# Subclassical Relevance: Broadening the Scope of Parikh's Concept*

Frederik Van De Putte
Centre for Logic and Philosophy of Science
Ghent University
Blandijnberg 2, 9000 Gent, Belgium
`frvdeput.vandeputte@UGent.be`

August 31, 2011

**Abstract**

In this paper, Parikh's finest splitting result [17] and Kourousias and Makinson's theorem of parallel interpolation [14] are generalized to 10 subclassical logics, including intuitionistic logic and the maximal paraconsistent logic **CLuNs** from [5]. It is shown how this allows us to define subclassical relevance, and more specifically, a sensible axiom of relevance for inconsistent theories and belief bases. This axiom is compared to the concept of local change in [13] and [10]. Numerous technical results from [14] and [19] are succesfully generalized from classical logic to the subclassical logics under consideration. In addition, an elegant proof is provided for a generalized version of the least letter-set theorem from [15].

## 1 Aim of this Paper

Since the publication of [17], Parikh's definition of the finest splitting of a set of beliefs and the related axiom of relevance have received quite some attention in the literature on belief revision – see e.g. [14, 16, 23, 26, 19]. Although this axiom has not yet the same status as the AGM postulates for belief revision, many authors find it useful to prove that the revision operations they define obey this additional axiom – see e.g. [6, 18, 7, 27, 20].

Nevertheless, a lot of issues still remain open for debate in this (relatively young) area. One of them is the question whether, and if so, how the relevance axiom may be applied to inconsistent beliefs. The current paper adresses this question. Its main results are: (i) as it stands, Parikh's relevance axiom trivializes inconsistent belief bases; (ii) however, if we weaken our standard of deduction to a paraconsistent logic, we obtain a new, fairly strong yet non-trivializing axiom of relevance; (iii) more generally, we may replace the standard of deduction by various subclassical logics, and extend many important results from the literature to the resulting axioms of relevance. Besides these basic results, the

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*I am greatly indebted to Peter Verdée and David Makinson for their helpful comments and suggestions.

paper briefly discusses the relation between (paraconsistent) relevance and the notion of local change from [10, 13].

**Outline of this paper.** I will recapitulate the definition of the finest splitting and the axiom of relevance in Section 3.1. Intuitively, the relevance axiom states that whenever a proposition is in the set of initial beliefs, and the new information you receive is not "relevant" – in a very well-specified sense, see Section 3.1 – to this proposition, then you should hold on to this proposition. However, since relevance is a function of the classical logic consequences of the set of initial beliefs, the relevance axiom trivializes (mutually) inconsistent beliefs. This problem will be explained in Section 3.2. The solution, as outlined in Section 3.3, is to replace classical logic as the standard of deduction by a paraconsistent logic.

I will argue that this solves the problem of inconsistent relevance, using the well-known paraconsistent logic **CLuNs** [5, 4, 3, 25] to illustrate this point (Sections 4.1 and 4.2). In Section 4.3, the resulting axiom of relevance is compared to Hansson & Wasserman's notion of Local Change [13] and to Fuhrmann's requirement that not every inconsistency is removed whenever one performs a revision or contraction [10, 21].

The remainder of the paper is more of a technical nature. In Section 5, nine other subclassical logics are defined, including intuitionistic logic. In the subsequent section, it is shown that many results from [14], [19] and [15] can be easily generalized to each of these systems and **CLuNs**. First and foremost, where **L** is one of these logics, every set of formulas has a finest **L**-splitting. As a result, it is possible to define an axiom of **L**-relevance for each of these logics **L** (Section 6.1). Second, the notion of a canonical form is generalized to that of an **L**-canonical form, and it is proven that a specific partial meet contraction of any **L**-canonical form of a set of formulas obeys the axiom of **L**-relevance (Section 6.2). Third, for all the logics I consider and for every set $\Gamma$, one can determine a unique set $Min_{\mathbf{L}}(\Gamma)$, such that $Min_{\mathbf{L}}(\Gamma)$ is a **L**-canonical form of $\Gamma$ (Section 6.3). Finally, the least letter-set theorem from [15] is generalized to the logics **L**, and an elegant proof is presented which relies on the definition of $Min_{\mathbf{L}}(\Gamma)$ (Section 6.4).

## 2 Preliminaries

All results from this paper are situated at the propositional level. The language $\mathcal{L}$ is built up from the set of elementary letters $\mathcal{E} = \{p, q, r, \ldots, p_1, \ldots\}$ and the connectives $\sim, \wedge, \vee, \supset, \equiv$. Using the regular formation rules of classical logic (henceforth **CL**), we obtain the set of formulas $\mathcal{W}$. Let $\mathcal{W}^l = \mathcal{E} \cup \{\sim A \mid A \in \mathcal{E}\}$. $\mathcal{L}^{\perp}$ is obtained by adding $\perp$ to $\mathcal{L}$; I use $\mathcal{W}^{\perp}$ for the associated set of formulas. As usual, $\top =_{\mathsf{def}} \perp \supset \perp$.

Unless specified differently, I will use $A, B, C, \ldots$ as metavariables for formulas in $\mathcal{W}^{\perp}$, $\Gamma, \Delta, \Theta, \ldots$ as metavariables for subsets of $\mathcal{W}^{\perp}$ and $\mathbb{A}, \mathbb{B}, \mathbb{C}, \ldots$ as metavariables for sets of subsets of $\mathcal{W}^{\perp}$. Where $\mathbb{N}$ is the set of naturals numbers, I will use $i, j, k, \ldots$ as metavariables for members of $\mathbb{N}$, and $I, J, K, \ldots$ as metavariables for initial subsequences of $\mathbb{N}$. $E(A)$, $E(\Delta)$ are used to denote the set of elementary letters that occur in $A$, resp. $\Delta$.

I use $\vdash_{\mathbf{L}} A$ to denote that $A$ is a **L**-theorem; let $\Gamma \vdash_{\mathbf{L}} A$ iff there are $B_1, \ldots, B_n \in \Gamma$ such that $\vdash_{\mathbf{L}} (B_1 \wedge \ldots \wedge B_n) \supset A$. Let $Cn_{\mathbf{L}}(\Gamma) = \{A \mid \Gamma \vdash_{\mathbf{L}} A\}$. Slightly abusing notation, I write $\Gamma \vdash_{\mathbf{L}} \Delta$ whenever $\Gamma \vdash_{\mathbf{L}} A$ for every $A \in \Delta$. I write $\Gamma \dashv\vdash_{\mathbf{L}} \Delta$ as an abbreviation for ($\Gamma \vdash_{\mathbf{L}} \Delta$ and $\Delta \vdash_{\mathbf{L}} \Gamma$), and $A \vdash_{\mathbf{L}} B$ for $\{A\} \vdash_{\mathbf{L}} B$. A logic **L** will be called *paraconsistent* iff there are $A, B \in \mathcal{W}^{\perp}$ such that $A \wedge {\sim}A \nvdash_{\mathbf{L}} B$. **L** is *fully paraconsistent* iff there are no $A \in \mathcal{W}$ such that $A \wedge {\sim}A \vdash_{\mathbf{L}} B$ for every $B \in \mathcal{W}$. A monotonic logic **L** is *maximally paraconsistent* iff adding a **CL**-axiom $A$ for which $\nvdash_{\mathbf{L}} A$ to **L** yields full **CL**.

As customary, a distinct set of metavariables $\Upsilon, \Upsilon_1, \ldots, \Upsilon', \ldots$ is used to refer to sets of beliefs. In this notation, $\Upsilon$ may be closed under a logic **L** or not. Where $\Upsilon$ is closed under a logic **L**, we say that it is a **L**-*theory*; otherwise, $\Upsilon$ is called a *base*.

# 3 Relevance and Inconsistency

## 3.1 Parikh's Relevance Axiom

Belief revision became a subject of intensive research since the middle of the 1980s. I refer to [12] for a more gentle introduction, and will only mention some of the basic concepts here. The most common starting point for the logic of belief revision is the following question: given a set of initial beliefs $\Upsilon$, and some piece of new information $A$ that possibly contradicts $\Upsilon$, how are we to revise $\Upsilon$ such that $A$ can be incorporated? This is typically done by defining a revision operation $\oplus$, which is a function that maps every couple $\langle \Upsilon, A\rangle$ to a set of formulas $\Upsilon \oplus A$, called the revision set of $\Upsilon$ by $A$.

In the standard approach, as initiated by Alchourron, Gärdenfors and Makinson, belief revision is reformulated as a combination of belief *contraction* and belief *expansion*. To contract $\Upsilon$ by $B$ means to select a $\Upsilon' \subseteq \Upsilon$ (or a $\Upsilon' \subseteq Cn_{\mathbf{CL}}(\Upsilon)$) which maximally approximates $\Upsilon$, but such that $\Upsilon' \nvdash_{\mathbf{CL}} B$. To expand $\Upsilon$ by $C$ simply means to add $C$ to $\Upsilon$ – for theory-based contraction, the resulting set is closed under **CL**. The revision of $\Upsilon$ by $A$ is then reformulated as follows: we first contract $\Upsilon$ by ${\sim}A$ – this gives us the contraction set $\Upsilon \ominus {\sim}A$ – and next we expand the latter set by $A$.

In [1], a number of rationality postulates for belief revision and contraction of **CL**-theories are presented. As Parikh remarks in [17], these postulates are still too weak, in that they allow for the "trivial contraction operation" (henceforth $\ominus_{\mathbf{T}}$). This operation is defined as follows: if $\Upsilon \vdash_{\mathbf{CL}} A$, then $\Upsilon \ominus_{\mathbf{T}} A = Cn_{\mathbf{CL}}(\emptyset)$; otherwise, $\Upsilon \ominus_{\mathbf{T}} A = Cn_{\mathbf{CL}}(\Upsilon)$ . Likewise, a trivial revision operation – or in Parikhs terms, "the trivial update" – gives extremely poor consequences in all interesting applications, yet still obeys the AGM postulates for rational belief revision. As Parikh notes, 'this is unsatisfactory, because we would like to keep as much of the old information as possible [even when it contradicts the new information]. Hence the above list [= the list of postulates] needs to be supplemented to rule out the trivial update' [17, p. 3].

As Kourousias and Makinson explain in [14], this problem is easily transposable to belief bases, i.e. (in their framework) belief sets that are not closed under **CL**. For example, when revising the base $\Upsilon_1 = \{p \wedge q\}$ by ${\sim}p$, there are "rational" (in the sense of [11]) belief operators that yield ${\sim}p$ as the only resulting belief, hence removing both the implicit beliefs $p$ and $q$.

Parikh's positive contribution consists in the formulation of an additional postulate, i.e. the axiom of relevance **P**. To spell out this axiom, he defines the finest splitting of a set of formulas. This requires some notational preparation. A *partition* $\mathbb{A} = \{\Lambda_i\}_{i\in I}$ of a set $\Theta$ is a set of non-empty, pairwise disjoint sets such that $\bigcup_{i\in I}\{\Lambda_i\} = \Theta$. In this notation, the sets $\Lambda_i$ are called the *cells* of $\mathbb{A}$.

**Definition 1 ([16]: Def. 3.1)** *Let $\mathbb{E} = \{\Lambda_i\}_{i\in I}$ be a partition of $\mathcal{E}$. We say that $\mathbb{E}$ is a* splitting *of $\Gamma$ iff there is a $\Delta = \bigcup_{i\in I}\Delta_i$ such that each $E(\Delta_i) \subseteq \Lambda_i$ and $\Delta \dashv\vdash_{\mathbf{CL}} \Gamma$.*[1]

**Example 1** *Let $\Upsilon_2 = \{(p \vee q) \wedge r, {\sim}r \vee s, q \vee t, r \vee u\}$. Note that this set is* **CL***-equivalent to $\Upsilon_2' = \{p \vee q, q \vee t, r, s\}$. From the latter, we may generate the following splittings of $\Upsilon_2$:*

$\mathbb{E}_1(\Upsilon_2) = \{\mathcal{E}\}$
$\mathbb{E}_2(\Upsilon_2) = \{\{p,q,t\},\{r,s\}\} \cup \{\{A\} \mid A \in \mathcal{E} - \{p,q,r,s,t\}\}$
$\mathbb{E}_3(\Upsilon_2) = \{\{p,q,t\},\{r\},\{s\}\} \cup \{\{A\} \mid A \in \mathcal{E} - \{p,q,r,s,t\}\}$

$\mathbb{E}$ is *at least as fine as* $\mathbb{E}'$ iff every cell of $\mathbb{E}'$ is the union of cells of $\mathbb{E}$; $\mathbb{E}$ is finer than $\mathbb{E}$ iff it $\mathbb{E}$ is at least as fine as $\mathbb{E}'$ but the converse fails. Note that if $\mathbb{E}$ is a splitting of $\Gamma$, and $\mathbb{E}$ is finer than the partition $\mathbb{E}'$ of $\mathcal{E}$, it immediately follows that $\mathbb{E}'$ is also a splitting of $\Gamma$ (see [17, pp. 4-5]). We say that $\mathbb{E}$ is a *finest splitting* of $\Gamma$ iff there is no splitting $\mathbb{E}'$ of $\Gamma$ that is finer than $\mathbb{E}$.

**Example 2** *Take $\Upsilon_2$ from Example 1. Note that $\mathbb{E}_2(\Upsilon_2)$ is finer then $\mathbb{E}_1(\Upsilon_2)$, and $\mathbb{E}_3(\Upsilon_2)$ is finer then $\mathbb{E}_2(\Upsilon_2)$. Provably, $\mathbb{E}_3(\Upsilon_2)$ is a finest splitting of $\Upsilon_2$.*

Note that if $\Upsilon \dashv\vdash_{\mathbf{CL}} \Upsilon'$, and $A \in \mathcal{E} - E(\Upsilon')$, then $\{A\}$ is a cell of a splitting of $\Upsilon$ – see Example 1: $\{u\}$ is a cell in $\mathbb{E}_3(\Upsilon_2)$. To avoid clutter, I will henceforth only mention the letters that are non-redundant in $\Upsilon$ when I represent splittings of $\Upsilon$. E.g. $\mathbb{E}_3(\Upsilon_2)$ will be represented as $\{\{p,q,t\},\{r\},\{s\}\}$.

**Theorem 1 ([14]: Th. 2.4)** *Every $\Gamma \subseteq \mathcal{W}^{\perp}$ has a unique finest splitting.*

Parikh then uses the finest splitting to define his notion of relevance in the context of belief revision.

**Definition 2** *Let $\mathbb{E}$ be the finest splitting of $\Upsilon$. We say that a formula $B$ is irrelevant to the revision (contraction) of $\Upsilon$ by $A$ iff for every cell $\Lambda_i \in \mathbb{E}$: $\Lambda_i \cap E(A) = \emptyset$ or $\Lambda_i \cap E(B) = \emptyset$.*

Note that relevance modulo a revision of $\Upsilon$ by $A$ is equivalent to relevance modulo a contraction of $\Upsilon$ by $A$ – both are a function of $E(A)$ and the finest splitting of $\Upsilon$. Also, if $\Upsilon$ and $\Upsilon'$ are **CL**-equivalent, then the finest splitting of $\Upsilon$ equals the finest splitting of $\Upsilon'$, whence relevance modulo a revision (contraction) of $\Upsilon$ by $A$ will be equivalent to relevance modulo a revision (contraction) of $\Upsilon'$ by $A$. Hence relevance is independent of the way we represent $\Upsilon$.

The concept of relevance finally allows us to state Parikh's axiom of relevance. His original formulation of this axiom is the following:

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[1] The idea of a splitting originates in [17]. I use Makinson's definition because it includes the case where $\Gamma$ is infinite.

$\mathbf{P}_{\mathsf{or}}$ *Relevance:* If $B \in \Upsilon$ is irrelevant to the revision (contraction) of $\Upsilon$ by $A$, then $B \in \Upsilon \oplus A$ ($B \in \Upsilon \ominus A$)

However, Parikh only intends to apply this axiom to **CL**-theories – for bases, it would not solve the above problem. Consider again the example $\Upsilon_1$: the formula $p \wedge q \in \Upsilon$ is *relevant* to the revision of $\Upsilon$ by $\sim\!p$. Hence if we take $\mathbf{P}_{\mathsf{or}}$ literally, there is no problem in dropping $p \wedge q$, which is the only belief in $\Upsilon$. In order to deal with both belief bases and **CL**-theories, we may restate the axiom as follows:[2]

**P** *Relevance:* If $B \in Cn_{\mathbf{CL}}(\Upsilon)$ is irrelevant to the revision (contraction) of $\Upsilon$ by $A$, then $B \in Cn_{\mathbf{CL}}(\Upsilon \oplus A)$ ($B \in Cn_{\mathbf{CL}}(\Upsilon \ominus A)$)

**Example 3** *Consider the contraction of $\Upsilon_2$ by $r$. If* **P** *is obeyed, then this implies that $p \vee q$, $p \vee t$ and $s$ are in the contraction set of $\Upsilon_2$ by $r$.*

## 3.2 The Problem with Inconsistent Beliefs

As pointed out by several authors, inconsistent belief bases are a fact of life – see e.g. [13, 10, 21, 8]. Especially when large databases are constructed, it becomes very hard to avoid inconsistencies altogether. Likewise, it is commonly acknowledged that even our most reliable scientific theories can turn out to be inconsistent. It is therefore reasonable to try to adapt achievements in the field of belief revision to a paraconsistent setting, i.e. a setting in which the presence of inconsistencies is taken seriously, and does not lead to absurd outcomes. Examples of this paraconsistent turn can be found in [10, 21, 8], where results from the AGM approach are generalized to approaches based on paraconsistent, relevantist and inconsistency-adaptive logics respectively.

In a similar vein, one may ask whether the idea of relevant belief change can be reasonably applied to inconsistent belief bases or belief sets. However, if we take the relevance axiom literally, the outcome seems fairly negative. Let me briefly explain why this is the case, for both Parikhs original axiom $\mathbf{P}_{\mathsf{or}}$ and my alternative formulation **P**.

Consider a $\Upsilon$ such that $\Upsilon = Cn_{\mathbf{CL}}(\Upsilon)$ and $\Upsilon \vdash_{\mathbf{CL}} \bot$. Hence also $A \in \Upsilon$ for every $A \in \mathcal{W}^l$, whence $\mathcal{W}^l \dashv\vdash_{\mathbf{CL}} \Upsilon$. It follows that the finest splitting of $\Upsilon$ is $\mathbf{E} = \{\{A\} \mid A \in \mathcal{E}\}$. This means that relevance modulo $\Upsilon \oplus A$ reduces to mere letter-sharing: $B \in \Upsilon$ is relevant to $\Upsilon \oplus A$ iff $E(B) \cap E(A) \neq \emptyset$. As a result, a revision operation $\Upsilon \oplus A$ that obeys $\mathbf{P}_{\mathsf{or}}$ would result in (a superset of) the set $\{B \in \mathcal{W}^\perp \mid E(B) \cap E(A) = \emptyset\}$. Hence, such a revision operation would result in something close to plain triviality.[3]

So how about the alternative **P**? Suppose again that $\Upsilon \vdash_{\mathbf{CL}} \bot$ — this time, we need not assume that $\Upsilon$ is a **CL**-theory. By the same reasoning as in the previous paragraph, $\mathcal{W}^l \dashv\vdash_{\mathbf{CL}} \Upsilon$ and the finest splitting of $\Upsilon$ is $\mathbf{E} = \{\{A\} \mid A \in \mathcal{E}\}$. If **P** is obeyed, this means that for every $B \in \mathcal{W}^l$, $B$ has to be in the **CL**-consequence set of $\Upsilon \oplus A$. Hence it is required that $\Upsilon \oplus A$ is inconsistent; but more importantly, it suffices to take *any* inconsistent set $\Upsilon'$, in order to

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[2] Whenever $\Upsilon$, $\Upsilon \oplus A$ and $\Upsilon \ominus A$ are closed under **CL**, as in the traditional AGM-approach, this formulation reduces to Parikh's original axiom.

[3] By "plain triviality" I mean that every $B \in \mathcal{W}^\perp$ is an element of $\Upsilon \oplus A$, resp. $\Upsilon \ominus A$. The triviality that **P** yields in the face of an inconsistent belief base is slightly weaker: for every $B \in \mathcal{W}^\perp$, if $E(B) \cap E(A) = \emptyset$, then $B \in \Upsilon \oplus A$, resp. $B \in \Upsilon \ominus A$.

obey the axiom **P**. Arguably, this requirement is far too liberal to receive the status of a rationality postulate.

This does not mean that the intuition behind Parikh's relevance axiom is not applicable to inconsistent beliefs. Consider the belief set $\Upsilon_3 = \{p \wedge q, r, {\sim} r\}$, and suppose we have to contract this set by $q \vee r$. Even though one has to remove both $q$ and $r$, one can readily argue that $p$ is not relevant to this particular contraction. This has little to do with the fact that $\Upsilon_3$ is inconsistent.

How can we get more formal grip on this? Note that even a very weak paraconsistent logic will usually validate Simplification (from $A \wedge B$, infer $A$ and $B$).[4] Hence in the above example, every such logic will allow us to derive $p$ from $p \wedge q$, and hence consider $p$ as separable from $q$ in $\Upsilon_3$. More generally, a logic can be (fully) paraconsistent, yet still allow us to analyse our set of initial beliefs to some extent, and hence obey a certain degree of relevance. So it seems at least plausible that we can obtain a strong, but also non-trivializing relevance axiom, if we weaken our standard of deduction in such a way that inconsistencies do not cause us to believe anything.

## 3.3 Outline of a Solution

Before we move over to a concrete solution for the problem of relevance in an inconsistent setting, let me briefly spell out the basic ingredients we need, using **L** as a metavariable for any subclassical logic. We first generalize the definition of a splitting, obtaining the concept of a **L**-splitting:

**Definition 3 (L-splitting)** *Let $\mathbb{E} = \{\Lambda_i\}_{i \in I}$ be a partition of $\mathcal{E}$. We say that $\mathbb{E}$ is a* **L**-splitting *of $\Gamma$ iff there is a $\Delta = \bigcup_{i \in I} \Delta_i$ such that each $E(\Delta_i) \subseteq \Lambda_i$ and $\Delta \dashv\vdash_{\mathbf{L}} \Gamma$.*

Suppose that for a specific **L**, every $\Gamma \subseteq \mathcal{W}^{\perp}$ has a finest **L**-splitting. Then we can use the notion of a finest splitting to define **L**-relevance, just as in the case where classical logic was the underlying logic:

**Definition 4 (L-relevance)** *Let $\mathbb{E}$ be the finest* **L***-splitting of $\Upsilon$. We say that a formula $B$ is* **L***-irrelevant to the revision of $\Upsilon$ by $A$ iff for every cell $\Lambda_i \in \mathbb{E}$: $\Lambda_i \cap E(A) = \emptyset$ or $\Lambda_i \cap E(B) = \emptyset$.*

Finally, this allows us to define an axiom of **L**-relevance:

$\mathbf{P_L}$ If $B \in Cn_{\mathbf{L}}(\Upsilon)$ is **L**-irrelevant to the revision (contraction) of $\Upsilon$ by $A$, then $B$ is an element of $Cn_{\mathbf{L}}(\Upsilon \oplus A)$ $(Cn_{\mathbf{L}}(\Upsilon \ominus A))$.

By these three basic steps, we obtain a notion of subclassical relevance. As is clear from the above definitions, the crucial property we need to arrive at this result, is that every $\Gamma \subseteq \mathcal{W}^{\perp}$ has a finest **L**-splitting. Of course, to allow for a sensible notion of relevance in the context of inconsistent beliefs, we have to use a *paraconsistent* logic **L**. As announced, I will first focus on one particular such logic. In the subsequent sections, I will move to a more general level, and show that the above strategy can be applied to a number of other subclassical systems, including intuitionistic logic.

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[4] For some examples of logics that do *not* validate Simplification, see [3] – Chapter 8 of this book contains a survey of logics in which some connectives other than $\sim$ behave non-classically in various ways.

# 4 CLuNs-relevance and Local Consistency

## 4.1 The Paraconsistent Logic CLuNs

To explain the idea behind a subclassical relevance axiom, I will use the paraconsistent logic **CLuNs**, as axiomatized in [5]. This choice is motivated by two properties of the logic: (i) **CLuNs** is maximally paraconsistent, which means that its analytic power is very close to that of **CL**; (ii) nevertheless, **CLuNs** is also fully paraconsistent, whence **CLuNs**-relevance will not trivialize any belief set $\Gamma \subseteq \mathcal{W}$. Each of these advantages will be illustrated below.

The propositional fragment of **CLuNs** is one of the three systems devised by Schütte in [22], the two others are **CLaNs** and **CLoNs** and will be presented in Section 5. All three of these systems are particularly strong in that they allow us to drive the paraconsistent negation inwards; e.g. it is possible to derive $\sim A, \sim B$ from $\sim(A \vee B)$, and similariy to derive $A \wedge \sim B$ from $\sim(A \supset B)$. A distinctive feature of **CLuNs** is that it is paraconsistent but not paracomplete (unlike the other Schütte systems): it can model cases where both $A$ and $\sim A$ are true, but it cannot model cases in which both are false.

**CLuNs** is axiomatized by the rule **MP** (from $A, A \supset B$ to infer $B$), the axioms of full positive **CL**:

**A**$\supset$**1** $A \supset (B \supset A)$
**A**$\supset$**2** $((A \supset B) \supset A) \supset A)$
**A**$\supset$**3** $(A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))$
**A**$\perp$ $\perp \supset A$
**A**$\wedge$**1** $(A \wedge B) \supset A$
**A**$\wedge$**2** $(A \wedge B) \supset B$
**A**$\wedge$**3** $A \supset (B \supset (A \wedge B))$
**A**$\vee$**1** $A \supset (A \vee B)$
**A**$\vee$**2** $B \supset (A \vee B)$
**A**$\vee$**3** $(A \supset C) \supset ((B \supset C) \supset ((A \vee B) \supset C))$
**A**$\equiv$**1** $(A \equiv B) \supset (A \supset B)$
**A**$\equiv$**2** $(A \equiv B) \supset (B \supset A)$
**A**$\equiv$**3** $(A \supset B) \supset ((B \supset A) \supset (A \equiv B))$

the rule of excluded middle:

**EM** $A \vee \sim A$

and the following axioms that drive negation inwards:

**A**$\sim\sim$ $\sim\sim A \equiv A$
**A**$\sim\supset$ $\sim(A \supset B) \equiv (A \wedge \sim B)$
**A**$\sim\wedge$ $\sim(A \wedge B) \equiv (\sim A \vee \sim B)$
**A**$\sim\vee$ $\sim(A \vee B) \equiv (\sim A \wedge \sim B)$
**A**$\sim\equiv$ $\sim(A \equiv B) \equiv \sim(A \supset B) \vee \sim(B \supset A))$

For reasons of space, I will not discuss the various semantic characterizations of **CLuNs** – see e.g. [5, 4, 3, 25]. Note that since $\supset$ and $\perp$ behave classically in **CLuNs**, it is possible to define a classical negation $\neg$ in this system by $\neg A =_{\mathsf{def}} (A \supset \perp)$.

To see how **CLuNs** behaves, consider $\Upsilon_4 = \{\sim(p \supset (q \vee r)), (\sim s \vee (t \wedge \sim\sim u)) \wedge p, \sim(\sim q \wedge p), v, \sim v \wedge \sim q\}$. Each of the following holds:

(1) $\Upsilon_4 \vdash_{\mathbf{CLuNs}} p \wedge \sim(q \vee r)$ (by **A**$\sim\supset$)

(2) $\Upsilon_4 \vdash_{\mathbf{CLuNs}} p, {\sim}q, {\sim}r$ (by (1) and $\mathbf{A}\wedge\mathbf{1}$, $\mathbf{A}\wedge\mathbf{2}$)
(3) $\Upsilon_4 \vdash_{\mathbf{CLuNs}} {\sim}s \vee (t \wedge u)$ (by $\mathbf{A}\wedge\mathbf{1}$ and $\mathbf{A}{\sim}{\sim}$)
(4) $\Upsilon_4 \vdash_{\mathbf{CLuNs}} {\sim}s \vee t, {\sim}s \vee u$ (by (3) and $\mathbf{A}\wedge\mathbf{1}$, $\mathbf{A}\wedge\mathbf{2}$)
(5) $\Upsilon_4 \vdash_{\mathbf{CLuNs}} {\sim}{\sim}q \vee {\sim}p$ (by $\mathbf{A}{\sim}\wedge$)
(6) $\Upsilon_4 \vdash_{\mathbf{CLuNs}} q \vee {\sim}p$ (by (5) and $A{\sim}{\sim}$)
(7) $\Upsilon_4 \vdash_{\mathbf{CLuNs}} (p \wedge {\sim}p) \vee (q \wedge {\sim}q)$ (by (2), (6))

$\Upsilon_4$ is clearly inconsistent. Since **CLuNs** invalidates disjunctive syllogisme, it is not possible to **CLuNs**-derive e.g. ${\sim}p$ from ${\sim}q$ and $q \vee {\sim}p$. Hence $\Upsilon_4 \nvdash_{\mathbf{CLuNs}} {\sim}p$.

## 4.2 CLuNs-relevance

I will now illustrate the three steps we need to obtain a subclassical relevance axiom, using **CLuNs** as an example: the definition of a finest **CLuNs**-splitting, the definition of **CLuNs**-relevance, and finally the axiom of **CLuNs**-relevance itself. The set $\Upsilon_4$ from Section 4.1 will be used to illustrate each of these notions.

Consider $\Upsilon'_4 = \{p, {\sim}q, {\sim}r, {\sim}s \vee t, {\sim}s \vee u, {\sim}p \vee q, v, {\sim}v\}$. In view of (1)-(7), it follows immediately that $\Upsilon_4 \vdash_{\mathbf{CLuNs}} \Upsilon'_4$. I leave it as an exercise to the reader to prove that also $\Upsilon'_4 \vdash_{\mathbf{CLuNs}} \Upsilon_4$. As a result, $\Upsilon_4$ and $\Upsilon'_4$ are **CLuNs**-equivalent.

Note that $\Upsilon'_4$ can be partitioned into three subsets: $\Delta_1 = \{p, {\sim}q, {\sim}p \vee q\}$, $\Delta_2 = \{{\sim}r\}$, $\Delta_3 = \{{\sim}s \vee t, {\sim}s \vee u\}$ and $\Delta_4 = \{v, {\sim}v\}$. Note also that the letter sets of each $\Delta_i$ are pairwise disjoint. Hence we can obtain a **CLuNs**-splitting of $\Upsilon_4$, i.e. $\mathbb{E}(\Upsilon_4) = \{\{p, q\}, \{r\}, \{s, t, u\}, \{v\}\}$.

In Section 6.1, it is proven that every set $\Upsilon$ has a finest **CLuNs**-splitting, and in Section 6.3 I show that one can obtain this splitting by the construction of a specific set $Min_{\mathbf{CluNs}}(\Upsilon) \subset Cn_{\mathbf{CLuNs}}(\Upsilon)$. In the current case, these results warrant that $\mathbb{E}(\Upsilon_4)$ is in fact the *finest* **CLuNs**-splitting of $\Upsilon_4$.

Let us go back to the example. Suppose that we contract $\Upsilon_4$ by $p \vee {\sim}s$. Note that $\{p, q\}$ and $\{s, t, u\}$ are the only sets $\Lambda$ in $\mathbb{E}(\Upsilon_4)$ for which $\Lambda \cap E(p \vee {\sim}s) \neq \emptyset$. The axiom of **CLuNs**-relevance tells us the following: a formula $A \in Cn_{\mathbf{CLuNs}}(\Upsilon_4)$ is relevant to the contraction of $\Upsilon_4$ by $p \vee {\sim}s$ iff $E(A) \cap \{p, q\} \neq \emptyset$ or $E(A) \cap \{s, t, u\} \neq \emptyset$. Hence the following **CLuNs**-consequences of $\Upsilon_4$ are *not* relevant to the contraction of $\Upsilon_4$ by $p$: ${\sim}r, v, {\sim}v$.

This immediately brings us to the axiom of relevance. In the current case, this axiom stipulates that the beliefs ${\sim}r, v, {\sim}v$ should be upheld. Note that this means that a contradiction has to be upheld, in order to obey $\mathbf{P_{CLuNs}}$ – I will return to this fact in Section 4.3. However, the axiom does not require us to believe just anything: e.g. if we remove $p$ from $\Upsilon'_4$, we obtain a non-trivial yet fairly rich belief set that does not **CLuNs**-entail $p \vee r$.

So, on the one hand, we are able to separate e.g. ${\sim}r$ from $p$, notwithstanding the fact that in the initial formulation of $\Upsilon_4$, these formulas are tied to each other. On the other hand, some beliefs are still considered relevant to the new information, and removing some of these results in a reasonable contraction set. In short, we obtain a non-trivial, yet also non-trivializing relevance axiom for inconsistent belief sets.

## 4.3 Local Change and CLuNs-relevance

Consider again the contraction operation of $\Upsilon_4$ by $p \vee {\sim}s$, as described in the preceding section. One could ask oneself: should the beliefs $v$ and ${\sim}v$ be upheld? If so, the resulting contraction set will remain inconsistent. But is this a sufficient reason to remove (either of) these beliefs from $\Upsilon_4$? Clearly, they have little to do with the formula by which we are contracting, no matter whether we consider $\Upsilon_4$ in its initial formulation, or a more analysed version of it, such as $\Upsilon'_4$.

According to the standard AGM approach, inconsistencies cannot occur in any contraction or revision set. This also applies to the more recent approaches in terms of belief bases: in both cases, it is required that $\Upsilon \ominus A \nvdash_{\mathbf{CL}} A$, and that $\Upsilon \oplus A$ is a consistent set. Hence any inconsistency is removed from $\Upsilon$ whenever this set is contracted or revised.

At least some authors seem to suggest that it should be possible to leave certain inconsistencies in $\Upsilon$ undisturbed – to uphold the belief in these inconsistencies, while focusing on other problematic parts of $\Upsilon$. Such an approach may perhaps best be understood as a kind of "local consolidation" or "local revision", as described by Hansson and Wasserman in their [13, p. 51]:

> *Local Consolidation.* Inconsistencies are removed from some part of the belief base. The rest of the agent's beliefs may well be inconsistent. For instance, I can make my beliefs about biological evolution consistent, while retaining global inconsistency between biological and religious beliefs.
>
> *Local Revision.* A new belief is added to the belief base in such a way that a certain part of the resulting base is made (kept) consistent. If I see, for example, that it is a sunny day in Amsterdam, then this contradicts my belief that it is always raining in Holland, and leads to revision. This can be done without checking whether my beliefs about Brazilian politics are consistent with the new belief.

Similar ideas can be found in [21, p. 10], where it is argued that a relevantist approach to belief contraction allows us to model processes in which inconsistencies are removed one by one, such that the intermediary belief states remain inconsistent. The authors quote Fuhrmann, who writes the following in a section of his [10] titled "Local Inconsistency":

> [...] Thus, in the face of inconsistent theories we should want two things:
>
> (a) *localise inconsistencies* – an inconsistent theory should not be rendered totally corrupt just because some inconsistency has crept into the theory; and
>
> (b) *locally restore consistency* – we should be able to resolve one inconsistency at a time by contracting an inconsistent theory such that other inconsistencies, which we cannot yet resolve, may be carried over into the contraction theory.

In order to obtain (a) and (b), Fuhrmann recommends that "theories be generated from bases by means of a consequence operation induced by some parconsistent logic."[10, p. 187]

Recall that in our example, the axiom stipulated that the inconsistency $v, {\sim}v$ is upheld, since it is not relevant to the formula $p \vee {\sim}s$ by which we ought to contract. More generally, the axiom of **CLuNs**-relevance does not distinguish between formulas that behave consistently and those that behave inconsistently; all that matters is whether formulas are relevant to the revision or contraction. If an inconsistency is *not* relevant to this operation, then it is upheld.

In Furhmanns terms, the axiom of **CLuNs**-relevance requires that we should only locally restore consistency. Moreover, it does so in very clear and precise logical terms, and in a way that is perhaps much stronger than what Fuhrmann and the later advocates of local consolidation had in mind – recall my remark in Section 3.1 that relevance is invariant under different equivalent formulations of $\Upsilon$.

# 5 Some Subclassical Logics

In this section, I define 9 subclassical logics, and list some generic properties of these systems. These properties make it possible to apply the strategy spelled out in Section 3.3 to these logics. As a result, they provide a sufficient condition for the properties proven in Section 6.

## 5.1 The Logics

**The Schütte Logics.** Let us start with the two systems **CLoNs** and **CLaNs**, which were already mentioned in Section 4.1. **CLoNs** is obtained by removing **EM** from **CLuNs**. As a result, the standard negation displays both gluts and gaps: $A$ and $\sim A$ can both be true, but they can also both be false. **CLaNs** is the counterpart of **CLuNs**, in that its negation displays gaps but not gluts. **CLaNs** and can be obtained by adding the rule of ex falso quodlibet to **CLoNs**:

**EFQ** $A \supset (\sim A \supset B)$

Note that, since **EM** is not valid in **CLaNs**, this logic invalidates the rule $(A \supset B) \wedge (\sim A \supset B) \vdash B$, which makes it somewhat similar to intuitionistic logic. However, Peirce's axiom **A⊃2** remains valid in **CLaNs**, unlike intuitionistic logic (see below).

In view of their axiomatization, it follows immediately that **CLuNs** is stronger than **CLoNs**, that **CLaNs** is stronger than **CLoNs** and that **CLuNs** and **CLaNs** are incommensurable.

**The Basic Paralogics.** The three basic paralogics **CLoN**, **CLuN** and **CLaN** are the weaker nephews of **CLoNs**, **CLuNs** and **CLaNs** respectively, in that they invalidate the axioms that drive negation inwards. Hence e.g. **CLaN** is obtained by closing the set of **CLaNs**-axioms, minus **A∼∼**, **A∼ ⊃**, **A∼∧**, **A∼∨** and **A∼ ≡** under **MP**.

Alternatively, we may say that **CLuN**,**CLaN** and **CLoN** boil down to the full positive part of **CL** (see page 7), with gl**u**ts, g**a**ps, and respectively b**o**th gluts and gaps for the **N**egation. As before, we can easily infer that **CLuN** and **CLaN** are incommensurable, but that both are stronger than **CLoN**. Also, each of the basic logics is weaker than its respective Schütte-variant.

The advantage of the basic logics is that they maximally localise inconsistencies, or in the case of **CLaN**, $\sim$-incompleteness. For example, if $A$ is not itself contradictory and $A \neq B$, then $A \wedge \sim A \nvdash_{\mathbf{CLuN}} B \wedge \sim B$. This makes **CLuN** a very good candidate to serve as the lower limit logic of inconsistency-adaptive logics.[5]

**The Vasil'ev Logics.** A third class of logics are the Vasil'ev systems **CLoNv**, **CLuNv** and **CLaNv**. These are peculiarly strong, in that in them, the negation $\sim$ is assumed to behave classically in front of all complex formulas. Hence e.g. $\sim(A \wedge B)$ is equivalent to $(A \wedge B) \supset \bot$.

The Vasil'ev systems are obtained by adding the following axiom schema to **CLoN**, **CLuN** and **CLaN** respectively:

**A$\sim$V** Where $A \in \mathcal{W}^{\bot} - \mathcal{E}$: $\sim A \supset (A \supset \bot)$

Note that in view of this axiom schema, none of the Vasil'ev systems are fully paraconsistent. For example, $(p \vee q) \wedge \sim(p \vee q) \vdash_{\mathbf{CLuNv}} A$ for any $A \in \mathcal{W}^{\bot}$. As a result, these systems are only useful in a context where inconsistency or $\sim$-incompleteness is restricted to the level of propositional letters, as e.g. in $\Gamma = \{p, q \wedge \sim p, r \vee s, \sim r, \sim s\}$.

**Intuitionistic Logic.** The last system we will consider is of a rather different nature: intuitionistic logic. I will assume familiarity with this system and its properties – I merely present an axiomatization for the sake of self-containedness. The system **I** can be obtained as follows: (i) take the positive fragment of **CL**; (ii) remove axiom **A$\supset$2**, and (iii) add the following two axioms that characterize $\sim$ in **I**:

**A$\sim$I1** $(A \supset B) \supset ((A \supset \sim B) \supset \sim A)$
**A$\sim$I2** $\sim A \supset (A \supset B)$

## 5.2 The Properties

Let in the remainder **L** be a metavariable for all logics axiomatized in the preceding sections plus **CL**. In view of my definition of consequence (see Section 2), the following is immediate:

**Theorem 2** *$\Gamma \vdash_{\mathbf{L}} A$ iff there are $B_1, \ldots, B_n \in \Gamma$ such that $\{B_1, \ldots, B_n\} \vdash_{\mathbf{L}} A$. (Compactness)*

The next property on the list is the Deduction Theorem. For **I**, a proof can be found in [9, Chapter 4]. For the 9 paralogics from Section 5.1, it follows immediately from the fact that $\supset$ behaves classically in each of these systems. For the current purposes, it is convenient to rewrite this property as follows:

**Theorem 3** *$A \wedge B \vdash_{\mathbf{L}} C$ iff $A \vdash_{\mathbf{L}} B \supset C$ (Deduction Theorem)*

Note that this theorem follows from the regular deduction theorem for **L**, whenever $\wedge$ behaves classically in **L**, i.e. whenever $A \wedge B \dashv\vdash_{\mathbf{L}} \{A, B\}$.

In view of the deduction theorem and the definition of $\Gamma \vdash_{\mathbf{L}} A$, each of the logics **L** is reflexive, transitive and monotonic:

---

[5] These are logics that try to minimize inconsistencies and thereby strengthen a paraconsistent logic non-monotonically – see [2] for a recent survey of inconsistency-adaptive logics.

**Theorem 4** *Each of the following holds:*

1. $\Gamma \subseteq Cn_{\mathbf{L}}(\Gamma)$ *(Reflexivity)*
2. *if* $\Gamma' \subseteq Cn_{\mathbf{L}}(\Gamma)$*, then* $Cn_{\mathbf{L}}(\Gamma') \subseteq Cn_{\mathbf{L}}(\Gamma)$ *(Transitivity)*
3. $Cn_{\mathbf{L}}(\Gamma) \subseteq Cn_{\mathbf{L}}(\Gamma \cup \Gamma')$ *(Monotonicity)*

In view of the above properties, it can be easily shown that $\Gamma \subseteq Cn_{\mathbf{L}}(\Gamma')$ and $\Gamma' \subseteq Cn_{\mathbf{L}}(\Gamma)$ iff $\Gamma \dashv\vdash_{\mathbf{L}} \Gamma'$. The last property I will need is interpolation:

**Theorem 5** *If* $\Gamma \vdash_{\mathbf{L}} A$*, then there is a* $B$ *such that* $\Gamma \vdash_{\mathbf{L}} B$*,* $B \vdash_{\mathbf{L}} A$*, and* $E(B) \subseteq E(\Gamma) \cap E(A)$*. (Standard Interpolation)*

For all considered logics except **I** and **CL**, standard interpolation was proven in [5]. For **I**, I refer to [9, Chapter 4].

# 6 Generic Results for L-relevance

In this section, I establish a number of theoretic results concerning **L**-splittings and **L**-relevance – recall that **L** is used as a metavariable for any of the logics axiomatized in the preceding sections, and **CL**. The proofs only rely on the properties mentioned in Section 5.2, whence the current results may be easily generalized to a yet broader class of logics. Eventually, this yields a partial answer to the question posed in the concluding section of [14]:

> [...] how far can the results [of our paper] be established for subclassical, (e.g. intuitionistic) consequence relations or supraclassical ones (e.g., preferential consequence relations or the relation of logical friendliness of Makinson [8])?

As it turns out, each of the results referred to by Kourousias and Makinson are transferable to the subclassical logics which I consider here – see Sections 6.1 and 6.2. Moreover, in Section 6.3, a nice generalization is obtained for Theorem 3 from [19], by the introduction of the notion of **L**-minimal formulas relative to a set $\Gamma$. Finally, using this same notion, the least letter-set theorem from [15] is generalized to all logics **L** from this paper.

## 6.1 Finest L-splitting

A crucial theorem for Kourousias and Makinson's finest splitting result in [14] is that of parallel interpolation for **CL**, which is a strengthening of standard interpolation. The proof for Theorem 6 is readily obtained through a variation on the proof for Theorem 1.1 in [14].

**Theorem 6** *Let* $\Delta = \bigcup_{i \in I}\{\Delta_i\}$ *where the letter sets* $E(\Delta_i)$ *are pairwise disjoint, and suppose* $\Delta \vdash_{\mathbf{L}} A$*. Then there are formulas* $B_i$ *such that (1) each* $E(B_i) \subseteq E(\Delta_i) \cap E(A)$*, (2) each* $\Delta_i \vdash_{\mathbf{L}} B_i$*, and (3)* $\bigcup_{i \in I}\{B_i\} \vdash_{\mathbf{L}} A$*. (Parallel Interpolation)*

*Proof.* Suppose the antecedent holds. By the compactness of **L**, there is a finite subfamily of finite subsets of the $\Delta_i$, the conjunction of whose elements implies $A$. Let these subsets be $\Delta'_{j_1}, \ldots, \Delta'_{j_n}$, and let for every $k \leq n$, $B_k$ be the conjunction of the members of $\Delta'_{j_k}$. It follows that $B_1 \wedge \ldots \wedge B_n \vdash_{\mathbf{L}} A$. By the Deduction Theorem, $B_1 \vdash_{\mathbf{L}} (B_2 \wedge \ldots \wedge B_n) \supset A$, which implies, by standard interpolation, that there is a formula $C_1$ such that (1) $B_1 \vdash_{\mathbf{L}} C_1$ and $C_1 \vdash_{\mathbf{L}} (B_2 \wedge \ldots \wedge B_n) \supset A$ and (2) $E(C_1) \subseteq E(B_1) \cap E((B_2 \wedge \ldots \wedge B_n) \supset A)$. Since the sets $E(B_i)$ are pairwise disjoint, (2) implies that $E(C_1) \subseteq E(B_1) \cap E(A)$.

By (1) and the Deduction Theorem, $C_1 \wedge B_2 \wedge \ldots \wedge B_n \vdash_{\mathbf{L}} A$ and the sets $E(C_1), E(B_2), \ldots, E(B_n)$ are pairwise disjoint. Hence we may repeat the procedure for $B_2$, obtaining a suitable interpolant $C_2$, and so on. After $n$ applications of standard interpolation, we have obtained $C_1, \ldots, C_n$, where each $E(C_i) \subseteq E(B_i) \cap E(A) \subseteq E(\Delta_{j_i}) \cap E(A)$ and $C_1 \wedge \ldots \wedge C_n \vdash_{\mathbf{L}} A$. ■

By a similar variation on the proofs for Lemma 2.3 and Theorem 2.4 of [14], we may derive the following:

**Theorem 7** *Every* $\Gamma \subseteq \mathcal{W}^{\perp}$ *has a finest* **L**-*splitting.*

I will not provide the proof for this Theorem here. Just as the proof for Theorem 6, it is almost identical to the proof in [14]. It suffices to merely replace the $\vdash$ in Lemma 2.3 and Theorem 2.4 from that paper by $\vdash_{\mathbf{L}}$. More importantly, in Section 6.3 from the current paper, it is explained how an alternative proof for Theorem 7 can be obtained, relying on the notion of **L**-minimal formulas.

Once the notion of a finest splitting is generalized to a class of logics **L**, a question that immediately springs to mind is when and how the finest **L**-splitting relates to the finest **L**′-splitting, for two logics **L** and **L**′. In fact, the stronger a logic, the finer the associated finest splitting of $\Gamma$:

**Theorem 8** *If* **L** *is at least as strong as* **L**′, *then the finest* **L**′-*splitting of* $\Gamma$ *is a* **L**-*splitting of* $\Gamma$.

*Proof.* Suppose the antecedent holds and $\mathbb{E} = \{E_i\}_{i \in I}$ is the finest **L**′-splitting of $\Gamma$. Hence there is a $\Delta = \bigcup_{i \in I} \Delta_i$ such that $\Delta \dashv\vdash_{\mathbf{L}'} \Gamma$ and each $E(\Delta_i) \subseteq E_i$. It follows from the supposition that $\Delta \dashv\vdash_{\mathbf{L}} \Gamma$. But then $\mathbb{E}$ is a **L**-splitting of $\Gamma$. ■

Note that this proof works for any logic **L** and **L**′, on the assumption that every $\Gamma$ has a finest **L**′-splitting. Trivial as its proof is, this is a noteworthy result. Recall that in order to avoid that relevance results in triviality, it was necessary to weaken the standard of deduction, hence to define a notion of subclassical finest splittings and an associated relevance criterion. Theorem 8 indicates that the stronger the subclassical logic of our choice, the better we may approximate the finest **CL**-splitting without ending up with triviality in the case of an inconsistency.

By Theorem 8, we may infer that e.g. the finest **CLuNs**-splitting of $\Upsilon$ is always at least as fine as the finest **CLuN**-splitting of $\Upsilon$, and likewise that the finest **CLaNv**-splitting of $\Upsilon$ is always at least as fine as the finest **CLaN**-splitting. Also, we may infer that each of the subclassical splittings is further refined by the finest **CL**-splitting of $\Upsilon$.

To illustrate this point, we may consider the following splittings of $\Upsilon_4$ from page 7:

$\mathbb{E}_1(\Upsilon_4) = \{\{p, q, r\}, \{s, t, u\}, \{v\}\}$
$\mathbb{E}_2(\Upsilon_4) = \{\{p, q\}, \{r\}, \{s, t, u\}, \{v\}\}$

$\mathbb{E}_3(\Upsilon_4) = \{\{p\}, \{q\}, \{r\}, \{s\}, \{t\}, \{u\}, \{v\}\}$

$\mathbb{E}_1(\Upsilon_4)$ is the finest **CLuN**-splitting of $\Upsilon_4$, $\mathbb{E}_2(\Upsilon_4)$ the finest **CLuNs**-splitting of $\Upsilon_4$ and $\mathbb{E}_3(\Upsilon_4)$ the finest **CL**-splitting of $\Upsilon_4$.

From the preceding observations, we can also infer that whenever $B \in Cn_{\mathbf{CL}}(\Gamma)$ is **L**-irrelevant to the revision (contraction) of $\Upsilon$ by $A$, then it is also **CL**-irrelevant to the revision (contraction) of $\Upsilon$ by $A$. In other words, there are just as many means to show that a formula is **CL**-irrelevant to a particular revision or contraction operation, as there are subclassical logics for which Theorem 7 holds.

### 6.2 L-canonical Forms

In [14], Kourousias and Makinson introduce the notion of a *canonical form* of the belief set $\Upsilon$, a notion that is already implicit in the definition of a finest **CL**-splitting. Where $\mathbb{E} = \{\Lambda_i\}_{i \in I}$ is the finest **CL**-splitting of $\Upsilon$, $\Upsilon' = \bigcup_{i \in I} \Upsilon_i$ is a canonical form of $\Upsilon$ iff (i) it is **CL**-equivalent to $\Upsilon$, and (ii) each $E(\Upsilon_i) \subseteq E_i$.

In [19], it is explained that there may in fact be several canonical forms $\Upsilon'$ for one and the same $\Upsilon$, whence it is better to speak of the *set* of canonical forms of $\Upsilon$ instead of "the" canonical form of $\Upsilon$. If we generalize this notion in order to include subclassical logics, we obtain the following definition:

**Definition 5 (Set of L-canonical Forms)** *Where* $\mathbb{E} = \{\Lambda_i\}_{i \in I}$ *is the finest* **L***-splitting of* $\Upsilon$*:* $\mathbb{C}_{\mathbf{L}}(\Upsilon) = \{\Delta = \bigcup_{i \in I}\{\Delta_i\} \mid \Delta \dashv\vdash_{\mathbf{L}} \Upsilon$ *and for every* $i \in I : E(\Delta_i) \subseteq \Lambda_i\}$.

Kourousias and Makinson proceed to show that if $\Upsilon$ is consistent, then every partial meet contraction of a canonical form of $\Upsilon$ by $A$ obeys the axiom of relevance. To obtain a similar result for the subclassical logics from Section 5, we first have to define subclassical contraction and revision operations on belief bases.[6] These are obtained by a generalization of partial meet contraction and revision for bases, as defined in [11]. It suffices to replace all references to **CL** in the definition of partial meet contraction and revision by **L**:

**Definition 6 (Set of L-remainders)** $\Upsilon \curlywedge_{\mathbf{L}} A$ *is the set of all* $\Delta \subseteq \Upsilon$ *such that:*

*(i)* $\Delta \nvdash_{\mathbf{L}} A$*, and*
*(ii) for no* $\Delta' \subseteq \Upsilon$*:* $\Delta \subset \Delta'$ *and* $\Delta' \nvdash_{\mathbf{L}} A$.

**Definition 7 (L-contraction)** $\Upsilon \ominus_{\mathbf{L}} A = \bigcap \gamma(\Upsilon \curlywedge_{\mathbf{L}} A)$.

**Definition 8 (L-revision)** $\Upsilon \oplus_{\mathbf{L}} A = \Upsilon \ominus_{\mathbf{L}} {\sim}A \cup \{A\}$.

Theorems 9 and 10 below state that Kourousias and Makinson's result can be generalized to all 10 subclassical logics from Section 5. Note that in Section 3.1, I strengthened the axiom of relevance such that it holds for all $B \in Cn_{\mathbf{L}}(\Upsilon)$, not just for all $B \in \Upsilon$. As a result, Theorem 9 is also stronger than Kourousias and Makinson's Theorem 4.1 when applied to belief bases. To prove it, I first establish a lemma that generalizes Kourousias and Makinson's Theorem 4.1 to the logics **L**:

---

[6] Where $\Upsilon$ is a theory, we may obtain the contracted resp. revised theory by closing the result of the contraction (revision) operation defined here under **L**.

**Lemma 1** *Where $\Upsilon$ is not* **L**-*trivial and* $\Upsilon' \in \mathbb{C}_{\mathbf{L}}(\Upsilon)$*: if* $B \in \Upsilon'$ *is not* **L**-*relevant to the contraction of* $\Upsilon$ *by* $A$*, then* $B \in \Upsilon' \ominus_{\mathbf{L}} A$.

*Proof.* Suppose $B \in \Upsilon'$ but $B \notin \Upsilon' \ominus_{\mathbf{L}} A$, whereas $B$ is not **L**-relevant to the contraction of $\Upsilon$ by $A$ — I derive a contradiction. Let $\mathbb{E}_{\mathbf{L}}(\Upsilon) = \{E_i\}_{i \in I}$ be the finest **L**-splitting of $\Upsilon$, such that $\Upsilon' = \bigcup_{i \in I} \Upsilon_i$ and for each $i \in I$, $E(\Upsilon_i) \subseteq E_i$. Let $\{E_j\}_{j \in J}$ be the subfamily of cells in $\mathbb{E}_{\mathbf{L}}(\Upsilon)$ that share some elementary letter with $E(A)$. By the irrelevance, $\bigcup_{j \in J}\{E_j\} \cap E(B) = \emptyset$.

Since $B \notin \Upsilon' \ominus_{\mathbf{L}} A$, by Definition 7, there is a $\Delta \in \Upsilon' \curlywedge_{\mathbf{L}} A$ such that $B \notin \Delta$. By Definition 6, $\Delta \cup \{B\} \vdash_{\mathbf{L}} A$. Put $\Upsilon_a = \bigcup_{j \in J} \Upsilon_j$ and $\Upsilon_b = \bigcup_{i \in I-J} \Upsilon_i$. Then since $\Delta \subseteq \Upsilon' = \Upsilon_a \cup \Upsilon_b$, we have $(\Delta \cap \Upsilon_a) \cup (\Delta \cap \Upsilon_b) \cup \{B\} = \Delta \cap (\Upsilon_a \cup \Upsilon_b) \cup \{B\} = \Delta \cup \{B\} \vdash_{\mathbf{L}} A$. Hence by compactness, $\{C_1, \ldots, C_n\} \cup \{D_1, \ldots, D_m\} \cup \{B\} \vdash_{\mathbf{L}} A$, where $C_1, \ldots, C_n$ are elements of $\Upsilon_a$ and $D_1, \ldots, D_m$ are elements of $\Upsilon_b$.

By the Deduction Theorem: $\{D_1, \ldots, D_m\} \cup \{B\} \vdash_{\mathbf{L}} (C_1 \wedge \ldots \wedge C_n) \supset A$. In view of the construction, the formulas on the left side and those on the right side have no letters in common. But this means that either $\{D_1, \ldots, D_m\} \cup \{B\}$ is **L**-trivial, or $(C_1 \wedge \ldots \wedge C_n) \supset A$ is a **L**-theorem. In the former case, since $D_1, \ldots, D_m, B \in \Upsilon'$, it follows that $\Upsilon'$ is **L**-trivial whence also $\Upsilon$ is **L**-trivial — a contradiction. In the latter case, since $C_1, \ldots, C_n \in \Delta$, by the Deduction Theorem, $\Delta \vdash_{\mathbf{L}} A$, which contradicts the fact that $\Delta \in \Upsilon' \curlywedge_{\mathbf{L}} A$. ■

The following lemma is easily derivable from the parallel interpolation theorem. It provides the link between the weak relevance from the preceding lemma, and the strong relevance as defined in Section 3.1.

**Lemma 2** *If* $\Delta = \bigcup_{i \in I} \Delta_i \vdash_{\mathbf{L}} A$*,* $\Delta$ *is not* **L**-*trivial, and the letter sets* $E(\Delta_i)$ *are pairwise disjoint, then* $\bigcup_{i \in I}\{\Delta_i \mid E(\Delta_i) \cap E(A) \neq \emptyset\} \vdash_{\mathbf{L}} A$.

*Proof.* Suppose the antecedent holds. By parallel interpolation, there are $B_i$ (with $i \in I$) such that (1) each $\Delta_i \vdash_{\mathbf{L}} B_i$, (2) each $E(B_i) \subset E(\Delta_i) \cap E(A)$ and (3) $\{B_i\}_{i \in I} \vdash_{\mathbf{L}} A$. Suppose that for an $i \in I$, $E(\Delta_i) \cap E(A) = \emptyset$. By (3), $E(B_i) = \emptyset$, whence (4) $B_i \dashv\vdash_{\mathbf{L}} \bot$ or (5) $B_i \dashv\vdash_{\mathbf{L}} \top$. In view of the supposition and (1), (4) is false. In view of (5), it follows that $\{B_1, \ldots, B_{i-1}, B_{i+1}, \ldots, B_n\} \vdash_{\mathbf{L}} A$.

Hence for every $i$ such that $E(\Delta_i) \cap E(A) = \emptyset$, we may remove the formula $B_i$ from the set $\{B_i\}_{i \in I}$, without losing $A$ as a **L**-consequence. Hence $\{B_i \mid E(B_i) \cap E(A) \neq \emptyset\} \vdash_{\mathbf{L}} A$. By (1), (3) and the transitivity of **L**, $\bigcup_{i \in I}\{\Delta_i \mid E(\Delta_i) \cap E(A) \neq \emptyset\} \vdash_{\mathbf{L}} A$. ■

**Theorem 9** *Where* $\Upsilon$ *is not* **L**-*trivial and* $\Upsilon' \in \mathbb{C}_{\mathbf{L}}(\Upsilon)$*: if* $B \in Cn_{\mathbf{L}}(\Upsilon)$ *and* $B$ *is not* **L**-*relevant to the contraction of* $\Upsilon$ *by* $A$*, then* $B \in Cn_{\mathbf{L}}(\Upsilon' \ominus_{\mathbf{L}} A)$.

*Proof.* Suppose the antecedent holds. Note that since $\Upsilon' \dashv\vdash_{\mathbf{L}} \Upsilon$, $\Upsilon' \vdash_{\mathbf{L}} B$. Let $\mathbb{E}_{\mathbf{L}}(\Upsilon) = \{E_i\}_{i \in I}$ be the finest **L**-splitting of $\Upsilon$, such that $\Upsilon' = \bigcup_{i \in I} \Upsilon_i$ and for each $i \in I$, $E(\Upsilon_i) \subseteq E_i$. By Lemma 2, it follows that (†) $\bigcup_{i \in I}\{\Upsilon_i \mid E(\Upsilon_i) \cap E(B) \neq \emptyset\} \vdash_{\mathbf{L}} B$.

Note that for every $i \in I$ such that $E_i \cap E(B) \neq \emptyset$, $E_i \cap E(A) = \emptyset$, in view of the supposition and Definition 2. Hence for every $C \in \Upsilon_i$ such that $E(\Upsilon_i) \cap E(B) = \emptyset$, $C$ is not relevant to the contraction of $\Upsilon$ by $A$. By Lemma 1, $C \in \Upsilon' \ominus_{\mathbf{L}} A$. Hence, $\{\Upsilon_i \mid E(\Upsilon_i) \cap E(B) \neq \emptyset\} \subseteq \Upsilon' \ominus_{\mathbf{L}} A$. By (†) and the monotonicity of **L**, $B \in Cn_{\mathbf{L}}(\Upsilon' \ominus_{\mathbf{L}} A)$. ■

**Theorem 10** *Where $\Upsilon$ is not* **L***-trivial and $\Upsilon' \in \mathbb{C}_{\mathbf{L}}(\Upsilon)$: if $B \in Cn_{\mathbf{L}}(\Upsilon)$ and $B$ is not* **L***-relevant to the revision of $\Upsilon$ by $A$, then $B \in Cn_{\mathbf{L}}(\Upsilon' \oplus_{\mathbf{L}} A)$.*

*Proof.* Immediate in view of (i) the fact that relevance modulo a revision by $A$ is equivalent to relevance modulo a contraction by $\sim A$, (ii) Definition 8, (iii) Theorem 9, and (iv) the monotonicity of **L**. ■

## 6.3 The Set of L-minimal Formulas

In [19], it is proven that the set of prime implicates of $\Upsilon$ is a **CL**-canonical form of $\Upsilon$. In the current section, I define a unique set $Min_{\mathbf{L}}(\Gamma)$ for every $\Gamma$, and prove that this set is a **L**-canonical form of $\Gamma$.[7] In view of Theorem 9 and Theorem 10, we may use this set in order to obtain a relevant belief contraction or revision. Moreover, the result below forms the basis of a proof for Theorem 7: it is shown how we may obtain the finest splitting of $\Upsilon$ from the set $Min_{\mathbf{L}}(\Gamma)$.

**Definition 9** *$A$ is a minimal* **L***-consequence of $\Gamma$, $A \in Min_{\mathbf{L}}(\Gamma)$ iff $A \in Cn_{\mathbf{L}}(\Gamma)$ and there is no $\Gamma' \subseteq Cn_{\mathbf{L}}(\Gamma)$ such that (i) $\Gamma' \vdash_{\mathbf{L}} A$ and for every $B \in \Gamma'$, $E(B) \subset E(A)$.*

Intuitively, the set $Min_{\mathbf{L}}(\Gamma)$ corresponds to the maximal level of analysis (in terms of the seperation of letters) the logic **L** allows us to perform.

**Lemma 3** $Min_{\mathbf{L}}(\Gamma) \dashv\vdash_{\mathbf{L}} \Gamma$

*Proof.* In view of Definition 9, it suffices to prove the left-right direction. Suppose $A \in \Gamma$, whence by the reflexivity of **L**, $A \in Cn_{\mathbf{L}}(\Gamma)$. I prove by an induction that $A \in Cn_{\mathbf{L}}(Min_{\mathbf{L}}(\Gamma))$. If $A \in Min_{\mathbf{L}}(\Gamma)$, then by the reflexivity of **L**, $A \in Cn_{\mathbf{L}}(Min_{\mathbf{L}}(\Gamma))$. If $A \notin Min_{\mathbf{L}}(\Gamma)$, then since $A \in Cn_{\mathbf{L}}(\Gamma)$ and by Definition 9, there is a $\Gamma' \subseteq Cn_{\mathbf{L}}(\Gamma)$, such that (i) $\Gamma' \vdash_{\mathbf{L}} A$ and (ii) for every $B \in \Gamma'$, $E(B) \subset E(A)$. For every $B \in \Gamma'$ such that $B \notin Min_{\mathbf{L}}(\Gamma)$, we repeat the same reasoning: since $B \in Cn_{\mathbf{L}}(\Gamma)$, there is a $\Gamma'' \subseteq Cn_{\mathbf{L}}(\Gamma)$ such that (i) for every $C \in \Gamma''$, $E(C) \subset E(B) \subset E(A)$ and (ii) $\Gamma'' \vdash_{\mathbf{L}} B$, whence by the transitivity and monotonicity of **L**, $(\Gamma' - \{B\}) \cup \Gamma'' \vdash_{\mathbf{L}} A$. Since $A$ contains finitely many letters, we will at a finite point arrive at a set $\Delta \subseteq Min_{\mathbf{L}}(\Gamma)$ such that $\Delta \vdash_{\mathbf{L}} A$. By the monotonicity of **L**, $Min_{\mathbf{L}}(\Gamma) \vdash_{\mathbf{L}} A$. ■

In the remainder of this section, I will prove the following:

**Theorem 11** *$Min_{\mathbf{L}}(\Gamma)$ is an* **L***-canonical form of $\Gamma$*

In view of Lemma 3, it suffices to prove that there is a splitting $\mathbb{E} = \{E_i\}_{i \in I}$ of $\mathcal{E}$, and a partition $\{\Delta_i\}_{i \in I}$ of $Min_{\mathbf{L}}(\Gamma)$ such that (1) each $E(\Delta_i) \subseteq E_i$ and (2) $\mathbb{E}$ is the finest splitting of $\Gamma$. In the remainder of this section, I will define an $\mathbb{E}$ for which it is quite easy to show that it fulfills requirement (1); by a slightly longer proof, I will arrive at (2), as stated in Theorem 12.

I first define a relation $\sim_\Delta$ over the members of $\Delta$, for every $\Delta \subseteq \mathcal{W}^\perp$:

**Definition 10** *$A$ is path-relevant to $B$ modulo $\Delta$ ($A \sim_\Delta B$) iff there are $C_1, \ldots, C_n \in \Delta$ such that $E(A) \cap E(C_1) \neq \emptyset$, $E(C_1) \cap E(C_2) \neq \emptyset$, $\ldots$, and $E(C_n) \cap E(B) \neq \emptyset$.*

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[7] The precise formulation in the definition of $Min_{\mathbf{L}}(\Gamma)$ greatly benefited from a suggestion made by David Makinson (personal correspondence).

It will be convenient to rely on the following property specific to $\sim_\Delta$ defined only over the members of $\Delta$:

**Fact 1** *$\sim_\Delta$ is transitive, reflexive and symmetric with respect to all $A, B, C \in \Delta$, whence $\sim_\Delta$ is an equivalence relation on the members of $\Delta$.*

**Definition 11** *$\mathbb{M}_{\mathbf{L}}(\Gamma)$ is the quotient set of $Min_{\mathbf{L}}(\Gamma)$ by $\sim_{Min_{\mathbf{L}}(\Gamma)}$.*[8] *Where $\mathbb{M}_{\mathbf{L}}(\Gamma) = \{\Delta_i\}_{i\in I}$, $\mathbb{E}_{\mathbf{L}}(\Gamma) = \{E(\Delta_i)\}_{i\in I} \cup \{\{A\} \mid A \in \mathcal{E} - E(Min_{\mathbf{L}}(\Gamma))\}$.*

Since $\sim_{Min_{\mathbf{L}}(\Gamma)}$ is an equivalence relation on $Min_{\mathbf{L}}(\Gamma)$, $\mathbb{M}_{\mathbf{L}}(\Gamma)$ is a partition of $Min_{\mathbf{L}}(\Gamma)$. Also, note that for no $\Delta_i \in \mathbb{M}_{\mathbf{L}}(\Gamma) : \Delta_i = \emptyset$, whence also for no $E_i \in \mathbb{E}_{\mathbf{L}}(\Gamma)$: $E_i = \emptyset$. It remains prove that $\mathbb{E}_{\mathbf{L}}(\Gamma)$ is the finest **L**-splitting of $\Gamma$.

I first prove that $\mathbb{E}_{\mathbf{L}}(\Gamma)$ is a partition of $\mathcal{E}$. This follows immediately from (1) the fact that every $E_i$ is non-empty, (2) the fact that $\bigcup \mathbb{E}_{\mathbf{L}}(\Gamma) = \mathcal{E}$, and the following lemma:

**Lemma 4** *For every $E_i, E_j \in \mathbb{E}_{\mathbf{L}}(\Gamma)$: $E_i \neq E_j$ iff $E_i \cap E_j = \emptyset$.*

*Proof.* Let $E_i, E_j \in \mathbb{E}_{\mathbf{L}}(\Gamma)$. The right-left direction is obvious since no $E_i \in \mathbb{E}_{\mathbf{L}}(\Gamma)$ is empty. For the left-right direction, suppose that for $E_i, E_j \in \mathbb{E}_{\mathbf{L}}(\Gamma)$, $E_i \cap E_j \neq \emptyset$. I only consider the case where $E_i = E(\Delta_i)$ and $E_j = E(\Delta_j)$ for $\Delta_i, \Delta_j \in \mathbb{M}_{\mathbf{L}}(\Gamma)$ – in the other case, it follows immediately that $E_i \cap E_j = \emptyset$. Suppose that $E(\Delta_i) \cap E(\Delta_j) \neq \emptyset$. This implies that there are $A \in \Delta_i, B \in \Delta_j$: $E(A) \cap E(B) \neq \emptyset$, whence $A \sim_{Min_{\mathbf{L}}(\Gamma)} B$, hence $A$ and $B$ are in the same equivalence class. As a result, $\Delta_i = \Delta_j$, whence $E_i = E_j$. ■

**Theorem 12** *$\mathbb{E}_{\mathbf{L}}(\Gamma)$ is the finest **L**-splitting of $\Gamma$.*

*Proof.* Suppose there is a splitting $\mathbb{E} = \{E_j\}_{j\in J}$ of $\Gamma$, such that $\mathbb{E}$ is finer than $\mathbb{E}_{\mathbf{L}}(\Gamma)$. Hence for some $E_k \in \mathbb{E}_{\mathbf{L}}(\Gamma)$, there is a $j \in J$: $\emptyset \subset E_j \subset E_k$. This means that $E_k$ cannot be a singleton, whence $E_k = E(\Delta_k)$ for some $\Delta_k \in \mathbb{M}_{\mathbf{L}}(\Gamma)$. So we have:

(†) For a $\Delta_k \in \mathbb{M}_{\mathbf{L}}(\Gamma)$, there is a $j \in J$: $\emptyset \subset E_j \subset E(\Delta_k)$

I will first prove that (‡) there is a $D \in \Delta_k$, for which $E(D) \cap E_j \neq \emptyset$, $E(D) \not\subseteq E_j$.

Suppose that for every $A \in \Delta_k, E(A) \subseteq E_j$. In that case, $E(\Delta_k) \subseteq E_j$, which contradicts (†). Hence there is a $A \in \Delta_k : E(A) \not\subseteq E_j$. Suppose that for every $B \in \Delta_k, E(B) \cap E_j = \emptyset$. In that case, $E(\Delta_k) \cap E_j = \emptyset$, which also contradicts (†). Hence there is a $B \in \Delta_k : E(B) \cap E_j \neq \emptyset$.

Since $A, B \in \Delta_k$, $A \sim_{Min_{\mathbf{L}}(\Gamma)} B$. Hence there are $C_1, \ldots, C_n \in \Delta_k$ such that $E(A) \cap E(C_1) \neq \emptyset, E(C_1) \cap E(C_2) \neq \emptyset, \ldots, E(C_n) \cap E(B) \neq \emptyset$. If $E(A) \cap E_j \neq \emptyset$, take $D = A$. If $E(A) \cap E_j = \emptyset$, we can infer that $E(C_1) \not\subseteq E_j$ from the fact that $E(A) \cap E(C_1) \neq \emptyset$ and that $E(C_1)$ is non-empty. We now start up recursive procedure, relying on the same reasoning:

If $E(C_l) \cap E_j \neq \emptyset$, then let $D = C_l$
If $E(C_l) \cap E_j = \emptyset$, then $E(C_{l+1}) \not\subseteq E_j$

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[8] This is the set of all equivalence sets of $Min_{\mathbf{L}}(\Gamma)$, given the equivalence relation $\sim_{Min_{\mathbf{L}}(\Gamma)}$ on $Min_{\mathbf{L}}(\Gamma)$.

This means that sooner or later, and to the latest at $B$, we arrive at a $D \in \Delta_k$, for which it holds that $E(D) \cap E_j \neq \emptyset$, $E(D) \not\subseteq E_j$.

I will now derive a contradiction from (‡). Note that since $\mathbb{E}$ is a splitting of $\Gamma$, $\mathbb{E}' = \{E_j, \bigcup \mathbb{E} - E_j\}$ is also a splitting of $\Gamma$. Hence there are $\Theta_j, \Theta$ such that $\Theta_j \cup \Theta \dashv\vdash_{\mathbf{L}} \Gamma$, $E(\Theta_j) \subseteq E_j$ and $E(\Theta) \subseteq \bigcup \mathbb{E} - E_j$. It follows that (‡) $E(\Theta_j) \cap E(\Theta) = \emptyset$. Moreover, since $\Gamma \vdash_{\mathbf{L}} D$, also $\Theta_j \cup \Theta \vdash_{\mathbf{L}} D$, whence by parallel interpolation, there are two formulae $F_j$ and $F$ such that (1) $E(F_j) \subseteq E(\Theta_j) \cap E(D)$, (2) $E(F) \subseteq (\Theta) \cap E(B)$ and (3) $\{F_j, F\} \vdash_{\mathbf{L}} D$.

Note that since $\Theta \cup \Theta_j \dashv\vdash_{\mathbf{L}} \Gamma$, also $\Gamma \vdash_{\mathbf{L}} F_j$ and $\Gamma \vdash_{\mathbf{L}} F$. By (‡), (1) and (2), $E(F_i) \subset E(D)$ and $E(F) \subset E(D)$. Hence by (3), $D \notin Min_{\mathbf{L}}(\Gamma)$ — a contradiction. ■

## 6.4 The Least Letter-set Theorem

The least letter-set theorem tells us that for every (possibly infinite) $\Gamma$, there is a unique least set of letters $\Delta \subseteq \mathcal{E}$, such that $\Gamma$ can be **CL**-equivalently expressed using only letters from $\Delta$.[9] Makinson makes the following remark on this property ([16, p. 378]):

> Intuitively, the least letter-set theorem is just what anyone would expect, but it needs proof. Getting minimal letter sets is trivial since every formula contains only finitely many letters. But getting a least one (which, by the antisymmetry of set-inclusion, will be unique) requires a bit more work.

I refer to the same paper for some more background on this theorem, and to [15, Appendix] for Makinson's (semantic) proof. Both papers are restricted to the case where $\mathbf{L} = \mathbf{CL}$. I will prove here that this theorem holds whenever $\mathbf{L}$ is reflexive, transitive, monotonic and obeys standard interpolation.[10] However, we must be careful: the exact formulation of the theorem in [15] is slightly different from the one in [16], because it is applied in a different context.[11] My formulation is a variation on the one in [16]. The proof I will present is very short, thanks to the introduction of the concept of **L**-minimality in the preceding section.

**Theorem 13** *For every $\Gamma \subseteq \mathcal{W}^{\perp}$, there is a $\Delta \subseteq \mathcal{E}$ such that (a) for every $\Gamma'$ that is **L**-equivalent to $\Gamma$: $\Delta \subseteq E(\Gamma')$ and (b) for a $\Gamma''$ that is **L**-equivalent to $\Gamma$, $\Delta = E(\Gamma'')$. (Least letter-set Theorem)*

*Proof.* Let $\Delta = E(Min_{\mathbf{L}}(\Gamma))$. (b) follows immediately by the construction and Lemma 3; hence it suffices to prove (a). Suppose (1) $\Gamma' \dashv\vdash_{\mathbf{L}} \Gamma$, but $\Delta \not\subseteq E(\Gamma')$. Hence there is a $A \in Min_{\mathbf{L}}(\Gamma)$: $E(A) \not\subseteq E(\Gamma')$, whence also (2) $E(A) \cap E(\Gamma') \subset$

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[9] For the finite case, the proof of the theorem is almost trivial whenever standard interpolation is available, as explained in the appendix of [15].

[10] As Makinson pointed out to me (personal correspondence), a weaker kind of interpolation suffices to obtain the least letter-set theorem along the lines of my proof: if $\Gamma \vdash_{\mathbf{L}} A$, then there is a $\Gamma'$ such that (i) $\Gamma \vdash_{\mathbf{L}} \Gamma'$, (ii) $\Gamma' \vdash_{\mathbf{L}} A$ and (iii) $E(\Gamma') \subseteq E(\Gamma) \cap E(A)$ *(Non-compact Interpolation)*.

[11] In [15], a specific set $\Gamma^*$ is defined for every $\Gamma$, and it is shown that this set is a least letter-set representation of $\Gamma$. $\Gamma^*$ is defined in semantic terms, and the proof proceeds likewise. On the other hand, the formulation of the least letter-set theorem in [16] is a "bare statement of existence" (Makinson, personal correspondence), without reference to any specific least letter-set representation.

$E(A)$. By (1) and Definition 9, $\Gamma' \vdash_{\mathbf{L}} A$. By interpolation, there is a $B$ such that (3) $\Gamma' \vdash_{\mathbf{L}} B$, (4) $B \vdash_{\mathbf{L}} A$ and (5) $E(B) \subseteq E(\Gamma') \cap E(A)$. By (1) and (3), it follows that $B \in Cn_{\mathbf{L}}(\Gamma)$, and by (2) and (5), it follows that $E(B) \subset E(A)$. But then by (3) and in view of Definition 9, $A \notin Min_{\mathbf{L}}(\Gamma)$ — a contradiction. ■

## 7 Further Research

In view of the setup of the current paper, the following topics for further research require little explanation:

- Is it possible to further generalize the results from this paper to other subclassical logics such as e.g. Priest's paracosistent logic **LP**, Brazilian anti-intuitionistic logic or the very powerful system $\mathbf{CL}^-$ from [24]?[12]

- Is it possible to develop adaptive logics for **L**-relevant belief revision or contraction, in a similar vain as the adaptive logics for **CL**-relevant belief revision from [20], but where **L** is a paraconsistent or paracomplete logic?

Another interesting topic would be the possibility of a non-monotonic (corrective) finest splitting. Within the adaptive logic programm, quite a few systems have been developed that allow one to interpret a set of beliefs "as consistently as possible", without trivializing inconsistent beliefs.[13] Some of these systems are equivalent to **CL** whenever the belief set is consistent, and most of them are usually much stronger than the monotonic systems from the current paper. It would hence be worthwhile to see whether such non-monotonic logics also yield a finest splitting for every belief set – in that case, the associated relevance axiom would be very strong, but it would still not trivialize inconsistent belief sets.

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[12] To the best of my knowledge, no interpolation results are available for these systems, which makes them tougher candidates to prove the finest splitting theorem for.

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