by $\vdash$. $\text{Cn}(A) = \{x | A \vdash x\}$. Slightly abusing notation, I write $A \vdash B$ whenever $A \vdash x$ for every $x \in B$. I write $A \vdash \neg B$ as an abbreviation for $(A \vdash B$ and $B \vdash A)$. I write $a \vdash b$ to abbreviate that $\{a\} \vdash b$.

I use the term ‘knowledge set’ to refer to both belief sets (or theories) and belief bases. A knowledge set $K$ can be any set of formulas. The set of sentential letters is $\mathcal{S} = \{p, q, r, \ldots\}$. The language $\mathcal{L}$ is obtained by adding the classical connectives $\bot, \neg, \lor, \land, \supset, \equiv$ to $\mathcal{S}$. Let $\mathcal{W}$ denote the set of all well-formed formulas of $\mathcal{L}$. I use $\mathcal{W}^d$ to refer to the set of literals: $\mathcal{W}^d = \mathcal{S} \cup \{\neg a | a \in \mathcal{S}\}$.

Let $E(a)$, $E(A)$ denote the set of sentential (or ‘elementary’) letters that occur in $a$, resp. $A$. I use $\bigvee A$ to denote the disjunction over the members of $A$, where in this notation, $A$ is always assumed to be finite and non-empty (if $A = \{a\}$, then $\bigvee A = a$).
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3 Belief revision and relevance: an overview

In this section, I explain the axiom and how normalized contraction operations deal with it. This section contains no new material but is necessary in order to understand the new results.

The axiom of relevance: belief revision has been a subject of intensive research since the middle of the 1980’s. I refer to [7] for a gentle introduction, and will henceforth assume familiarity with the basic concepts of the logic of belief revision.

A contraction function is a function \( f : (K, x) \rightarrow C(K, x) \), where \( K \) is a knowledge set, \( x \) is a formula, and \( C(K, x) \) is a set of formulas, also called the contraction set of \( K \) by \( x \). The goal of a contraction is to stay as close as possible to \( K \), but to make sure that \( x \) is not derivable from the contraction set. A revision function is a function \( g : (K, x) \rightarrow R(K, x) \), where \( K \) is a knowledge base, \( x \) is a formula, and \( R(K, x) \) is a set of formulas. The process of revision occurs when we learn some new fact \( x \) that contradicts \( K \), and adapt our knowledge base to this new fact.

The Levi identity states that for every belief set \( K \) and every formula \( x \): \( R(K, x) = Cn(C(K, \neg x) \cup \{ x \}) \): we first contract the belief set in view of \( x \), then we add \( x \) to the resulting contraction set, and finally we apply CL to obtain a revised belief set. For belief bases, Hansson restated the Levi identity as follows: \( R(K, x) = C(K, \neg x) \cup \{ x \} \).\(^1\) Given that in both cases, revision is defined in terms of contraction, I will focus on (the properties of) contraction operations in the current article.

In his [4] and [5], Gärdenfors formulates axioms that every operator for belief contraction should fulfill, widely known as the AGM axioms. As Rohit Parikh remarks in [11], these axioms are still too weak, in that they allow for the ‘trivial update’. For the case of contraction, this trivial update reads: whenever \( K \) implies \( x \), the contraction set of \( K \) by \( x \) equals \( \emptyset \); whenever \( K \) does not imply \( x \), then the contraction set of \( K \) by \( x \) equals \( K \). As Parikh notes, ‘this is unsatisfactory, because we would like to keep as much of the old information as possible [even when \( x \) follows from \( K \)]. Hence the AGM axioms need to be supplemented to rule out the trivial update’ [11, p. 3].

Parikh’s positive contribution consists of the formulation of an additional axiom, i.e. the axiom of relevance AR. This requires some notational preparation. A partition of a set \( \Delta \) is a family of non-empty, pairwise disjoint sets \( \Delta_i \) such that \( \bigcup_{i \in I} (\Delta_i) = \Delta \). In this notation, the sets \( \Delta_i \) are called the cells of \( \Delta \).

**Definition 1** ([9]: Definition 3.1)

Let \( E = \{E_i\}_{i \in I} \) be a partition of \( S \). We say that \( E \) is a splitting of \( K \) iff there is a family \( \{B_i\}_{i \in I} \) such that each \( E(B_i) \subseteq E_i \) and \( \bigcup_{i \in I} B_i \equiv K \).\(^2\)

**Example 1**

Let \( K_1 = \{(p \land q) \land r, \neg r \lor s, q \land t, r \lor u\} \). The following sets are splittings of \( K_1 \):

\[
\begin{align*}
E_1 &= \{S\} \\
E_2 &= \{(p, q, t), \{r, s\}\} \cup \{\{a\} | a \in S - (p, q, r, s, t)\} \\
E_3 &= \{(p, q, t), \{r\}\} \cup \{\{a\} | a \in S - (p, q, r, s, t)\}
\end{align*}
\]

Note that where \( K^0 \) is a randomly chosen least-letter set representation of \( K \), and \( a \in S \) is not a member of \( E(K^0) \), there is a splitting of \( K \) that contains the cell \{a\} as member.\(^3\) An example is the

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\(^1\) See e.g. [6, p. 19] where the Levi identity is adapted to the case of belief bases.

\(^2\) The original idea of the definition of a splitting originates in [11]. I use Makinson’s definition because it includes the case where \( K \) is infinite.

\(^3\) The least-letter set theorem states: for every set \( A \) of formulae, there is a unique least set of elementary letters \( \Delta \) such that there is a \( B: (i) E(B) = \Delta \) and \( B \equiv A \) (see e.g. [9, p. 378]). A least-letter set representation of \( K \) is a \( K' \) such that (i) \( K' \equiv K \) and (ii) every elementary letter that occurs in \( K' \) is a member of the least-letter set of \( K \).
sentential letter $u$, which does not occur in the least-letter set representation of $K_1$. To avoid clutter, I will not explicitly mention those $a \not\in E(K^0)$ in subsequent examples.

$E$ is at least as fine as $E'$ iff every cell of $E'$ is the union of cells of $E$. Note that if $E$ is a splitting of $K$, and $E$ is finer than the partition $E'$ of $S$, it immediately follows that $E'$ is also a splitting of $K$. We say that $E$ is the finest splitting of $K$ iff $E$ is a splitting of $K$ and there is no splitting $E' \neq E$ of $K$ such that $E'$ is at least as fine as $E$.

**Example 2**
Take $K_1$ from Example 1. Note that $E_2$ is finer then $E_1$, and $E_3$ is finer then $E_2$. Moreover, $E_3$ is the finest splitting of $K_1$.

**Theorem 1** (for the finite case; [8] for the general case)
Every set $K$ of formulas has a unique finest splitting.

Parikh then uses the finest splitting to define propositional relevance:

**Definition 2**
Let $E$ be the finest splitting of $K$. We say that a formula $a$ is irrelevant to the contraction (revision) of $K$ by $x$ iff for every cell $E_i \in E$: $E_i \cap E(a) = \emptyset$ or $E_i \cap E(x) = \emptyset$. Alternatively, $a$ is relevant to the contraction (revision) of $K$ by $x$ iff for some cell $E_i \in E$: $E_i \cap E(a) \neq \emptyset$ and $E_i \cap E(x) \neq \emptyset$.

**Axiom of Relevance (AR):** whenever $a \in K$ is irrelevant to the contraction (revision) of $K$ by $x$, then it should remain an element of the contraction (revision) set of $K$ by $x$.

**Example 3**
Consider the contraction of $K_1$ by $r$. If $AR$ is obeyed, then this implies that $p \lor q$, $p \lor t$ and $s$ are in the contraction set of $K_1$ by $r$.

Note that $AR$ only makes sense if we assume that $K$ is consistent. To see why, consider a $K$ such that $K \vdash_{CL} \bot$. Hence also $K \vdash_{CL} a$ for every $a \in W^d$, where $W^d \not\vdash_{CL} K$. It follows that the finest splitting of $K$ is $E = \{ \{a\} \mid a \in S \}$. In words: the finest splitting of $K$ is simply the set of all singletons $\{a\}$ where $a$ is an elementary letter.

This means that relevance modulo $K,x$ reduces to mere letter-sharing: $y$ is relevant to the contraction of $K$ by $x$ iff $E(y) \cap E(x) \neq \emptyset$. As a result, a contraction operation that obeys $AR$, would result in (a superset of) the set $\{ y \in W^d \mid E(y) \cap E(x) = \emptyset \}$. Such a contraction operation results in a set that is not $CL$-satisfiable.

It is easy to see that where a revision operation is defined by the application of the Levi identity to a contraction operation that obeys $AR$, this revision operation also obeys $AR$ -- see [8, p. 8] were this is explained to some detail.

**Partial meet contraction and relevance:** the best-known syntax-based contraction function is partial meet contraction, as defined in [1]. Let $K \perp x$ be the set of all maximal subsets $A$ of $K$ that do not imply $x$. For every $K,x$, let $\gamma$ be a function such that $\gamma(K \perp x)$ is a non-empty subset of $K \perp x$. $\gamma$ is called a selection function for $K$. Partial meet contraction is defined as follows:

**Definition 3**
$PMC(K,x) = \{ \gamma [K \perp x] \}$

The intuitive idea behind this definition is that, since we cannot choose between the different sets $A \in \gamma(K \perp x)$, we can only stick to the beliefs that are in each of these sets. A border case of partial meet contraction will henceforth be referred to as `(full) meet contraction' and used for illustrative purposes.
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Full meet contraction is defined by taking $\gamma(K \downarrow x) = K \downarrow x$ for all $K, x$. Henceforth, $MC(K, x)$ will be used to refer to the meet contraction of $K$ by $x$.

Although it was shown in [1] that partial meet contraction and revision obey all the AGM axioms, Parikh established the following fact:

**Fact 1**
Partial meet contraction (revision) does not obey $AR$.

**Example 4**
Let $K_2 = \{ p \land q \}$ and consider the meet contraction of $K_2$ by $p$. Obviously, $K_2 \downarrow p = \{ \emptyset \}$, where $MC(K_2, p) = \emptyset$. However, the finest splitting of $K_2$ is $E = \{ \{ p \}, \{ q \} \}$, where $q$ is irrelevant to the contraction of $K_2$ by $\neg p$.

This example illustrates that partial meet contraction on a belief base depends very heavily on the initial formulation of the belief base and therefore cannot obey $AR$. Partial meet contraction of belief sets does not fare much better in this respect: consider $Cn_{CL}(K_2)$. As this set contains not only $q$, but also $q \supset p$, neither of the two formulas will be in $MC(K_2, p)$.

**Normalized contraction**: in order to overcome the problem of relevance, Kourousias and Makinson propose that the contraction operation is performed on a so-called ‘normal form’ of $K$. The normal (or ‘canonical’) form of $K$ is defined using the finest splitting of $K$:

**Definition 4**
Where $E = \{ E_i \}_{i \in I}$ is the finest splitting of $K$, define the set of normal forms of $K$: $\mathbb{K}_N = \bigcup_{i \in I} \{ B_i \} \models K$ and for every $i \in I: E(B_i) \subseteq E_i$.

Kourousias and Makinson call the contraction of a $K_N \in \mathbb{K}_N$ a ‘normalized contraction operation’, and show that this solves the problem of relevance:

**Theorem 2** ([8], Theorem 4.1)
Where $K$ is consistent, partial meet contraction (revision) with respect to a $K \in \mathbb{K}_N$ obeys $AR$.

Not one but many normal forms

It should not be assumed that there is only one normal form of $K$, where ‘the’ normalized contraction of $K$ in view of $x$ is a unique operation. Many authors allow for some notational abuse, i.e. when they use the name ‘the finest splitting of $K$’ to refer to either a partition of the language or a set of formulas in normal form—see e.g. [8, 9, 15, 17]. A simple example shows that the definite article is not in place here.

Take $K_3 = \{ p \lor q, q \lor r, r \lor s \}$. The finest splitting of $K'$ is $E = \{ \{ p, q, r, s \} \}$. Hence both $K'_3 = K'$ and $K''_3 = \{ (p \lor q) \land (q \lor r) \land (r \lor s) \}$ are normal forms of $K_3$. This underdetermination of ‘the’ normal form of $K$ obviously carries over to the normalized contraction (and hence also revision) operations. Consider a meet contraction of the normal forms of $K_3$ by $p \lor q$. Contraction of $K'_3$ yields $\{ p \lor q, r \lor s \}$, whereas contraction of $K''_3$ yields $\emptyset$.

In summary, although every $K$ has a unique finest splitting, it may have several different normal forms, and the differences between them has a great impact on the contraction and revision sets obtained from them. So if we want to implement the result of Kourousias and Makinson (as stated in Theorem 2), we have to further specify which normalized contraction operation we are using.\(^4\)

\(^4\)Stolpe e.g. uses Theorem 2 when applying input–output logic to codes of laws, see [15].
The question then becomes: can we define a specific such set \( K_N \in K \) for any \( K \)? As I will now show, we can.

5 Prime implicates and relevance

Prime implicates: it is possible to define one specific set for each \( K \), of which we can prove that it is a normal form of \( K \). This set is known as the set of prime implicates of \( K \), or as the set of minimal disjunctions of literals that follow from \( K \).

**Definition 5**
For every belief base \( K \), define \( K_M = \{ \lor A \mid (i) A \subseteq W, (ii) \not\vdash \lor A, (iii) K \vdash \lor A \text{ and (iv) for no } A' \subset A : K \vdash \lor A' \}. \)

Note that tautologies are not allowed in \( K_M \). The underlying idea behind the definition of \( K_M \) can be stated as follows: \( K_M \) is the set that 'breaks down' the information in \( K \) as much as possible, i.e. into minimal disjunctions of literals. In contrast to the idea of a finest splitting, this is not done by a detour via the language.

**Theorem 3**
For every belief base \( K: K_M \in K_N \).

A proof of essentially this result was offered in [17]. However, it contains a flaw, which I analyse after giving my own proof in Section 5. The crucial motor behind the latter proof is the relation of path-relevance modulo a set \( A \), which is borrowed from [9]. This relation is defined as follows: \( x \sim_A y \) iff there are \( z_1, \ldots, z_n \in A \) such that \( E(x) \cap E(z_1) \neq \emptyset, E(z_1) \cap E(z_2) \neq \emptyset, \ldots, E(z_n) \cap E(y) \neq \emptyset \). Since \( \sim_K \) constitutes an equivalence relation on \( K_M \), it can be used to obtain a partition of \( K_M \) into \( \sim_K \)-connected subsets \( K_1, K_2, \ldots \). From this, we can obtain a partition of \( S: E_M = \{ E(K_1), E(K_2), \ldots \} \).

Finally, it is proven that \( E_M \) is the finest splitting of \( K \). This implies that \( K_M \) can be used to determine the finest splitting of \( K \).

The set \( K_M \) as defined above is used for a specific kind of model-based revision in [10]—the authors link their paper to Parikh’s and Makinson’s work on the notion of relevant belief change, but do not explicitly discuss the relation between \( K_M \) and \( K_N \). In [12], the same authors propose a solution to the problem of relevance in belief revision in terms of preferences over prime implicants, minimal conjunctions of literals that entail \( K \). The idea of defining revision of \( K \) in view of its prime implicate set was put forward in [3], where it is conjectured that this revision obeys relevance. However, so far no one seems to have made the distinction between \( K_M \) and other sets in \( K_N \).

By Theorem 2 and Theorem 3, meet contraction (revision) in view of \( K_M \) obeys AR whenever \( K \) is consistent. It also follows that there is an algorithm that can help us find the unique finest splitting of \( E \) by means of \( K_M \)—see [17].

**Proof of Theorem 3.** To obtain a set of letter sets from \( K_M \), I first define a relation \( \sim_A \) over the members of \( W \):

**Definition 6**
\( x \) is path-relevant to \( y \) modulo \( A \) (\( x \sim_A y \)) iff there are \( z_1, \ldots, z_n \in A \) such that \( E(x) \cap E(z_1) \neq \emptyset, E(z_1) \cap E(z_2) \neq \emptyset, \ldots, E(z_n) \cap E(y) \neq \emptyset \).
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Note that $x \sim_A y$ does not necessarily imply that $x$ and $y$ are members of $A$, only that there is a path from $x$ to $y$ through $A$. It will however also be convenient to rely on the following properties specific to $\sim_A$ defined only over the members of $A$:

**FACT 2**

$\sim_A$ is transitive, reflexive and symmetric with respect to all $x, y \in A$, where $\sim_A$ is an equivalence relation on the members of $A$.

**FACT 3**

If $x \sim_A y$, then $x \sim_{A \cup B} y$ for every $B$.

**Definition 7**

$\mathbb{K}_M$ is the quotient set of $K_M$ by $\sim_{K_M}$. Where $\mathbb{K}_M = \{K_1^M, K_2^M, \ldots\}$, $E_M = \{E(K_1^M), E(K_2^M), \ldots\} \cup \{\{a\} \mid a \in S - E(K_M)\}$.

Since $\sim_{K_M}$ is an equivalence relation on $K_M$, $\mathbb{K}_M$ is a partition of $K_M$. Note too that for no $K_i^M \in \mathbb{K}_M$, $K_i^M = \emptyset$, where also for no $E_i \in E_M$, $E_i = \emptyset$. It now remains to be proven that $E_M$ is the finest splitting of $K$.

I first prove that $E_M$ is a partition of $S$. This follows immediately from (1) the fact that every $E_i$ is non-empty, (2) the fact that $\bigcup E_M = S$ and the following lemma:

**Lemma 1**

For every $E_i, E_j \in E_M$: $E_i \neq E_j$ iff $E_i \cap E_j = \emptyset$.

**Proof.** Let $E_i, E_j \in E_M$. The right–left direction is obvious since no $E_i \in E_M$ is empty. For the left–right direction, suppose that for $E_i, E_j \in E_M$, $E_i \cap E_j \neq \emptyset$. I only consider the case where $E_i = E(K_i^M)$ and $E_j = E(K_j^M) = E(K_i^M)$ for $K_i^M, K_j^M \in \mathbb{K}_M$—in the other case, it follows immediately that $E_i \cap E_j = \emptyset$. Hence, suppose that $E(K_i^M) \cap E(K_j^M) \neq \emptyset$. This implies that there are $x \in K_i^M, y \in K_j^M: E(x) \cap E(y) \neq \emptyset$, where $x \sim_{K_M} y$, hence $x$ and $y$ are in the same equivalence class. As a result, $K_i^M = K_j^M$, where $E_i = E_j$.

To prove that $E_M$ is the finest splitting of $K$, I need two lemmas:

**Lemma 2** ([8], Theorem 1.1)

Let $A = \bigcup_{i \in I} \{A_i\}$ where the letter sets $E(A_i)$ are pairwise disjoint, and suppose $A \vdash x$. Then there are formulas $b_i$ such that each $E(b_i) \subseteq E(A_i) \cap E(x), A_i \vdash b_i$, and $\bigcup_{i \in I} \{b_i\} \vdash x$.

**Lemma 3**

If (1) $[a, b] \vdash c \vee d$, (2) $E(a) \subseteq E(c)$, (3) $E(b) \subseteq E(d)$, and (4) $E(c) \cap E(d) = \emptyset$, then $[a] \vdash c$ or $[b] \vdash d$.

**Proof.** Suppose (1)–(4) holds, but $[a] \not\vdash c$ and $[b] \not\vdash d$. In that case, $a \wedge \neg c$ and $b \wedge \neg d$ are both $\text{CL}$-satisfiable. In view of (2), (3) and (4), $E(a \wedge \neg c) \cap E(b \wedge \neg d) = \emptyset$, where $E(a \wedge \neg c) \wedge E(b \wedge \neg d)$ is $\text{CL}$-satisfiable. This implies that $[a, b] \not\vdash c \vee d$, which contradicts (1).

**Theorem 4**

$E_M$ is the finest splitting of $K$.

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5 $\sim_A$ is not reflexive in general: in order for it to hold that $x \sim_A y$, $x$ has to share at least one elementary letter with a formula $a \in A$. Also, $\sim_A$ is not transitive in general. Let $A = \{p \supset q, r \supset s\}$ and let $x = p, y = q \lor r, z = s$. Then $x \sim_A y$ and $y \sim_A z$, whereas $x \not\sim_A z$.

6 This is the set of all equivalence sets of $K_M$, given the equivalence relation $\sim_{K_M}$ on $K_M$.

7 I rely on the fact that if $x$ and $y$ are $\text{CL}$-satisfiable and share no elementary letters, then $x \wedge y$ is $\text{CL}$-satisfiable.
Proof. Suppose there is a splitting $E = \{E_i\}_{i \in I}$ of $K$, such that $E$ is finer than $E_M$. Hence for some $E_j \not\subset E_M$, there is an $i \in I$: $\emptyset \subset E_i \subset E_j$. This means that $E_j$ cannot be a singleton, where we can derive that $E_j = E(K^j_M)$ for some $K^j_M \in \mathbb{K}_M$. So we have:

$(\dagger)$ For a $K^j_M \in \mathbb{K}_M$, there is an $i \in I$: $\emptyset \subset E_i \subset E(K^j_M)$

Part 1. I prove that there is a $v \in K^j_M$, for which $E(v) \cap E_i \neq \emptyset$, $E(v) \not\subset E_i$.

Suppose that for every $x \in K^j_M$, $E(x) \not\subset E_i$. In that case, $E(K^j_M) \not\subset E_i$, which contradicts $(\dagger)$. Hence there is an $x \in K^j_M$: $E(x) \not\subset E_i$. Suppose that for every $y \in K^j_M$, $E(y) \cap E_i = \emptyset$. In that case, $E(K^j_M) \cap E_i = \emptyset$, which again contradicts $(\dagger)$. Hence there is a $y \in K^j_M$: $E(y) \cap E_i \neq \emptyset$.

Since $x, y \in K^j_M$, $x \not\sim y$. Hence there are $z_1, \ldots, z_n \in K^j_M$ such that $E(x) \cap E(z_1) \neq \emptyset$, $E(z_1) \cap E(z_2) \neq \emptyset$, $E(z_2) \cap E(y) \neq \emptyset$. If $E(x) \cap E_i = \emptyset$, take $v = x$. If $E(x) \cap E_i = \emptyset$, we can infer from the fact that $E(x) \cap E(z_1) \neq \emptyset$ and that $E(z_1)$ is non-empty: $E(z_1) \not\subset E_i$. We now start up an induction, relying on the same reasoning:

If $E(z_k) \cap E_i \neq \emptyset$, then take $v = z_k$.
If $E(z_k) \cap E_i = \emptyset$, then $E(z_{k+1}) \not\subset E_i$.

This means that sooner or later, and to the latest at $y$, we arrive at a $v \in K^j_M$, for which it holds that $E(v) \cap E_i \neq \emptyset$, $E(v) \not\subset E_i$.

Part 2. I derive a contradiction. Note that since $E$ is a splitting of $K$, $E' = \{E_i, \bigcup (E - E_i)\}$ is also a splitting of $K$. Since $K \vdash v$, by Lemma 2, there are two formulae $b_i$ and $b$ such that (1) $E(b_i) \subseteq E(v)$ and (2) $E(b) \subseteq (\bigcup (E - E_i) \cap E(v))$, $K \vdash b_i$, $K \vdash b$ and $[b_i, b] \vdash v$.

Say $v = a_i \lor a_a$, where $a_i$ and $a_a$ are disjunctions of literals such that $\emptyset \subset E(a_i) \subseteq E_i$, $\emptyset \subset E(a_a) \subseteq (\bigcup (E - E_i))$, hence also $E(a) \cap E(a_i) = \emptyset$. By (1) and (2), we obtain: (1) $E(b_i) \subseteq E(a_i)$ and (2) $E(b) \subseteq E(a_a)$. By Lemma 3, this implies that $E(a) \vdash [b_i] \lor [b_a]$. Since $K \vdash b_i$ and $K \vdash b_a$, also $K \vdash [b_i] \lor [b_a]$. This however implies that $v$ is not a minimal disjunction of literals that is $CL$-derivable from $K$, where $v \not\in K_M$. \qed

Further comments on the proof: note that every set $K$ is associated with a unique set $K_M$, and from this $K_M$, we can uniquely obtain the set $E_M$. Hence Theorem 4 implies that every set $K$ has a unique finest splitting. Just as in [8], I needed the parallel interpolation theorem to arrive at this result.

In [17], Wu and Zhang also attempted to prove that the finest splitting of $K$ can be obtained from $K_M$—their set $C(K) \setminus K^g$ is almost identical to what I defined as $K_M$. The way they obtain the finest splitting from $K_M$ is also highly similar to the way I did it here: they define an equivalence relation $R^*$ on $K_M$, which is equivalent to my $\sim_{K_M}$, and obtain a set of $R^*$-connected subsets $K^1_M, K^2_M, \ldots$ of $K_M$. Finally, they claim that the set $E = \{E(K^1_M), E(K^2_M), \ldots\}$ is the finest splitting of $S$.

However, Wu and Zhang’s actual proof for this claim is mistaken. To see why, let me recapitulate their notion of indivisibility:

Definition 8

$A$ is indivisible iff for every partition $E = \{E_1, E_2\}$ of $E(A)$, there is an $a \in A$ such that: $E(a) \cap E_1 \neq \emptyset$ and $E(a) \cap E_2 \neq \emptyset$.

\[\text{The only difference is that tautologies are allowed for in } C(K) \setminus K^g, \text{ whereas I omitted them. This restriction is necessary in order to obtain Theorem 5 below, but it does not make a difference with respect to the argument here or the proof in the preceding section.} \]
Wu and Zhang seem to implicitly assume that the following holds:

\((\dagger)\) where \(A = \{A_i\}_{i \in I} \vdash_{\sim} K\): every cell of \(A\) is indivisible iff \(A\) is a canonical form of \(K\)

The authors prove that every \(\sim_{KM}\)-connected subset of \(KM\) is indivisible. By the left–right direction of \((\dagger)\), they infer that \(KM\) is ‘the finest splitting set of \(K\) [my emphasis]’. In view of Section 4 from the current article, this claim is slightly confusing. So I take it that they actually mean that \(KM\) is one of the normal forms of \(K\), and that \(E_M = \{E(K^1_M), E(K^2_M), ...\}\) is the finest splitting of \(K\). The crucial problem concerns the left–right direction of \((\dagger)\).

Let \(K_4 = \{p, q\}\). Note that every cell of \(A = \{\{p, q, p \supset q\}\}\) is indivisible. The finest splitting of \(K_4\) is obviously \(E = \{\{p\}, \{q\}\}\). Hence \(E_A = \{\{p, q\}\}\) is not the finest splitting of \(K_4\). More generally, it can be proven that for any \(K\), \(Cn(K)\) is indivisible, whereas \(Cn(K)\) is obviously not a canonical form of \(K\). As a result, the left–right direction of \((\dagger)\) fails. Put differently, the property of indivisibility has little bearing on the question whether a set of formulas \(A\) is a canonical form of \(K\).

My proof avoids this problem: it establishes that the language splitting, obtained from the sets \(K^1_M, K^2_M, ...\), is the finest splitting of \(K\). Of course, the proof relies on properties of \(KM\) as a set of formulas, but it does not use any intermediary property of sets of formulas in order to prove.

*Two derived results:* relying on Theorem 3, there is a way to define relevance without reference to the notion of a finest splitting. In [9], Makinson proves that \(x\) is path-relevant to \(y\) modulo a set \(K_N \in \mathbb{F}_N\) iff \(x\) is cell-relevant to \(y\) modulo the finest splitting of \(K\). In the current terminology, this means that \(x \sim_{KM} y\) iff \(y\) is relevant to the contraction of \(K\) by \(x\) (according to Definition 2). So we can define relevance in a way that it is independent from the finest splitting of \(K\):

**Corollary 1**

\(y\) is relevant to the contraction of \(K\) by \(x\) iff there are \(z_1, ..., z_n \in KM\) such that \(E(x) \cap E(z_1) \neq \emptyset, E(z_1) \cap E(z_2) \neq \emptyset, ..., E(z_n) \cap E(y) \neq \emptyset\).

As a result, we can define the set of formulas that are relevant to the contraction of \(K\) by \(x\) recursively.

A different way to determine relevance relies on the following property, which is proven in Section 5 below:

**Theorem 5**

If \(x \sim_{KM} y\), then \(x \sim_K y\).

The proof in Section 5 roughly goes as follows: for every \(z_i \in KM\) in the path from \(x\) to \(y\), a set \(A_i \subseteq K\) is picked such that \(A_i \vdash z_i\) and there is no subset \(B_i\) of \(A_i\) for which \(B_i \vdash z_i\). It is then proven that we can obtain a path through members of the sets \(A_i\), which connects \(x\) and \(y\).

Theorem 5 implies that if we can prove that \(x\) is not path-relevant to \(y\) modulo \(K\), then \(x\) is not path-relevant to \(y\) modulo \(KM\), where \(y\) is irrelevant to the contraction of \(K\) by \(x\). This means that in some cases, it suffices to prove a rather straightforward form of path-irrelevance—path-irrelevance modulo \(K\)—to argue that a belief must and can be upheld (in view of \(AR\)). If this does not work, one can then still turn to a more complex argument as to why a belief \(y\) is irrelevant to the contraction of \(K\) by \(x\).

**Proof of Theorem 5.**

**Definition 9**

\(A\) is \(\sim\)-connected iff for every \(x, y \in A\), \(x \sim_A y\).
Prime Implicates and Relevant Belief Revision

LEMMA 4
If $A$ and $B$ are \sim-connected and $E(A) \cap E(B) \neq \emptyset$, then $A \cup B$ is \sim-connected.

PROOF. Suppose $(1)$ $A$ and $B$ are \sim-connected and $(2)$ $E(A) \cap E(B) \neq \emptyset$. By Fact 3, for every $a_1, a_2 \in A$, $a_1 \sim_{A\cup B} a_2$ and for every $b_1, b_2 \in B$, $b_1 \sim_{A\cup B} b_2$. Hence it suffices to prove that for every $a \in A$ and every $b \in B$, $a \sim_{A\cup B} b$.

By $(2)$, we can infer that there is an $x \in A$ and an $y \in B$ such that $E(x) \cap E(y) \neq \emptyset$, where also $(\dagger)$ $x \sim_{A\cup B} y$. Consider an $a \in A$. By $(1)$, $a \sim_A x$, where by Fact 3, $a \sim_{A\cup B} x$. By the same reasoning, we can infer that for every $b \in B$, $b \sim_{A\cup B} y$. By the transitivity and symmetry of $\sim_{A\cup B}$, we can infer that $a \sim_{A\cup B} b$ for every $a \in A, b \in B$.

THEOREM 6
If $x \sim_{K_M} y$, then $x \sim_K y$.

PROOF. Suppose $x \sim_{K_M} y$. Then there are $z_1, \ldots, z_n \in K_M$ such that $E(x) \cap E(z_1) \neq \emptyset, E(z_1) \cap E(z_2) \neq \emptyset, \ldots,$ and $E(z_n) \cap E(y) \neq \emptyset$. Note that for every $i \in \{1, \ldots, n\}$, $(\dagger)$ there is a finite $A_i \subseteq K$ such that $A_i \vdash z_i$ and for all $B \subseteq A_i, B \not\subseteq z_i$.

Suppose that for some $i \in \{1, \ldots, n\}$, $E(z_i) \not\subseteq E(A_i)$. By Craig’s interpolation theorem, there is a $b_i$ such that $A_i \vdash b_i \vdash z_i$, and $E(b_i) \subseteq E(A_i) \cap E(z_i)$. Note that since $z_i \in K_M, \not\vdash z_i$, where also $b_i \not\vdash b_i$. However, this implies that $\emptyset \not\subseteq E(b_i) \subseteq E(z_i)$, where $z_i \not\in K_M$—a contradiction. So we obtain that $(\dagger)$ for all $i \in \{1, \ldots, n\}, E(z_i) \subseteq E(A_i)$.

Suppose that for some $i \in \{1, \ldots, n\}, A_i$ is not \sim-connected. Hence we can divide $A_i$ into two non-empty finite subsets $A_i^1, A_i^2$ such that $E(A_i^1) \cap E(A_i^2) = \emptyset$. Let $z_i = z_i^1 \lor z_i^2$, such that $E(z_i^1) \subseteq E(A_i^1)$ and $E(z_i^2) \subseteq E(A_i^2)$. By Lemma 3, it follows that $\land (A_i^1) \vdash z_i^1$ or $\land (A_i^2) \vdash z_i^2$. Hence $A_i^1 \vdash z_i$ or $A_i^2 \vdash z_i$, which contradicts $(\dagger)$.

Hence for every $i \in \{1, \ldots, n\}, A_i$ is \sim-connected. Also, for every $i \in \{1, \ldots, n-1\}, E(A_i) \cap E(A_{i+1}) \neq \emptyset$ in view of $(\dagger)$. By Lemma 4, we can derive that $A_1 \cup \ldots \cup A_n$ is \sim-connected. By $(\dagger)$, we have that $E(x) \cap E(A_1) \neq \emptyset$, where there is an $a_1 \in A_1$ such that $E(x) \cap E(a_1) \neq \emptyset$. By the same reasoning, there is a $a_n \in A_n$ such that $E(a_n) \cap E(y) \neq \emptyset$. Since $A_1 \cup \ldots \cup A_n$ is \sim-connected, we have that $a_1 \sim_K a_n$. As a result, also $x \sim_K y$.

6 Further research

In the preceding, it was shown that partial meet contraction (revision) of the prime implicate set of a set of beliefs $K$ is well-defined and that it obeys AR. This solves the problem of the underdetermination of the normal forms. Furthermore, it was explained how $K_M$ may help us to compute relevance without first computing the finest splitting of $K$. Finally, it was shown that path-relevance to $x$ modulo $K$ may already give some indications as to which beliefs in $K$ are relevant to the contraction of $K$ by $x$.

Whereas these results have been formulated in the context of classical propositional logic, inspection of the proofs makes it clear that they all carry over to the classical first-order context.

Let me briefly discuss three questions for future research: (1) is it possible to give a more dynamic account of (ir)relevance? (2) can we use subclassical finest splittings to approximate the finest splitting of $K$? and (3) is it possible to use a notion of subclassical relevance, in order to deal with inconsistent sets of beliefs in a more natural way?

9In the context of classical first-order logic, ‘elementary letters’ should be understood as elementary predicate and function symbols.
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(1) It is usually not immediately clear whether \(x\) is relevant to \(y\) modulo \(K\). If we move to the predicative level, there is no positive test for \(K \not\vdash a\) in general. But even for a finite \(K\) on the propositional level, it often requires a complex reasoning process to see whether \(x\) is (ir)relevant to \(y\) modulo \(K\). For instance, it might take a whole bunch of inferences to realize that \(K \vdash p\).

Before this is realized, one might think that \(p\) is relevant to \(q\) (since until that time, one only knows that \(K \vdash p \lor q\)). On the other hand, it might take a while before we can infer that a disjunction \(p \lor r\) is relevant to a disjunction \(q \lor s\) (e.g. because we first have to find out that \(K \vdash \{p \lor r_1, r_1 \lor r_2, \ldots, r_{n-1} \lor r_n, r_n \lor s\}\)). In both cases, previous revisions might have to be reconsidered in view of the newly obtained insights with regards to relevance.

Future work should therefore look for means to formally model this dynamic reasoning process itself. The formal framework I have in mind here, is that of adaptive logics. The proof theory of adaptive logics is such that it captures dynamic reasoning forms, in which certain conclusions may be retracted in view of new insights in the premises. See [2] for an introduction to adaptive logics and their meta-theory.

(2) As we have seen above, path-relevance modulo \(K\) can give some insights already as to which formulas are relevant to \(C(K, x)\). This implies that we can generate a splitting of \(K\) by means of \(K\) itself, which gives us some indications to what the finest splitting of \(K\) will be. It might be possible to take one step further, and define a continuum of finest \(L\)-splittings, where \(L\) is any subclassical logic.\(^{10}\)

Note that the logic \(L_0\) for which \(Cn_{L_0}(K) = K\) for all \(K\) is such a logic. Other, perhaps more interesting subclassical logics could be all sorts of paraconsistent and paracomplete logics.\(^{11}\) Yet another fragment \(L_1\) of \(CL\) could be defined as follows: \(A \vdash_{L_1} x\) iff there is some \(a \in A\) such that \(a \vdash x\) – note that such an operation would drastically reduce the number of calculations.

Although this requires further study, the working hypotheses is fairly straightforward: the stronger a logic, the more it enables one to split the language of a theory. Hence where \(L\) is stronger than \(L'\), the finest \(L\)-splitting of \(K\) will always be at least as fine as the finest \(L'\)-splitting of \(K\). If this can be established, it means that there is also a continuum of shortcuts to prove that a formula is irrelevant to \(C(K, x)\).

(3) As we saw in Section 3, Parikh’s axiom leads to triviality whenever the set of beliefs \(K\) is inconsistent. However, if we replace the underlying logic in the definition of relevance by a paraconsistent one, we may still be able to separate irrelevant from relevant information in \(K\)—to some extent—without trivializing inconsistent beliefs. This however relies on an assumption that is yet to be proven, i.e. that there are paraconsistent logics \(L\) that determine a finest splitting for every set \(K\).

Item (1) is studied in [14], both items (2) and (3) are studied in [13] (both papers are in preparation). The current article thus provides the necessary background from which several interesting lines of research depart.

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\(^{10}\)Kourousias and Makinson mention this as an interesting prospect for further research too, see [8, p. 9].

\(^{11}\)See [16] for a systematic overview of these.
References


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