Three Formats of Prioritized Adaptive Logics: 
a Comparative Study

Frederik Van De Putte & Christian Straßer
Centre for Logic and Philosophy of Science
Ghent University
Blandijnberg 2, 9000 Gent, Belgium
{frvdeput.vandeputte,christian.strasser}@UGent.be

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Abstract

A broad range of defeasible reasoning forms has been explicat ed by prioritized adaptive logics. However, the relative lack in meta-theory of many of these logics stands in sharp contrast to the frequency of their application. This paper presents the first comparative study of a large group of prioritized adaptive logics. Three formats of such logics are discussed: superpositions of adaptive logics, hierarchic adaptive logics from [20] and logics in the \textit{AL}_c-format from [24]. We restrict the scope to logics that use the strategy Minimal Abnormality. It is shown that the semantic characterizations of these systems are equivalent and that they are all sound with respect to either of these characterizations. Furthermore, sufficient conditions for the completeness and equivalence of the consequence relations of the three formats are established. Some attractive properties, including Fixed Point and the Deduction Theorem, are shown to hold whenever these conditions are obeyed.

1 Introduction

Prioritized Adaptive Logics. Adaptive logics (henceforth ALs) are formal systems that model and explicate various forms of human reasoning: reasoning with inconsistent premises [1], inductive generalization [6, 4], abduction [13, 11], reasoning on the basis of conflicting norms [10, 14], argumentation [19], belief revision [25], etc.\footnote{Unpublished papers in the reference section are available at the internet address http://logica.UGent.be/centrum/writings/} Many consequence relations from the literature have been reformulated as ALs, see e.g. [7, 9, 16, 27]. These achievements underline the strength of ALs as formal modeling tools and the unificatory power of the adaptive logic program.

ALs are first and foremost developed to capture defeasible reasoning forms (DRFs), i.e. reasoning forms in which certain inferences may be retracted in view of later insights. One of the most important developments within the AL program is the definition of a canonical format, the so-called standard format
for ALs. This format encompasses a dynamic proof theory and semantics. A rich and attractive meta-theory has been shown to hold generically for all ALs formulated in the standard format (see [3]): they are sound and complete, their consequence relation is idempotent, cautiously monotonic, etc. Most ALs have been successfully expressed within this format, whence it provides a good basis for a unifying study of DRFs.

Every AL in standard format is characterized by a triple: (i) a lower limit logic (henceforth LLL), (ii) a set of abnormalities \( \Omega \) and (iii) a strategy. The LLL is a monotonic logic, the rules of which are unconditionally valid in the AL. The AL strengthens its LLL by considering a certain set of formulas (the elements of \( \Omega \)) as abnormal, and by interpreting premises “as normally as possible”. Semantically, this is realized by means of a selection on the set of LLL-models, in the vein of Shoham [15]. How this selection proceeds, depends on the strategy of the AL, which can be either Reliability or Minimal Abnormality.

In this paper, we will confine ourselves to the latter strategy. In this case, the AL selects the LLL-models that verify a minimal set of abnormalities – the details will be spelled out in Section 2.

Notwithstanding its successes, the standard format does not incorporate prioritized ALs. These are elegant tools to model specific kinds of prioritized defeasible reasoning, i.e. the kind of defeasible reasoning in which defeasible assumptions with different degrees of priority play a role. Examples of such DRFs are: reasoning with prioritized belief bases [8] or imperatives [23]; inductive generalization with a preference for the strongest hypotheses [4]; ampliative reasoning supported by background knowledge [6]; inductive generalization of abduced hypotheses [22].

The most common way to deal with prioritized DRFs within the adaptive logic program is by the superposition of several ALs in standard format. Roughly speaking, this is done as follows: where \( \text{AL}_1, \text{AL}_2, \ldots \) are ALs in standard format that have the same LLL, and where \( Cn_{\text{AL}_i}(\Gamma) \) denotes the \( \text{AL}_i \)-consequence set of \( \Gamma \), we characterize the superposition of logics, \( \text{SAL} \), by

\[
Cn_{\text{SAL}}(\Gamma) = \ldots Cn_{\text{AL}_3}(Cn_{\text{AL}_2}(Cn_{\text{AL}_1}(\Gamma))) \ldots
\]

In [2], a semantics for these systems was proposed, in terms of a sequential selection of models. For Minimal Abnormality, this semantics is defined as follows: first we select the LLL-models of \( \Gamma \) that are minimally abnormal with respect to \( \text{AL}_1 \); from the resulting set of models, we select those that are minimally abnormal with respect to \( \text{AL}_2 \), etc (see Section 3 for the precise definitions).

Although for some concrete applications, superpositions of ALs were equipped with a proof theory (see e.g. [19, 6, 18]), there has been no attempt so far to formulate a generic proof theory for \( \text{SAL} \) and to prove that its derivability relation coincides with the consequence relation described above, or is sound and complete with respect to the intended semantics of \( \text{SAL} \).

More generally, there has been a substantial lack of meta-theory on these combinations of ALs. The results that have been obtained so far are fairly negative for Minimal Abnormality: superpositions of ALs that have the Minimal Abnormality Strategy lack many of the aforementioned properties which render

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2We refer to [3] for the definition of Reliability.

3In [5], a first proposal is made for such a generic proof theory. However, for superpositions of ALs that use the Minimal Abnormality Strategy, the derivability relation characterized by this proof theory does not coincide with the \( \text{SAL} \)-consequence relation as defined above.
the standard format so attractive. To mention but a few: their consequence relation lacks both soundness and completeness with respect to the above semantic characterization; it is neither idempotent, nor cautiously monotonic.\footnote{As we will see in Section 4.2, the logics $\mathbf{AL}_1, \mathbf{AL}_2, \ldots$ have to fulfill certain specific restrictions in order to get a well-behaved hierarchic logic – we refer to \cite{20} for more details.}

Another way to capture prioritized DRFs by means of a combination of ALs in standard format was put forward in \cite{20} under the name \textit{hierarchic adaptive logics}. There the central idea was to take the union of the consequence sets of several ALs. Where each logic $\mathbf{AL}_i$ has $\mathbf{LLL}$ as its lower limit logic, this combination is defined as follows:\footnote{In a sense, all formats for prioritized ALs are generalizations of the standard format: if there is only one priority level, the “prioritized” AL reduces to an AL in standard format. However, the generalization which leads to $\mathbf{AL}_{\preceq}$ is more fundamental, in that the crucial concepts of the standard format are generalized, in order to give the defeasibility a prioritized flavor – see \cite{24} for the details.}

\[
\text{Cn}_{\text{HAL}}(\Gamma) = \text{Cn}_{\text{LLL}}(\text{Cn}_{\text{AL}_1}(\Gamma) \cup \text{Cn}_{\text{AL}_2}(\Gamma) \cup \ldots)
\]

Semantically, the set of $\text{HAL}$-models of $\Gamma$ is the intersection of all the sets of $\mathbf{AL}_i$-models. A generic proof theory was defined for $\text{HAL}$, and proven to be adequate with respect to $\text{Cn}_{\text{HAL}}(\Gamma)$ in \cite{20}. Nevertheless, the hierarchic format suffers from the same meta-theoretic problems as superpositions of ALs: no fixed point, no completeness with respect to a non-redundant semantics, etc (see \cite{20} where these results are documented).

In \cite{24}, the format $\mathbf{AL}_{\preceq}$ for prioritized adaptive logics is presented, and it is proven that the meta-theoretic properties of the standard format easily generalize to this new format.\footnote{See \cite[Chapter 6]{5} for more details.} We refer to Section 2 for an overview of some technical results from \cite{24}. As for flat ALs, every logic in $\mathbf{AL}_{\preceq}$-format is characterized by a triple: an $\mathbf{LLL}$, a \textit{sequence of sets of abnormalities} $\langle \Omega_1, \Omega_2, \ldots \rangle$ and a strategy: $\sqsupset$-Reliability or $\sqsupset$-Minimal Abnormality. The semantics of $\mathbf{AL}_{\preceq}$-logics that use $\sqsupset$-Minimal Abnormality, is obtained by a lexicographic selection procedure – see Section 2.1 for more details.

In the remainder, we use the superscript $m$ to refer to the Minimally Abnormality-variants of the three aforementioned formats: $\mathbf{AL}_m^m$, $\mathbf{SAL}_m^m$ and $\mathbf{HAL}_m$.

\section*{Content and Outline of This Paper.} We will first provide a general characterization of flat adaptive logics and logics in the $\mathbf{AL}_m^m$-format, and introduce some notational conventions for the rest of the paper (Section 2). Next, we will show that three classes of prioritized ALs (logics in $\mathbf{AL}_m^m$-format, those in $\mathbf{HAL}_m$-format and logics in a specific subclass $\mathbf{SAL}_m^c$ of superposition-ALs) stand in a one-to-one relation, i.e. that for every triple of associated logics $\mathbf{AL}_m^m$, $\mathbf{HAL}_m$ and $\mathbf{SAL}_m^c$ each of the following holds:

\begin{enumerate}
\item The semantic consequence relations of all three systems are equivalent (Section 3, Corollary 1)
\item The consequence relation of $\mathbf{AL}_m^m$ is always at least as strong as that of $\mathbf{SAL}_m^c$ and $\mathbf{HAL}_m$ (Section 4, Corollaries 6 and 7)
\item $\mathbf{SAL}_m^c$ and $\mathbf{HAL}_m$ are complete and equivalent to $\mathbf{AL}_m^m$, given certain (weak) restrictions on the premise sets (Section 5, Corollaries 11 and 12)
\end{enumerate}
To arrive at (ii), we will prove that every logic in the format of \( \text{SAL}^m \) is sound with respect to its semantics (see Section 4, Theorem 15.2).

It is worthwhile to stress that all the above results are proven generically, i.e. the meta-proofs only rely on the properties of the formats, not on particularities of the logics defined in these formats. Property (i) contributes to the argument that the semantic consequence relation defined by these logics and, in view of the soundness and completeness of \( \text{AL}^m \), also the \( \text{AL}^m \)-consequence relation is a robust concept in the context of prioritized consequence relations.\(^7\) (ii) is of particular interest in view of the fact that \( \text{AL}^m, \text{SAL}^m \) and \( \text{HAL}^m \) are each characterized by their own peculiar semantics and proof theory. This means that we obtain a great variety of methods to prove that a formula is an \( \text{AL}^m \)-consequence of a set of premises.

In Appendix C, we show that (iii) fails in cases where the restrictions referred to in (iii) are not obeyed. However, as shown in Section 6, whenever the restrictions hold, properties such as Fixed Point and the Deduction Theorem can be easily transferred from \( \text{AL}^m \) to \( \text{HAL}^m \) and \( \text{SAL}^m \), relying on (i)-(iii).

The current paper focuses mainly on the semantic characterizations, resp. the properties of the consequence relations of \( \text{AL}^m, \text{SAL}^m \) and \( \text{HAL}^m \). Since consequence relation of flat ALs and logics in \( \text{AL}^m \)-format is defined in terms of their respective proof theory, we define the latter in the appendix to make this paper self-contained. Note however that the consequence relation of \( \text{SAL}^m \) and \( \text{HAL}^m \) are defined without any reference to proof theories. Hence, we will not spell out the generic proof theory of \( \text{HAL}^m \) in this paper (see [20]), and we consider that of \( \text{SAL}^m \) as a topic for future research. Also, we refer to [4, 5, 23, 17, 20, 24] for concrete applications of each of the aforementioned formats.

2 Flat and Prioritized Adaptive Logics

In this section, we introduce and define the format \( \text{AL}^m \) of flat ALs and the format \( \text{AL}^m \) of prioritized ALs from [24]. We restrict ourselves to the semantics of both formats (the proof theories are defined in Appendix A.1). We will first provide the official semantics of the systems (Section 2.1), and next define an alternative characterization of their sets of models (Section 2.2). The latter will turn out very useful for certain meta-proofs in the remainder of this paper. After that, we will briefly discuss normal premise sets, i.e. premise sets that do not give rise to any abnormalities (Section 2.3). Finally, we introduce some notions which will facilitate the meta-proofs in the remainder of this paper (Section 2.4). Unless specified differently, all results from this section are established and illustrated in detail in [3], [5] and [24] – all metaproofs can be found there.

Before we start, let us introduce some conventions.

Throughout this paper, all formulas are assumed to be finite strings in a given formal language. We will use \( A, B, C, \ldots \) as metavariables for formulas, and \( \Gamma, \Delta, \Theta, \ldots \) as metavariables for sets of formulas. Where \( \mathbb{N} \) is the set of naturals numbers without 0, we will use \( i, j, k, \ldots \) as metavariables for members of \( \mathbb{N} \), and \( I, J, K, \ldots \) as metavariables for initial sequences of \( \mathbb{N} \). Where \( I = \{1, \ldots, n\} \), let \( \vec{I} =_{\text{def}} n \); where \( I = \mathbb{N} \), let \( \vec{I} =_{\text{def}} \infty \). Let \( L \) be a logic with a characteristic

\(^7\)By the “robustness” of a concept, we mean that it can be found in many different contexts, under various names and/or characterizations.
semantics. Where $M$ is a $L$-model, we write $M \vDash A$ to denote that $A$ is verified by $M$. $M$ is a $L$-model of $\Gamma$ iff $M \vDash A$ for all $A \in \Gamma$. The set of $L$-models of $\Gamma$ is denoted by $M_L(\Gamma)$. We say that $A$ is a semantic $L$-consequence of $\Gamma$, $\Gamma \models L A$ if $A$ is verified by all $L$-models of $\Gamma$.

2.1 The Official Semantics

Every (flat or prioritized) adaptive logic is based on a lower limit logic $LLL^+$. $LLL^+$ is required to be compact, transitive, reflexive and monotonic. The superscript “+” refers to the fact that the lower limit logic is obtained by enriching a monotonic logic $LLL$ with classical connectives that are indicated by a check mark: $\neg, \vee, \wedge, \supset$, and for the predicative case also $\exists, \forall$. This enrichment is motivated by technical, meta-theoretic and philosophical reasons – we refer to [24, Section 2.1] where these are spelled out.

Where $W$ is the set of formulas in the language $L$ of $LLL$, $W_+$ is obtained by closing $W$ under the checked connectives. Intuitively, every AL models a reasoning process based on formulas in $W$, but uses formulas in $W_+$ to explain this reasoning process. Hence, although an AL is defined as a function $\wp(W_+) \to \wp(W_+)$, in concrete applications, premise sets are usually subsets of $W$. Nevertheless, for metatheoretic purposes (e.g. when considering the superposition of consequence relations), it is easier to let $\Gamma$ refer to any subset of $W_+$. Unless specified differently, we will do this in the remainder.

The AL strengthens $LLL^+$, by considering formulas of a certain form false “as much as possible”. These formulas are called abnormalities. So a necessary ingredient of any AL is a set of abnormalities $\Omega$, which is a set of formulas specified by a logical form in $L_+$. A flat AL considers each abnormality to be “equally bad”. For instance, if $\Gamma = \{A \vee B\}$, where $A, B \in \Omega$ and where $\vee$ behaves classically, then neither $\neg A$, nor $\neg B$ will be an adaptive consequence of $\Gamma$.

Prioritized ALs are also defined in terms of $LLL^+$ and a set of abnormalities $\Omega$, but this time $\Omega$ is further specified in terms of a sequence of sets, each associated with a certain priority level $i \in I$. In other words, we consider not one set of abnormalities, but a (possibly infinite) sequence of such sets: $\langle \Omega_i \rangle_{i \in I}$, which yields a prioritized set of abnormalities $\Omega = \bigcup_{i \in I} \Omega_i$. Abnormalities of level 1 are treated as the “worst” abnormalities by the prioritized AL, those of level 2 as the “second worst”, and so on. Hence the logic first tries to avoid the abnormalities from $\Omega_1$, next those from $\Omega_2$, etc. Suppose that in the above example, $A \in \Omega_1$ and $B \in \Omega_2 - \Omega_1$. In that case, $\neg A$, and hence also $B$ will be a consequence of $\Gamma$.

At the semantic level, every AL selects a subset of the set of $LLL^+$-models, in view of the abnormalities they verify. This requires some notational conventions. We first define the abnormal part of an $LLL^+$-model $M$: $Ab(M) = \{B \in \Omega \mid M \vDash B\}$. In $AL_{\subseteq}$, the abnormal parts of models are compared in terms of the subset-relation $\subseteq$. In $AL_{\subset}$, they are compared in terms of a lexicographic order $\subset$, which is defined as follows.\(^9\)

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\(^8\)This means that in $W_+$, checked connectives cannot occur in the scope of the connectives from $L$.

\(^9\)Lexicographic orders are well-known in mathematics (see e.g. [12, p. 1170]).
Definition 1 Where \( \Delta, \Delta' \subseteq \Omega \): \( \langle \Delta \cap \Omega \rangle_{i \in I} \sqsubset_{\text{lex}} \langle \Delta' \cap \Omega \rangle_{i \in I} \) iff (1) there is an \( i \in I \) such that for all \( j < i \), \( \Delta \cap \Omega_j = \Delta' \cap \Omega_j \), and (2) \( \Delta \cap \Omega_i \subset \Delta' \cap \Omega_i \). We write \( \Delta \sqsubset \Delta' \) iff \( \langle \Delta \cap \Omega \rangle_{i \in I} \sqsubset_{\text{lex}} \langle \Delta' \cap \Omega \rangle_{i \in I} \).

Let \( \prec \) be a metavariable for \( \sqsubset \) and \( \sqsubseteq \). Then the logic \( \text{AL}_m \) selects the \( \text{LLL}^+ \)-models of \( \Gamma \) whose abnormal part is \( \prec \)-minimal:

Definition 2 \( M \in \mathcal{M}_{\text{AL}_m}(\Gamma) \) iff \( M \in \mathcal{M}_{\text{LLL}^+}(\Gamma) \) and there is no \( M' \in \mathcal{M}_{\text{LLL}^+}(\Gamma) \) such that \( \text{Ab}(M') \prec \text{Ab}(M) \).

With the above definitions and the regular definition of semantic consequence \( (\Gamma \models \Delta \text{ if } A \text{ is true in every } M \in \mathcal{M}_{\Gamma}(\Delta)) \), we obtain the semantic consequence relations \( \models_{\text{AL}_m} \) and \( \models_{\text{AL}_m} \). As pointed out in Section 1, the standard format and the \( \text{AL}_m \)-format also encompass a proof theory which yields a syntactic consequence relation \( \vdash_{\text{AL}_m} \), that is sound and complete with respect to \( \models_{\text{AL}_m} \). We refer to Appendix A.1 for this proof theory. We define \( A \in Cn_{\text{AL}_m}(\Gamma) \) iff \( \Gamma \vdash_{\text{AL}_m} A \).

We continue this section with some theorems and a corollary, each of which will be referred to in the remainder of this paper. Recall that, unless stated differently, \( \Gamma \subseteq W_i \):

Theorem 1 Each of the following holds:

1. If \( \Gamma \vdash_{\text{AL}_m} A \), then \( \Gamma \models_{\text{AL}_m} A \).
2. Where \( \Gamma \subseteq W \): if \( \Gamma \models_{\text{AL}_m} A \), then \( \Gamma \vdash_{\text{AL}_m} A \).

Theorem 2 If \( M \in \mathcal{M}_{\text{LLL}^+}(\Gamma) \) and \( M \in \mathcal{M}_{\text{AL}_m}(\Gamma) \), then there is a \( M' \in \mathcal{M}_{\text{AL}_m}(\Gamma) \) such that \( \text{Ab}(M') \prec \text{Ab}(M) \). (Strong Reassurance)

Theorem 3 Where \( \Gamma \subseteq W \): \( Cn_{\text{AL}_m}(\Gamma) = Cn_{\text{LLL}^+}(\Gamma) \). (LLL-closure)

Theorem 4 Where \( \text{AL}_m \) is defined by \( (\text{LLL}^+, \bigcup_{i \in I} \Omega_i, m) \) and \( \text{AL}_m \) is defined by \( (\text{LLL}^+, \Omega_i)_{i \in I} \):

1. \( \mathcal{M}_{\text{AL}_m}(\Gamma) \subseteq \mathcal{M}_{\text{AL}_m}(\Gamma) \)
2. \( Cn_{\text{AL}_m}(\Gamma) \subseteq Cn_{\text{AL}_m}(\Gamma) \)

The last result which we mention here is proven in Appendix A.2. It states that for a very specific kind of premise sets \( \Gamma \subseteq W_i \), the logic \( \text{AL}_m \) is sound and complete as well:¹⁰

Theorem 5 Where \( \Gamma = Cn_{\text{LLL}^+}(\Gamma) \): \( \Gamma \vdash_{\text{AL}_m} A \) iff \( \Gamma \models_{\text{AL}_m} A \).

2.2 The Alternative Characterization of \( \mathcal{M}_{\text{AL}_m}(\Gamma) \)

The alternative characterization of the set of \( \text{AL}_m \)-models of a given \( \Gamma \) is based on the set of minimal disjunctions of abnormalities that are \( \text{LLL}^+ \)-derivable from \( \Gamma \). For flat ALs, this characterization is well-known from the metatheory of the standard format – see e.g. [3] where it is established and applied; it was generalized to the \( \text{AL}_m \)-format in [24]. Where \( \Delta \) is a finite subset of \( \Omega \), let \( \text{Dab}(\Delta) =_{\text{def}} \bigvee \Delta \). Where \( \Delta = \{ A \} \), \( \text{Dab}(\Delta) \) denotes \( A \); where \( \Delta = \emptyset \),

¹⁰We call upon Theorem 5 in Sections 4 (Lemma 2) and 5 (Lemma 6).
\[
\check{\vee} \text{Dab}(\Delta) \text{ denotes the empty string. Dab}(\Delta) \text{ is a minimal Dab-consequence of } \\
\Gamma \text{ if } \Gamma \vdash_{\text{LLL}^+} \text{Dab}(\Delta) \text{ and there is no } \Delta' \subset \Delta \text{ for which } \Gamma \vdash_{\text{LLL}^+} \text{Dab}(\Delta'). \\
\text{Where } \{\text{Dab}(\Delta_j) \mid j \in J \} \text{ are the minimal Dab-consequences of } \Gamma, \text{ let } \Sigma(\Gamma) = \{\Delta_j \mid j \in J\}.
\]

Let \(\Psi = \{\Delta_k \subseteq \Omega \mid k \in K\}\). We say that \(\varphi \subseteq \Omega\) is a choice set of \(\Psi\) iff for every \(k \in K\), \(\varphi \cap \Delta_k \neq \emptyset\). For the border case where \(\Psi = \emptyset\), this means that every set \(\varphi \subseteq \Omega\) is a choice set of \(\Psi\), including the empty set.

The following two facts give two salient properties of choice sets of \(\Sigma(\Gamma)\) and their relation to \(\text{LLL}^+\)-models of \(\Gamma\):

**Fact 1** Where \(M \in \mathcal{M}_{\text{LLL}^+}(\Gamma)\), \(\text{Ab}(M)\) is a choice set of \(\Sigma(\Gamma)\).

**Fact 2** If \(\Gamma\) has \(\text{LLL}^+\)-models, then for every choice set \(\varphi\) of \(\Sigma(\Gamma)\), there is an \(\text{LLL}^+\)-model \(M\) of \(\Gamma\) such that \(\text{Ab}(M) \subseteq \varphi\).

Just as we did with the abnormal parts of models, we may rank the choice sets of \(\Sigma(\Gamma)\) according to the partial orders \(\subset\) and \(\sqsubseteq\). As before, let \(\prec\) be a meta-variable for these partial orders. We say that \(\varphi\) is a \(\prec\)-minimal choice set of \(\Psi\) iff there is no choice set \(\psi\) of \(\Psi\) such that \(\psi \prec \varphi\).

**Definition 3** \(\Phi^\prec(\Gamma)\) is the set of \(\prec\)-minimal choice sets of \(\Sigma(\Gamma)\).

So, in view of the minimal disjunctions of abnormalities that are \(\text{LLL}^+\)-derivable from \(\Gamma\), we obtain a set of sets of abnormalities \(\Phi^\prec(\Gamma)\). The following theorem provides the link between \(\Phi^\prec(\Gamma)\) and the set of \(\text{AL}^m\)-models of \(\Gamma\):

**Theorem 6** Each of the following holds:

1. \(M \in \mathcal{M}_{\text{AL}^m}(\Gamma)\) iff \((M \in \mathcal{M}_{\text{LLL}^+}(\Gamma) \text{ and } \text{Ab}(M) \in \Phi^\prec(\Gamma))\)
2. If \(\Gamma\) has \(\text{LLL}^+\)-models, then \(\Phi^\prec(\Gamma) = \{\text{Ab}(M) \mid M \in \mathcal{M}_{\text{AL}^m}(\Gamma)\}\)

In view of this correspondence between \(\mathcal{M}_{\text{AL}^m}(\Gamma)\) and \(\Phi^\prec(\Gamma)\), Theorem 4.1 has the following counterpart:

**Theorem 7** \(\Phi^\subset(\Gamma) \subseteq \Phi^\prec(\Gamma)\).

In view of Theorem 6, we can establish a necessary and sufficient condition for the membership of \(\text{Cn}_{\text{AL}^m}(\Gamma)\), whenever \(\Gamma \subseteq \mathcal{W}\):

**Theorem 8** Where \(\Gamma \subseteq \mathcal{W}\): \(A \in \text{Cn}_{\text{AL}^m}(\Gamma)\) iff for every \(\varphi \in \Phi^\prec(\Gamma)\), there is a \(\Delta \subseteq \Omega - \varphi\) such that \(\Gamma \vdash_{\text{LLL}^+} A \check{\vee} \text{Dab}(\Delta)\).

The left-right direction of the above theorem also holds for the more general case where \(\Gamma \subseteq \mathcal{W}_+\):

**Theorem 9** If \(A \in \text{Cn}_{\text{AL}^m}(\Gamma)\), then for every \(\varphi \in \Phi^\prec(\Gamma)\), there is a \(\Delta \subseteq \Omega - \varphi\) such that \(\Gamma \vdash_{\text{LLL}^+} A \check{\vee} \text{Dab}(\Delta)\).
2.3 Normal Premise Sets

Every flat adaptive logic in standard format has a unique upper limit logic ULL, which is the logic that trivializes all abnormalities of the AL. Semantically, ULL is obtained as follows. We say that $M \in \mathcal{M}_{\text{ULL}^+}(\Gamma)$ is normal iff $M \not\models A$ for every $A \in \Omega$. We write $\Gamma \models_{\text{ULL}} A$ iff $A$ is true in every normal $\text{ULL}^+$-model of $\Gamma$. Similarly, for the prioritized logic $\text{AL}_{\text{m}}^c = (\text{LLL}^+, (\Omega_i)_{i \in I}, m)$, we obtain the associated ULL by defining normal models as those $\text{LLL}^+$-models that falsify every member of $\Omega = \bigcup_{i \in I} \Omega_i$. The following is immediate:

**Fact 3** $\mathcal{M}_{\text{ULL}}(\Gamma) \subseteq \mathcal{M}_{\text{AL}_{\text{m}}^c}(\Gamma)$

Where $\Theta \subseteq W_+$, let $\Theta^\frown = \{ \neg A \mid A \in \Theta \}$. Syntactically, we define $\text{Cn}_{\text{ULL}}(\Gamma) = \text{def} \ \text{Cn}_{\text{LLL}^+}(\Gamma \cup \Omega^\frown)$. In [3], it is shown that ULL is a monotonic logic that is sound and complete. Hence by Fact 3 and Theorem 1.1:

**Theorem 10** $\text{Cn}_{\text{AL}_{\text{m}}^c}(\Gamma) \subseteq \text{Cn}_{\text{ULL}}(\Gamma)$.

We say that a premise set $\Gamma$ is normal iff it has normal models; alternatively, iff $\Gamma \cup \Omega^\frown$ is $\text{LLL}^+$-satisfiable. The following is proven in [24]:

**Theorem 11** If $\Gamma$ is normal, then $\text{Cn}_{\text{AL}_{\text{m}}^c}(\Gamma) = \text{Cn}_{\text{ULL}}(\Gamma) = \text{Cn}_{\text{LLL}^+}(\Gamma \cup \Omega^\frown)$.

In other words, if it is possible to avoid all abnormalities, then the adaptive logic will do so. In Section 4.3, we will see that this result can be extended to the formats $\text{SAL}_{\text{m}}^c$ and $\text{HAL}_{\text{m}}^c$. There we will also formulate a stronger, prioritized variant of this theorem and prove it for each of $\text{AL}_{\text{m}}^c$, $\text{SAL}_{\text{m}}^c$ and $\text{HAL}_{\text{m}}^c$.

2.4 Some Notational Conventions

To close the gap between, on the one hand, combinations in terms of flat adaptive logics, and on the other hand logics in the $\text{AL}_{\text{m}}^c$-format, it will be convenient to define logics that only consider abnormalities up to a certain rank $i \in I$. For every $i \in I$, let $\Omega_{(i)} = \Omega_1 \cup \ldots \cup \Omega_i$. We define the lexicographic order up to level $i$ as follows:

**Definition 4** $(\Delta \cap \Omega_j)_{j \leq i} \subseteq \cap_{(i)\leq}(\Delta' \cap \Omega_j)_{j \leq i}$ iff (1) there is a $j \leq i$ such that for all $k < j$, $\Delta \cap \Omega_k = \Delta' \cap \Omega_k$, and (2) $\Delta \cap \Omega_j \subseteq \Delta' \cap \Omega_j$.

We write $\Delta \preceq (i) \Delta'$ iff $(\Delta \cap \Omega_j)_{j \leq i} \subseteq \cap_{(i)\leq}(\Delta' \cap \Omega_j)_{j \leq i}$.

We define the prioritized adaptive logic $\text{AL}_{\text{m}}^c_{(i)}$ just as $\text{AL}_{\text{m}}^c$, but replacing the whole sequence $(\Omega_i)_{i \in I}$ by $(\Omega_j)_{j \leq i}$, and replacing $\subseteq$ by $\preceq$. Similarly, we characterize the flat adaptive logic $\text{AL}_{\text{m}}^c_{(i)}$ by the triple $(\text{LLL}, \Omega_{(i)}, m)$. Note that since these logics are all defined in $\text{AL}_{\text{m}}^c$-format, all the theorems mentioned in Section 2 hold for them as well.

In a similar vain as for $\text{AL}_{\text{m}}^c$, we can characterize the $\text{AL}_{\text{m}}^c_{(i)}$-semantics in terms of minimal choice sets. We say that $\text{Dab}(\Delta)$ is a minimal $\text{Dab}_{(i)}$-consequence of $\Gamma$ iff $\Gamma \models_{\text{LLL}^+} \text{Dab}(\Delta)$ and there is no $\Delta' \subset \Delta$ such that $\Gamma \models_{\text{LLL}^+} \text{Dab}(\Delta)$. Where $\{ \text{Dab}(\Delta_j) \mid j \in J \}$ are the minimal $\text{Dab}_{(i)}$-consequences of $\Gamma$, let $\Sigma_{(i)}(\Gamma) = \{ \Delta_j \mid j \in J \}$. Let $\Phi_{(i)}(\Gamma)$ be the set of $\cap_{(i)}$-minimal choice sets of $\Sigma_{(i)}(\Gamma)$ and let $\Phi_{(i)}(\Gamma)$ be the set of $\cap$-minimal choice sets of $\Sigma_{(i)}(\Gamma)$.
Fact 4 Let \( \Delta, \Delta' \subseteq \Omega \). Then \( \Delta \cap \Omega \subseteq \Delta' \cap \Omega \) iff \( \Delta \subseteq (1) \Delta' \). Hence \( Cn_{\text{AL}_m^p}(\Gamma) = Cn_{\text{AL}_{1(1)}^m}(\Gamma) \), \( M_{\text{AL}_m^m}(\Gamma) = M_{\text{AL}_{1(1)}^m}(\Gamma) \), and \( \Phi(\Gamma) = \Phi_{\text{AL}_{1(1)}^m}(\Gamma) \).

In view of the definitions of \( \subseteq (i) \), \( \text{AL}_{1(1)}^m \) and \( \Phi_{\text{AL}_{1(1)}^m}(\Gamma) \), we have:

Fact 5 Each of the following holds for every \( i \in I \):

1. \( \Delta \subseteq (i) \Delta' \) iff \( \Delta \cap \Omega(i) \subseteq \Delta' \cap \Omega(i) \)
2. If \( \Delta \subseteq (i) \Delta' \), then \( \Delta \subseteq \Delta' \)
3. If \( \Delta \subseteq (i) \Delta' \), then \( \Delta \subseteq (j) \Delta' \) for every \( j \in I \), \( i \leq j \)
4. Where \( \Gamma \subseteq W \): \( Cn_{\text{AL}_{1(1)}^m}(\Gamma) \subseteq Cn_{\text{AL}_m^m}(\Gamma) \).

In the remainder of this paper, we will skip the sub- and superscript \( \subseteq \), in order to facilitate the reading and to stay as close as possible to the notational conventions of the adaptive logics program. Hence we will write \( \Phi(\Gamma) \) instead of \( \Phi_{\text{AL}_{1(1)}^m}(\Gamma) \) and \( \text{AL}_m^m \) instead of \( \text{AL}_{1(1)}^m \).

3 Two equivalent Semantic Characterizations

In this section we will define the semantics for superposed adaptive logics and for hierarchic adaptive logics. As explained in Section 1, both formats are defined in terms of a combination of a sequence of flat ALs \( \langle \text{AL}_1^m, \text{AL}_2^m, \ldots \rangle \). As a central result we will show that for a specific class of such sequences, the semantics of \( \text{SAL}_m^m \) and \( \text{HAL}_m^m \) are equivalent.

3.1 Superposition of Selections

In the introduction we have already informally explicated the idea of the sequential selections that constitute the semantics of \( \text{SAL}_m^m \). We will now make this idea formally precise. Where \( \text{SAL}_m^m \) is characterized by the sequence of flat adaptive logics \( \langle \text{AL}_1^m \rangle_{i \in I} \), its semantics is defined as follows:

Definition 5 Let \( M_{\text{SAL}_m^m}(\Gamma) = M_{\text{LLL}^+}(\Gamma) \). For every \( i \in I \), let \( M_{\text{SAL}_m^m}(\Gamma) = \{ M \in M_{\text{SAL}_{m+1}^m}(\Gamma) : \text{there is no } M' \in M_{\text{SAL}_{m+1}^m}(\Gamma) : Ab(M') \cap \Omega(i) \subseteq Ab(M) \cap \Omega(i) \} \). Let \( M_{\text{SAL}_m^m}(\Gamma) = \liminf_{i \to \Gamma} M_{\text{SAL}_m^m}(\Gamma) = \limsup_{i \to \Gamma} M_{\text{SAL}_m^m}(\Gamma) = \bigcap_{i \in I} M_{\text{SAL}_m^m}(\Gamma) \).

Fact 6 \( M_{\text{SAL}_m^m}(\Gamma) M_{\text{AL}_m^m}(\Gamma) = M_{\text{AL}_{1(1)}^m}(\Gamma) = M_{\text{AL}_{1(1)}^m}(\Gamma) \).

Fact 7 Each of the following holds for every \( i \in I \):

1. \( M_{\text{SAL}_m^m}(\Gamma) \subseteq M_{\text{SAL}_{1(1)}^m}(\Gamma) \)
2. \( M_{\text{SAL}_m^m}(\Gamma) = \bigcap_{i \leq I} M_{\text{SAL}_m^m}(\Gamma) \)

\( ^{11} \) For the proofs of Facts 5.1-5.3, it suffices to refer to Definitions 1 and 4. For Fact 5.4, we need Definition 2, Fact 5.2, resp. Fact 5.3, the soundness of \( \text{AL}_m^m \) and the completeness of \( \text{AL}_m^m \).

\( ^{12} \) Note that the sequence \( \langle M_{\text{SAL}_m^m}(\Gamma) \rangle_{i \in I} \) converges to its limes inferior resp. to its limes superior due to the fact that the sequence is anti-monotonic (see Fact 7.1).
Lemma 1 If \( M \in \mathcal{MSAL}^m(\Gamma) \) and \( M' \in \mathcal{M}_{LLL}^+(\Gamma) \) is such that \( Ab(M') \cap \Omega(i) \subseteq Ab(M) \cap \Omega(i) \), then \( M' \in \mathcal{MSAL}^m(\Gamma) \).

Proof. Assume that the antecedent holds, but \( M' \notin \mathcal{MSAL}^m(\Gamma) \). Let \( j \leq i \) be the smallest \( j \in I \) such that \( M' \notin \mathcal{MSAL}^m(\Gamma) \). By Definition 5, there is an \( M'' \in \mathcal{MSAL}^m_{(i-1)}(\Gamma) \) such that \( Ab(M'') \cap \Omega_j \subseteq Ab(M') \cap \Omega_j \). In view of the supposition, \( Ab(M'') \cap \Omega_j \subseteq Ab(M) \cap \Omega_j \), whence (i) \( Ab(M'') \cap \Omega_j \subseteq Ab(M) \cap \Omega_j \). By the supposition and Definition 5, \( M \in \mathcal{MSAL}^m_{(i-1)}(\Gamma) \). By (ii) and Definition 5, \( M \notin \mathcal{MSAL}^m(\Gamma) \), whence also \( M \notin \mathcal{MSAL}^m(\Gamma) \) — a contradiction.

In the remainder we will often restrict our focus to a specific class of superposed adaptive logics, namely of sequences of the type \( \langle \mathcal{AL}^m_{(1)}, \mathcal{AL}^m_{(2)}, \ldots \rangle \). In what follows we write \( \mathcal{SAL}^m_{(i)} \) for the superposed adaptive logic characterized by the sequence \( \langle \mathcal{AL}^m_{(1)}, \ldots, \mathcal{AL}^m_{(i)}, \ldots \rangle \), and \( \mathcal{SAL}^m \) for the superposed adaptive logic characterized by \( \langle \mathcal{AL}^m_{(i)} \rangle_{i \in I} \).

From Theorem 12 below, we can infer that the semantics of \( \mathcal{SAL}^m \) is equivalent to that of the logic \( \mathcal{HAL}^m \) characterized by the triple \( \langle \mathcal{LLL}, (\Omega_i)_{i \in I}, m \rangle \).

Theorem 12 \( \mathcal{MSAL}^m(\Gamma) = \mathcal{MSAL}^m(\Gamma) \).

Proof. (\( \mathcal{MSAL}^m(\Gamma) \subseteq \mathcal{MSAL}^m(\Gamma) \)) Assume that \( M \in \mathcal{MSAL}^m_{(i)}(\Gamma) \). By Definition 5, \( M \in \mathcal{M}_{LLL}^+(\Gamma) \). Hence by Definition 1 and Definition 2, there is an \( i \in I \) and an \( M' \in \mathcal{M}_{LLL}^+(\Gamma) \) such that (1) for every \( k < i \) : \( Ab(M') \cap \Omega(k) = Ab(M) \cap \Omega(k) \) and (2) \( Ab(M') \cap \Omega_i \subseteq Ab(M) \cap \Omega_i \). It follows that (1) \( Ab(M') \cap \Omega(k) = Ab(M) \cap \Omega(k) \) and (2) \( Ab(M') \cap \Omega_i \subseteq Ab(M) \cap \Omega_i \). By Definition 5, \( M \in \mathcal{MSAL}^m_{(i-1)}(\Gamma) \). But then also \( M' \in \mathcal{MSAL}^m_{(i-1)}(\Gamma) \), which implies that \( M \notin \mathcal{MSAL}^m_{(i-1)}(\Gamma) \). By Definition 5, \( M \notin \mathcal{MSAL}^m(\Gamma) \) — a contradiction.

(\( \mathcal{MSAL}^m(\Gamma) \subseteq \mathcal{MSAL}^m(\Gamma) \)) Assume that \( M \in \mathcal{MSAL}^m_{(i)}(\Gamma) \). By Definition 5, there is an \( M' \in \mathcal{MSAL}^m_{(i-1)}(\Gamma) \) such that \( Ab(M') \cap \Omega(i) \subseteq Ab(M) \cap \Omega(i) \). Note that \( M' \in \mathcal{M}_{LLL}^+(\Gamma) \). Also, it follows that there is a \( k < i \) such that (1) for every \( j \in I, j < k \) : \( Ab(M') \cap \Omega(j) = Ab(M) \cap \Omega(j) \) and (2) \( Ab(M') \cap \Omega_k \subset Ab(M) \cap \Omega_k \). But then \( M \notin \mathcal{MSAL}^m(\Gamma) \) by Definitions 1 and 2.

3.2 Intersection of Selections

Similar to superpositions of ALs, hierarchic ALs are characterized on the basis of sequences of flat adaptive logics: \( \langle \mathcal{AL}_1, \mathcal{AL}_2, \ldots \rangle \). In the hierarchic case however, the logics in the sequence have to fulfill the condition that \( \Omega_i \subseteq \Omega_{i+1} \). Thus, the sequences that characterize hierarchic adaptive logics using the minimal abnormality strategy can be written as \( \langle \mathcal{AL}^m_{(1)}, \mathcal{AL}^m_{(2)}, \ldots \rangle \). Their semantics is characterized by the following set of models:

Definition 6 \( \mathcal{M}_{HAL}^m(\Gamma) = \bigcap_{i \in I} \mathcal{M}_{AL}^m_{(i)}(\Gamma) \).

As for \( \mathcal{SAL}^m \), we can prove that the semantics of every logic \( \mathcal{HAL}^m \) is equivalent to that of \( \mathcal{AL}^m \), where the latter is defined by \( \langle \mathcal{LLL}, (\Omega_i)_{i \in I}, m \rangle \):

Theorem 13 \( \mathcal{M}_{HAL}^m(\Gamma) = \mathcal{M}_{AL}^m(\Gamma) \)
such that Ab

In this section, we provide a definition of 2. Where

If

Each of the following holds:

(1) for every j < i

(2) Ab(M') \cap \Omega_j \subset Ab(M) \cap \Omega_j.

It follows that (2)' Ab(M') \cap \Omega_i \subset Ab(M) \cap \Omega_i. Thus M \not\in M_{\text{HAL}^m}(\Gamma), whence by Definition 6, M \not\in M_{\text{HAL}^m}(\Gamma).

(M_{LLL}^m(\Gamma) \subseteq M_{\text{HAL}^m}(\Gamma)) Suppose M \in M_{\text{LLL}^+}(\Gamma) - M_{\text{HAL}^m}(\Gamma). Hence there is an i \in I: M \not\in M_{\text{HAL}^m}(\Gamma), whence by Definition 2, there is an M' \in M_{\text{LLL}^+}(\Gamma): Ab(M') \cap \Omega_i \subset Ab(M) \cap \Omega_i. It follows that there is a k \leq i such that (1) for every j < k: Ab(M') \cap \Omega_j = Ab(M) \cap \Omega_j, and (2) Ab(M') \cap \Omega_k \subset Ab(M) \cap \Omega_k. Thus M \not\in M_{\text{HAL}^m}(\Gamma). □

3.3 Some Corollaries

The following corollary is one of the central results presented in this paper. It shows that the semantics of hierarchic adaptive logics and logics in the AL^m-formalism define the same consequence relation. Moreover, there is a class of superposed adaptive logics for which the semantics is equivalent to that of HAL^m and AL^m as well. This class of superposed adaptive logics is characterized by the same sequences of flat adaptive logics as hierarchical adaptive logics, namely sequences of the form (AL^m, ..., ). The fact that three different paths of devising selection semantics for prioritized logics lead to the same semantic consequence relation demonstrates the centrality, robustness and usefulness of the latter.

Corollary 1 M_{\text{AL}^m}(\Gamma) = M_{\text{SAL}^m}(\Gamma) = M_{\text{HAL}^m}(\Gamma). Hence, \Gamma \models_{\text{AL}^m} A \iff \Gamma \models_{\text{SAL}^m} A \iff \Gamma \models_{\text{HAL}^m} A.

In the remainder of this section, let PAL \in \{AL^m, SAL^m, HAL^m\}. By Fact 3 and Corollary 1, we have:

Corollary 2 M_{\text{ULL}}(\Gamma) \subseteq M_{\text{PAL}}(\Gamma).

Since AL^m is sound and complete with respect to \models_{\text{AL}^m} (see Theorem 1), the corollary equips us with alternative semantic selection procedures for AL^m:

Corollary 3 Each of the following holds:

1. If \Gamma \models_{\text{AL}^m} A, then \Gamma \models_{\text{PAL}} A.
2. Where \Gamma \subseteq W: if \Gamma \models_{\text{PAL}} A, then \Gamma \models_{\text{AL}^m} A.

Theorems 2, 12 and 13 imply that the Strong Reassurance property holds for SAL^m and HAL^m:

Corollary 4 If M \in M_{\text{LLL}^+}(\Gamma) - M_{\text{PAL}}(\Gamma), then there is an M' \in M_{\text{PAL}}(\Gamma) such that Ab(M') \subset Ab(M).

4 Two Consequence Relations

In this section, we provide a definition of Cn_{\text{SAL}^m}(\Gamma) and Cn_{\text{HAL}^m}(\Gamma). We will show that the SAL^m-consequence relation is both reflexive and closed under LLL^+. Next, we will show that each logic in the specific class of logics SAL^m
is sound with respect to the semantic characterization from Section 3.1. We will recapitulate the soundness result for HAL\textsuperscript{m} from [20] and derive from this that \( AL\textsuperscript{m} \) is always at least as strong as HAL\textsuperscript{m}. Finally, we will show that whenever it is possible in view of the lower limit logic and \( \Gamma \), to falsify all abnormalities up to a certain level \( i \), then each of SAL\textsuperscript{m}, HAL\textsuperscript{m} and AL\textsubscript{C} will do so.

4.1 Superpositions of Consequence Relations

Before we turn to the definition of SAL\textsuperscript{m}, recall that this format is more general than SAL\textsubscript{C} – see Section 3.1. The consequence relation of SAL\textsuperscript{m}, obtained by the superposition of the logics \( \langle AL\textsuperscript{m} \rangle \in I \), is defined as follows:

**Definition 7** Let SAL\textsubscript{m} \( = \) LLL\textsuperscript{+}. For every \( i \in I \), let

\[
Cn_{SAL\textsuperscript{m}}(\Gamma) = Cn_{AL\textsuperscript{m}}(\ldots(Cn_{AL\textsuperscript{m}}(Cn_{AL\textsuperscript{m}}(\Gamma)))\ldots)
\]

Moreover, let \( Cn_{SAL\textsuperscript{m}}(\Gamma) = \liminf_{i \to \infty} Cn_{SAL\textsuperscript{m}}(\Gamma) = \limsup_{i \to \infty} Cn_{SAL\textsuperscript{m}}(\Gamma) = \bigcup_{i \in I} Cn_{SAL\textsuperscript{m}}(\Gamma). \)

**Fact 8** \( Cn_{SAL\textsuperscript{m}} = Cn_{SAL\textsuperscript{m}_1} = Cn_{AL\textsuperscript{m}} = Cn_{AL\textsuperscript{m}_1} \).

Let us first consider some properties that hold for SAL\textsuperscript{m} in general. Since every logic AL\textsuperscript{m} is reflexive, we immediately get:

**Fact 9** Each of the following holds:

1. For every \( i \in I \), \( \Gamma \subseteq Cn_{SAL\textsuperscript{m}}(\Gamma) \).
2. \( \Gamma \subseteq Cn_{SAL\textsuperscript{m}}(\Gamma) \). (Reflexivity)
3. For every \( i \in I \) : \( Cn_{SAL\textsuperscript{m}_1}(\Gamma) \subseteq Cn_{SAL\textsuperscript{m}}(\Gamma) \).

The following lemma requires a bit more explanation. As Theorem 3 states, every logic AL\textsuperscript{m} is closed under LLL\textsuperscript{+}. However, this theorem is restricted to the case where \( \Gamma \subseteq W \).\textsuperscript{14} In order to generalize it to SAL\textsuperscript{m}, we first need to establish the LLL\textsuperscript{+}-closure of AL\textsuperscript{m}, for a specific kind of premise sets \( \Gamma \subseteq W_\ast \):

**Lemma 2** If \( \Gamma = Cn_{LLL\textsuperscript{+}}(\Gamma) \), then \( Cn_{AL\textsuperscript{m}}(\Gamma) = Cn_{LLL\textsuperscript{+}}(Cn_{AL\textsuperscript{m}}(\Gamma)) \).

**Proof.** That \( Cn_{AL\textsuperscript{m}}(\Gamma) \subseteq Cn_{LLL\textsuperscript{+}}(Cn_{AL\textsuperscript{m}}(\Gamma)) \) is immediate in view of the reflexivity of LLL\textsuperscript{+}. For the other direction, suppose that \( \langle \rangle \Gamma = Cn_{LLL\textsuperscript{+}}(\Gamma) \) and \( A \in Cn_{LLL\textsuperscript{+}}(Cn_{AL\textsuperscript{m}}(\Gamma)) \). Hence \( A \) is true in every \( M \in M_{LLL\textsuperscript{+}}(Cn_{AL\textsuperscript{m}}(\Gamma)) \). By the soundness of LLL\textsuperscript{+}, \( \langle \rangle \Gamma \) is true in every LLL\textsuperscript{+}-model of \( Cn_{AL\textsuperscript{m}}(\Gamma) \). By Definition 2 and the soundness of \( AL\textsuperscript{m} \), every AL\textsuperscript{m}-model of \( \Gamma \) is an LLL\textsuperscript{+}-model of \( Cn_{AL\textsuperscript{m}}(\Gamma) \). Hence by \( \langle \rangle \Gamma \), \( \Gamma \models_{AL\textsuperscript{m}} A \). By \( \langle \rangle \Gamma \) and Theorem 5, \( \Gamma \vdash_{AL\textsuperscript{m}} A \). □

**Theorem 14** Where \( \Gamma \subseteq W \), each of the following holds:

1. for every \( i \in I \), \( Cn_{SAL\textsuperscript{m}}(\Gamma) = Cn_{LLL\textsuperscript{+}}(Cn_{SAL\textsuperscript{m}_i}(\Gamma)) \).
2. \( Cn_{SAL\textsuperscript{m}}(\Gamma) = Cn_{LLL\textsuperscript{+}}(Cn_{SAL\textsuperscript{m}}(\Gamma)) \). (LLL-closure)

\textsuperscript{13}Note that the sequence \( \langle Cn_{SAL\textsuperscript{m}}(\Gamma) \rangle \in I \) converges to its limes inferior resp. to its limes superior due to the fact that the sequence is monotonic (see Fact 9.3).

\textsuperscript{14}There are \( \Gamma \subseteq W \), for which it fails – see [21, Chapter 2, Section 8] for an example.
Proof. Ad 1. (i = 1) Immediate in view of Theorem 3 and Fact 8.

(i ⇒ i + 1) By Definition 7, \(Cn_{\text{AL}_{i+1}^m}(\Gamma) = Cn_{\text{AL}_{i+1}^m}(Cn_{\text{AL}_i^m}(\Gamma))\). Hence by the induction hypothesis and Lemma 2,

\[
Cn_{\text{AL}_{i+1}^m}(\Gamma) = Cn_{\text{LLL}^+}(Cn_{\text{AL}_{i+1}^m}(Cn_{\text{AL}_i^m}(\Gamma)))
\]  

By (1) and Definition 7, \(Cn_{\text{AL}_{i+1}^m}(\Gamma) = Cn_{\text{LLL}^+}(Cn_{\text{AL}_i^m}(\Gamma))\).

Ad 2. That \(Cn_{\text{AL}_{i+1}^m}(\Gamma) \subseteq Cn_{\text{LLL}^+}(Cn_{\text{AL}_i^m}(\Gamma))\) is immediate in view of the reflexivity of \(\text{LLL}^+\). Suppose that \(A \in Cn_{\text{LLL}^+}(Cn_{\text{AL}_i^m}(\Gamma))\). By Definition 7 and the compactness of \(\text{LLL}^+\), there is an \(i \in I\) such that \(A \in Cn_{\text{LLL}^+}(\bigcup_{i \in I} Cn_{\text{AL}_i^m}(\Gamma))\), whence by Fact 9.3, \(A \in Cn_{\text{LLL}^+}(Cn_{\text{AL}_i^m}(\Gamma))\). By item 1, \(A \in Cn_{\text{AL}_i^m}(\Gamma)\), whence by Definition 7, \(A \in Cn_{\text{AL}_{i+1}^m}(\Gamma)\).

For the proof of Theorem 15 below, we need to establish a specific property of the \(\text{AL}_i^m\)-semantics. It states that if every member of a set \(\Gamma\) is true in all \(\text{AL}_i^m\)-models of \(\Gamma\), then \(\mathcal{M}_{\text{AL}_i^m}(\Gamma) = \mathcal{M}_{\text{AL}_i^m}(\Gamma)\):

**Lemma 3** If \(\Gamma' \subseteq \{A \mid \Gamma \models_{\text{AL}_i^m} A\}\), then \(\mathcal{M}_{\text{AL}_i^m}(\Gamma \cup \Gamma') = \mathcal{M}_{\text{AL}_i^m}(\Gamma)\).

Proof. Suppose \((\forall) \Gamma' \subseteq \{A \mid \Gamma \models_{\text{AL}_i^m} A\}\).

\((\mathcal{M}_{\text{AL}_i^m}(\Gamma \cup \Gamma') \subseteq \mathcal{M}_{\text{AL}_i^m}(\Gamma))\). Consider an \(M \in \mathcal{M}_{\text{AL}_i^m}(\Gamma \cup \Gamma')\). By Definition 2, \(M \in \mathcal{M}_{\text{LLL}^+}(\Gamma' \cup \Gamma')\) and hence \(M \in \mathcal{M}_{\text{LLL}^+}(\Gamma)\). Assume that \(M \not\in \mathcal{M}_{\text{AL}_i^m}(\Gamma)\). By Theorem 2, there is an \(M' \in \mathcal{M}_{\text{AL}_i^m}(\Gamma)\) such that \(Ab(M') \prec Ab(M)\). However, in view of \((\forall)\), \(M' \models A\) for every \(A \in \Gamma'\), whence also \(M' \in \mathcal{M}_{\text{LLL}^+}(\Gamma' \cup \Gamma')\). By Definition 2, \(M \not\in \mathcal{M}_{\text{AL}_i^m}(\Gamma)\). A contradiction.

\((\mathcal{M}_{\text{AL}_i^m}(\Gamma \cup \Gamma') \supseteq \mathcal{M}_{\text{AL}_i^m}(\Gamma))\). Consider an \(M \in \mathcal{M}_{\text{AL}_i^m}(\Gamma)\). By Definition 2, \(M\) is an \(\text{LLL}^+\)-model of \(\Gamma\). We thus obtain that \(M\) is a \(\text{LLL}^+\)-model of \(\Gamma \cup \Gamma'\). Assume that \(M \not\in \mathcal{M}_{\text{AL}_i^m}(\Gamma \cup \Gamma')\). By Theorem 2, there is an \(M' \in \mathcal{M}_{\text{LLL}^+}(\Gamma \cup \Gamma')\): \(Ab(M') \prec Ab(M)\). Hence \(M' \in \mathcal{M}_{\text{LLL}^+}(\Gamma)\). By Definition 2, \(M \not\in \mathcal{M}_{\text{AL}_i^m}(\Gamma)\) — a contradiction.

The preceding observations allow us to establish the main result of this section. This concerns the more specific class of logics \(\text{AL}_i^m\), defined in terms of the sequence of logics \(\langle \text{AL}_i^m \rangle_{i \in I}\). In the remainder, we prove that these logics are sound with respect to the sequential superposition of semantic selections from Section 3.1, whence by Corollary 1 they are also sound with respect to the other semantic selections featured in this paper.\(^\text{16}\)

**Theorem 15** Each of the following holds for every \(i \in I\):

1. For every \(M \in \mathcal{M}_{\text{AL}_i^m}(\Gamma)\), \(Ab(M) \cap \Omega(i) \subseteq \Phi(i)(Cn_{\text{AL}^m(i-1)}(\Gamma))\)
2. If \(A \in Cn_{\text{AL}_i^m}(\Gamma)\), then \(\Gamma \models_{\text{AL}_i^m} A\).

Proof. (i = 1) Item 1 is immediate in view of Fact 8 and Theorem 6.1; item 2 is immediate in view of Fact 6, Fact 8 and the Soundness of \(\text{AL}_i^m\).

(i ⇒ i + 1) Ad 1. By the induction hypothesis (2.), Fact 7 and Theorem 12 respectively, \(Cn_{\text{AL}_i^m}(\Gamma) \subseteq \{A \mid \Gamma \models_{\text{AL}_i^m} A\}\) \(\subseteq \{A \mid \Gamma \models_{\text{AL}_i^{(i+1)}_m} A\}\) = \(\{A \mid \Gamma \models_{\text{AL}_i^{(i+1)}_m} A\}\).

---

\(^{15}\)This lemma generalizes Lemma 34 from [24] – there it was shown that the consequent of the lemma holds for \(\text{AL}_i^m\) whenever \(\Gamma' \subseteq Cn_{\text{AL}_i^m}(\Gamma)\).

\(^{16}\)Logics in \(\text{AL}_i^m\)-format are not in general sound with respect to their semantics – see [21, Chapter 3, Section 3] for an example.
By Lemma 3, $M_{\operatorname{AL}}^{m_{i+1}}(\Gamma) = M_{\operatorname{AL}}^{m_{i+1}}(\Gamma \cup \mathcal{C}_{n})$, whence by the Reflexivity of $\mathcal{SAL}$, $M_{\operatorname{AL}}^{m_{i+1}}(\Gamma) = M_{\operatorname{AL}}^{m_{i+1}}(\mathcal{C}_{n})$.

By Theorem 12,

$$M_{\mathcal{SAL}}^{m_{i+1}}(\Gamma) = M_{\mathcal{SAL}}^{m_{i+1}}(\mathcal{C}_{n}) \quad (2)$$

Suppose $\Gamma \models M_{\mathcal{SAL}}^{m_{i+1}}(\Gamma)$. By (2), $M \in M_{\mathcal{SAL}}^{m_{i+1}}(\mathcal{C}_{n})$. By Definition 5, $M$ is an $\mathcal{LLL}^+$-model of $\mathcal{C}_{n}$, whence by Fact 1, $\mathcal{M}(\Gamma)$ is a choice set of $\mathcal{SAL}$.

Suppose $\mathcal{M}(\Gamma)$ is a choice set of $\mathcal{SAL}$. By Definition 3, there is a choice set $\psi$ of $\mathcal{SAL}$ such that $\psi \subseteq M(\Gamma)$. By Fact 2, there is an $\mathcal{LLL}^+$-model $\mathcal{M}'$ of $\mathcal{SAL}$ such that $\mathcal{M}(\Gamma)$ is a choice set of $\mathcal{SAL}$.

Let $M \in M_{\mathcal{SAL}}^{m_{i+1}}(\Gamma)$. By item 1, there is a $\mathcal{SAL}$ such that $\mathcal{M}(\Gamma)$ is a choice set of $\mathcal{SAL}$. By Definition 5, $M \models \mathcal{SAL}$ for every such $\mathcal{SAL}$.

Note that if $A \in C_{n}^{m}(\Gamma)$, then by Definition 7, there is an $i \in I$ such that $A \in C_{n}^{m}(\Gamma)$. Also, by Definition 5, if $\Gamma \models C_{n}$, then $\Gamma \models A$. Hence in view of Theorem 15, we immediately obtain:

**Corollary 5** If $A \in C_{n}^{m}(\Gamma)$, then $\Gamma \models A$.

By Theorems 1.2 and 12, we obtain:

**Corollary 6** Where $\Gamma \subseteq \mathcal{W}$: $C_{n}^{m}(\Gamma) \subseteq C_{n}^{m}(\Gamma)$

### 4.2 Unions of Consequence Sets

In the notational conventions from the current paper – see also Section 3.3 –, we can define the $\mathcal{HAL}^{m}$-consequence relation as follows:

**Definition 8** $C_{n}^{\mathcal{HAL}^{m}}(\Gamma) = C_{n}^{\mathcal{LLL}^{+} \bigcup_{i \in I} C_{n}^{m}}(\Gamma)$

We refer to [20] for illustrations of how this combination leads to a prioritized treatment of the sets of abnormalities $(\Omega_{i})_{i \in I}$. Theorem 14 from that paper states that $\mathcal{HAL}^{m}$ is sound with respect to the semantics defined in Section 3.2.
Theorem 16 If $A \in Cn_{\text{HAL}^m}(\Gamma)$, then $\Gamma \models_{\text{HAL}^m} A$.

By Corollary 1 and Theorem 1.2, we immediately have:

Corollary 7 Where $\Gamma \subseteq W$: $Cn_{\text{HAL}^m}(\Gamma) \subseteq Cn_{\text{AL}^m}(\Gamma)$.

4.3 Normal Premise Sets Revisited

In Section 2.3, we saw that whenever $\Gamma \cup \Omega^+$ has $\text{LLL}^+$-models, then $\text{AL}^m$ is identical to its upper limit logic $\text{ULL}$. As we will see below, this result can be generalized to sequential superpositions of ALs, as well as to hierarchic ALs—see Theorem 19 below. However, in the case of prioritized ALs, one may also wonder whether a slightly stronger property holds. That is, suppose that for some $i \in I$, it is possible to verify all members of $\Gamma$, yet also falsify all abnormalities up to level $i$. In that case, it seems a desirable property for the prioritized logic indeed considers all the members of $\Omega(i)$ to be false.

To formally express this property, we introduce the concept of normality at level $i$, resp. up to level $i$:

Definition 9 $\Gamma$ is normal at level $i$ iff $\Gamma \cup \Omega^+_i$ has $\text{LLL}^+$-models. $\Gamma$ is normal up to level $i$ iff $\Gamma \cup \Omega^+_i$ has $\text{LLL}^+$-models.

The following is immediate in view of Definition 9:

Fact 10 If $\Gamma$ is normal up to level $i$, then each of the following holds:

1. $\Gamma$ is normal at level $j$, for every $j \leq i$
2. $\Gamma \cup \Omega^+_i$ is normal at level $j+1$, for every $j < i$
3. $Cn_{\text{LLL}^+}(\Gamma \cup \Omega^+_i)$ is normal at level $j+1$, for every $j < i$

In the remainder, we use $\text{ULL}_i$ to refer to the upper limit logic of $\text{AL}^m_i$, i.e. $\text{ULL}_i$ is the monotonic logic that trivializes all abnormalities of level $i$. Likewise, $\text{ULL}_i(\Omega)$ denotes the upper limit logic of $\text{AL}^m_i(\Omega)$ and trivializes all abnormalities up to level $i$. Finally, let $\text{PAL} \in \{\text{AL}^m, \text{SAL}^m, \text{HAL}^m\}$.

Theorem 17 If $\Gamma$ is normal up to level $i$, then $Cn_{\text{ULL}_i(\Gamma)} \subseteq Cn_{\text{PAL}(\Gamma)}$.

Proof. ($\text{PAL} = \text{AL}^m$) This is Theorem 25 in Appendix A.3.\(^{17}\)

($\text{PAL} = \text{SAL}^m$) Suppose $\Gamma$ is normal up to level $i$. We show by an induction that for all $j \leq i$, $Cn_{\text{SAL}^m}(\Gamma) = Cn_{\text{LLL}^+}(\Gamma \cup \Omega^+_i)$—the rest follows immediately.

1. ($j = 1$) By Fact 10.1 and the supposition, $\Gamma$ is normal at level 1. By Fact 8 and Theorem 11 respectively, $Cn_{\text{SAL}^m}(\Gamma) = Cn_{\text{AL}^m}(\Gamma) = Cn_{\text{LLL}^+}(\Gamma \cup \Omega^+_1)$.
2. ($j \Rightarrow j + 1$): By the induction hypothesis, $Cn_{\text{SAL}^m}(\Gamma) = Cn_{\text{LLL}^+}(\Gamma \cup \Omega^+_j)$. By Fact 10.3 and the supposition, $Cn_{\text{LLL}^+}(\Gamma \cup \Omega^+_j)$ is normal at level $j + 1$. But then by Theorem 11, $Cn_{\text{SAL}^m}(\Gamma) = Cn_{\text{AL}^m}(\Gamma \cup \Omega^+_j) = Cn_{\text{LLL}^+}(\Gamma \cup \Omega^+_j) = Cn_{\text{LLL}^+}(\Gamma \cup \Omega^+_j) = Cn_{\text{LLL}^+}(\Gamma \cup \Omega^+_{j+1})$.

\(^{17}\)Where $\Gamma \subseteq W$, this theorem can be proven by semantic means. For the more general case ($\Gamma \subseteq W_i$), the proof of this theorem requires reference to syntactic notions, whence we put it in the appendix.
\textbf{(PAL = HAL)} Immediate in view of the fact that $C_n^{\text{AL}(m)}(\Gamma) \subseteq C_n^{\text{HAL}(m)}(\Gamma)$ (see Definition 8), Definition 9, and Theorem 11. ■

**Theorem 18** \( C_n^{\text{PAL}}(\Gamma) \subseteq C_n^{\text{NULL}}(\Gamma) \).

**Proof.** Suppose \( A \in C_n^{\text{PAL}}(\Gamma) \). By the soundness of \text{PAL} (see Theorem 1.1, Corollary 6 and Theorem 16 respectively), \( \Gamma \models_{\text{PAL}} A \). By Corollary 2, \( \Gamma \models_{\text{ULL}} A \). By the completeness of \text{ULL}, \( A \in C_n^{\text{NULL}}(\Gamma) \). ■

In view of Theorem 17, Theorem 18 and the monotonicity and compactness of \( \text{LLL}^+ \), the proof of the following can be safely left to the reader:

**Theorem 19** If \( \Gamma \) is normal, then \( C_n^{\text{PAL}}(\Gamma) = C_n^{\text{NULL}}(\Gamma) \).

## 5 Equivalence Results

In this section, we establish the third major result we promised in the introduction, i.e. that given certain weak conditions, the logics \( \text{SAL}_m^c \) and \( \text{HAL}_m^m \) (defined from the sequence of flat ALs \( \langle \text{AL}_m^m(\iota) \rangle_{i \in I} \)) are complete and equivalent to \( \text{AL}_m^m \) (defined by the triple \( \langle \text{LLL}^+, (\Omega_i)_{i \in I}, m \rangle \)).

### 5.1 The Basic Criteria for Equivalence

Note that the following is the case:

**Theorem 20** Where \( \Gamma \subseteq W \) and \( \text{PAL} \in \{ \text{SAL}_m^c, \text{HAL}_m^m \} \): if \( M_{\text{LLL}^+}(C_n^{\text{PAL}}(\Gamma)) = M_{\text{PAL}}(\Gamma) \) (3) then

1. \( \Gamma \models_{\text{PAL}} A \) iff \( A \in C_n^{\text{PAL}}(\Gamma) \), and
2. \( C_n^{\text{PAL}}(\Gamma) = C_n^{\text{AL}_m^m}(\Gamma) \).

**Proof.** Ad 1. \((\Rightarrow)\) If \( \Gamma \models_{\text{PAL}} A \) then \( A \) is true in every \( M \in M_{\text{LLL}^+}(C_n^{\text{PAL}}(\Gamma)) \). By the completeness of \( \text{LLL}^+ \) and the fact that \( C_n^{\text{PAL}}(\Gamma) \) is closed under \( \text{LLL}^+ \) (in case \( \text{PAL} = \text{SAL}_m^c \) see Theorem 14.2, in case \( \text{PAL} = \text{HAL}_m^m \) this holds by definition), \( A \in C_n^{\text{PAL}}(\Gamma) \). \((\Leftarrow)\) See Corollary 5, resp. Theorem 16.

Ad 2. Immediate in view of item 1, Corollary 1 and the soundness and completeness of \( \text{AL}_m^m \). ■

Equation (3) expresses that the set of \text{PAL}-models is characterized by means of the \text{PAL}-consequence set: the models of the prioritized adaptive logic are exactly those \( \text{LLL}^+ \)-models that verify the \text{PAL}-consequences. This is a central criterion since it is sufficient for both, the soundness and completeness of \text{PAL} (point 1.), and for the equivalence of the consequence relation of the three prioritized adaptive logics that are presented in this paper (point 2.).

The criteria for soundness and equivalence are defined by means of sets of complements of minimal choice sets. Where \( \prec \in \{ \subset, \subseteq \} \), let \( ^c \Phi^\prec(\Gamma) = \{ \Omega - \varphi \mid \varphi \in \Phi^\prec(\Gamma) \} \). Likewise, let \( ^c \Phi^{\prec(\iota)}(\Gamma) = \{ \Omega^{(\iota)} - \varphi \mid \varphi \in \Phi^{\prec(\iota)}(\Gamma) \} \) and let \( ^c \Phi^{(i)}(\Gamma) = \{ \Omega^{(i)} - \varphi \mid \varphi \in \Phi^{(i)}(\Gamma) \} \).

In Sections 5.2 and 5.3 we will give syntactic criteria in terms of these sets, that warrant (3) for \( \text{SAL}_m^c \), resp. \( \text{HAL}_m^m \). But first, let us show that this holds for flat ALs and ALs in \( \text{AL}_m^m \)-format.
Lemma 4  Where $\Gamma \subseteq W$: if $^c\Phi^\prec(\Gamma)$ has no infinite minimal choice sets, then $M_{LLL^+}(\mathcal{CN}_{AL^m}(\Gamma)) = M_{AL^m}(\Gamma)$.

Proof. Suppose $^c\Phi^\prec(\Gamma)$ has no infinite minimal choice sets. That $M_{AL^m}(\Gamma) \subseteq M_{LLL^+}(\mathcal{CN}_{AL^m}(\Gamma))$ is immediate in view of Definition 2 and the soundness of $\mathcal{AL}^m$. So assume that $M \in M_{LLL^+}(\mathcal{CN}_{AL^m}(\Gamma)) - M_{AL^m}(\Gamma)$. By Theorem 6, for every $\varphi \in \Phi^\prec(\Gamma)$, there is an $A_\varphi \in \Omega - \varphi$ such that $M \not\vdash A_\varphi$. Note that $\{A_\varphi \mid \varphi \in \Phi^\prec(\Gamma)\}$ is a choice set of $^c\Phi^\prec(\Gamma)$. Hence by the supposition, there is a finite $\Theta \subseteq \{A_\varphi \mid \varphi \in \Phi^\prec(\Gamma)\}$, such that $\Theta$ is a choice set of $^c\Phi^\prec(\Gamma)$. It follows by Theorem 6 that $\Gamma \not\models_{AL^m} \wedge \Theta$, and hence by Theorem 1.2, also $\Gamma \not\models_{AL^m} \wedge \Theta$. Hence the lemma follows.

Lemma 5  If $M_{LLL^+}(\mathcal{CN}_{AL^m}(\Gamma)) = M_{AL^m}(\Gamma)$, then $^c\Phi^\prec(\Gamma)$ has no infinite minimal choice sets.

Proof. Let $\Theta$ be an infinite minimal choice set of $^c\Phi^\prec(\Gamma)$. Assume there is no $LLL^+$-model of $\mathcal{CN}_{AL^m}(\Gamma) \cup \Theta$. By the compactness of $LLL^+$, there is a finite $\{A_j \mid j \in J\} \subseteq \Theta$ such that $\mathcal{CN}_{AL^m}(\Gamma) \not\models_{LLL^+} \forall j \in J \wedge A_j$. Hence by the soundness of $\mathcal{AL}^m$, every $\mathcal{AL}^m$-model of $\Gamma$ falsifies an $A_j$ ($j \in J$). By Theorem 6, for every $\varphi \in \Phi^\prec(\Gamma)$, there is a $j \in J$ such that $A_j \not\models \varphi$. But then $\{A_j \mid j \in J\}$ is a choice set of $^c\Phi^\prec(\Gamma)$ — a contradiction to the minimality of $\Theta$. So there is a $LLL^+$-model $M$ of $\mathcal{CN}_{AL^m}(\Gamma) \cup \Theta$.

Assume $M \in M_{AL^m}(\Gamma)$. By Theorem 6, there is a $\varphi \in \Phi^\prec(\Gamma)$ such that $Ab(M) = \varphi$. However, since $\Theta$ is a choice set of $^c\Phi^\prec(\Gamma)$, there is an $A \in (\Omega - \varphi) \cap \Theta$ — a contradiction. Hence $M \in M_{LLL^+}(\mathcal{CN}_{AL^m}(\Gamma)) - M_{AL^m}(\Gamma)$.

Corollary 8  Where $\Gamma \subseteq W$: $^c\Phi^\prec(\Gamma)$ has no infinite minimal choice sets iff $M_{LLL^+}(\mathcal{CN}_{AL^m}(\Gamma)) = M_{AL^m}(\Gamma)$.

Where $\prec \subseteq \subseteq$, the same result can be obtained for a specific class of premise sets $\Gamma \subseteq W$.

Lemma 6  Where $\Gamma = Cn_{LLL^+}(\Gamma)$: $^c\Phi(\Gamma)$ has no infinite minimal choice sets iff $M_{LLL^+}(\mathcal{CN}_{AL^m}(\Gamma)) = M_{AL^m}(\Gamma)$.

Proof. ($\Rightarrow$) Immediate in view of the proof for Lemma 4 — replace Theorem 1.2 by Theorem 5. ($\Leftarrow$) Immediate in view of Lemma 5.

The above results are of crucial importance for the completeness and equivalence results of both $\mathcal{SAL}_c^m$ and $\mathcal{HAL}_c^m$, which we shall present subsequently. The following additional lemmas will also be useful in the remainder:

Lemma 7  For every $\varphi \in \Phi^\prec(\Gamma)$, there is a $\psi \in \Phi^\prec(\Gamma)$ such that $\psi \cap \Omega_{\{i\}} = \varphi$.

Proof. Case 1. $\Gamma$ is not $LLL^+$-satisfiable. In that case, $\Gamma \not\models_{LLL^+} A$ for every $A \in \Omega$, whence $\Phi^\prec(\Gamma) = \{\Omega_{\{i\}}\}$ and $\Phi^\prec(\Gamma) = \{\Omega\}$. Hence the lemma follows immediately.

Case 2. $\Gamma$ is $LLL^+$-satisfiable. Suppose $\varphi \in \Phi^\prec(\Gamma)$ for an $i \in I$. By Theorem 6.2, there is an $M \in M_{AL^m}(\Gamma)$ such that $Ab(M) \cap \Omega_{\{i\}} = \varphi$. Note that $M \in M_{LLL^+}(\Gamma)$. If $M \in M_{AL^m}(\Gamma)$, then by Theorem 6.1, $Ab(M) \in \mathcal{AL}^m$-model.
\[ \Phi^c(\Gamma), \text{ whence the lemma follows immediately. So suppose } M \notin \mathcal{M}_{\text{AL}}(\Gamma). \]

Then by Theorem 2, there is an \( M' \in \mathcal{M}_{\text{AL}}(\Gamma) \) such that \( Ab(M') \subset Ab(M) \).

Assume (1) \( Ab(M') \cap \Omega(\iota) \neq Ab(M) \cap \Omega(\iota) \). In view of Definitions 1 and 4, there is a \( j \leq i \) such that \( Ab(M') \subset (j, \iota) \). By Fact 5.3, \( Ab(M') \subset (j, \iota) \). But then \( M \notin \mathcal{M}_{\text{AL}}(\Gamma) \) — a contradiction. Hence (1) fails: \( Ab(M') \cap \Omega(\iota) = Ab(M) \cap \Omega(\iota) \). Since by Theorem 6.1, \( Ab(M') \in \Phi^c(\Gamma) \), the lemma follows immediately.

**Lemma 8** For every \( \varphi \in \Phi^c(\Gamma) \), \( \varphi \cap \Omega(\iota) \in \Phi^c(\Gamma) \).

Proof. Assume that \( \varphi \in \Phi^c(\Gamma) \), but \( \varphi \cap \Omega(\iota) \notin \Phi^c(\Gamma) \). Note that since \( \varphi \) is a choice set of \( \Sigma(\Gamma) \), \( \varphi \cap \Omega(\iota) \) is a choice set of \( \Sigma(\Gamma) \). Hence there is a \( \psi \in \Phi^c(\Gamma) \) such that \( \psi \cap (\iota, \varphi) \). By Lemma 7, there is a \( \psi' \in \Phi^c(\Gamma) \) such that \( \psi' \cap \Omega(\iota) = \psi \). But then by Fact 5.2, \( \psi' \subset (\iota, \varphi) \) — a contradiction.

**Corollary 9** \( \Phi^c(\Gamma) = \{ \varphi \cap \Omega(\iota) \mid \varphi \in \Phi^c(\Gamma) \} \).

**Lemma 9** If \( ^c\Phi^c(\Gamma) \) has no infinite minimal choice sets, then for every \( i \in I \), \( ^c\Phi^c(\Gamma) \) has no infinite minimal choice sets.

Proof. Let \( \Theta \) be an infinite minimal choice set of \( ^c\Phi^c(\Gamma) \). By Corollary 9,

\[ (\dagger_1) \quad \Theta \text{ is a minimal choice set of } \{ \Omega(\iota) - (\varphi \cap \Omega(\iota)) \mid \varphi \in \Phi^c(\Gamma) \} = \{ \Omega(\iota) - \varphi \mid \varphi \in \Phi^c(\Gamma) \} \]

Assume that for some \( \varphi \in \Phi^c(\Gamma) \), \( \Omega(\iota) - \varphi \neq \emptyset \). But then \( \varphi = \Omega(\iota) \) and \( \Phi^c(\Gamma) = \{ \Omega(\iota) \} \). Hence \( ^c\Phi^c(\Gamma) = \{ \emptyset \} \), which is a contradiction to the minimality of \( \Theta \). Thus:

\[ (\dagger_2) \quad \text{for all } \varphi \in \Phi^c(\Gamma), \Omega(\iota) - \varphi \neq \emptyset \]

By (\dagger_1) and (\dagger_2), for all \( \varphi \in \Phi^c(\Gamma) \), \( \Omega(\iota) - \varphi \neq \emptyset \). By (\dagger_1) and since \( \Omega(\iota) - \varphi \subseteq \Omega \), \( \Theta \) is a choice set of \( ^c\Phi^c(\Gamma) \).

Assume there is a \( \Theta' \subset \Theta \) which is a choice set of \( ^c\Phi^c(\Gamma) \). Since \( \Theta \subset \Omega(\iota) \), also \( (\dagger_3) \Theta' \subset \Omega(\iota) \). Note that (\dagger) for each \( \varphi \in \Phi^c(\Gamma) \), \( ((\Omega - \varphi) - (\Omega(\iota) - \varphi)) \cap \Omega(\iota) = \emptyset \). By (\dagger_3) and (\dagger), \( \Theta' \) is a choice set of \( ^c\Phi^c(\Gamma) \), which contradicts the minimality of \( \Theta \). Hence \( \Theta \) is a minimal choice set of \( ^c\Phi^c(\Gamma) \).

5.2 Restricted Completeness and Equivalence for \( \text{SAL}_c^m \)

The basic completeness/equivalence criterion for \( \text{SAL}_c^m \) reads as follows:

\[ (\star_{\text{SAL}_c^m}) \quad ^c\Phi^c(\Gamma) \text{ has no infinite minimal choice sets} \]

In the remainder, we will show that \( \star_{\text{SAL}_c^m} \) is equivalent to equation (3) from Theorem 20, for \( \text{SAL}_c^m \).

**Lemma 10** If for all \( i \in I \), \( \mathcal{M}_{\text{LLL}}(C_{\text{SAL}_c^m}(\Gamma)) = \mathcal{M}_{\text{SAL}_c^m}(\Gamma) \), then

\[ \mathcal{M}_{\text{LLL}}(C_{\text{SAL}_c^m}(\Gamma)) = \mathcal{M}_{\text{SAL}_c^m}(\Gamma) \]

Proof. We have: \( \mathcal{M}_{\text{LLL}}(C_{\text{SAL}_c^m}(\Gamma)) = \mathcal{M}_{\text{LLL}}(\bigcup_{i \in I} C_{\text{SAL}_c^m}(\Gamma)) = \bigcap_{i \in I} \mathcal{M}_{\text{LLL}}(C_{\text{SAL}_c^m}(\Gamma)) = \bigcap_{i \in I} \mathcal{M}_{\text{SAL}_c^m}(\Gamma) = \mathcal{M}_{\text{SAL}_c^m}(\Gamma) \). ■
Lemma 11 Where $\Gamma \subseteq W$: if $\Gamma$ satisfies $\star_{\text{SAL}^m}$, then $\mathcal{M}_{\text{LLL}}^+(Cn_{\text{SAL}^m}(\Gamma)) = M_{\text{SAL}^m}(\Gamma)$.

Proof. Suppose $\Gamma \subseteq W$ and $\Gamma$ satisfies $\star_{\text{SAL}^m}$. Thus, $\mathcal{C}_c(\Gamma)$ has no infinite minimal choice sets.

If $\Gamma$ is not $\text{LLL}^+$-satisfiable, then by Fact 9.1 and the monotonicity of $\text{LLL}^+$, $Cn_{\text{SAL}^m}(\Gamma)$ is not $\text{LLL}^+$-satisfiable for every $i \in I$. Also, by Definition 5, $\Gamma$ is not $\text{SAL}^m_{(i)}$-satisfiable for every $i \in I$. Hence $M_{\text{LLL}}^+(Cn_{\text{SAL}^m}(\Gamma)) = M_{\text{SAL}^m}(\Gamma) = \emptyset$, whence the lemma follows immediately. So suppose that $\Gamma$ is $\text{LLL}^+$-satisfiable. We will prove by induction that for every $i \in I$, $M_{\text{LLL}}^+(Cn_{\text{SAL}^m}(\Gamma)) = M_{\text{SAL}^m}(\Gamma)$, whence by Lemma 10, the property follows immediately.

(i = 1) By the supposition, Fact 4 and Lemma 9, $\mathcal{C}_c(\Gamma)$ has no infinite minimal choice sets. Hence by Lemma 4, $M_{\text{LLL}}^+(Cn_{\text{AL}^m_1}(\Gamma)) = M_{\text{AL}^m_1}(\Gamma)$. The rest follows immediately in view of Facts 6 and 8.

(i $\Rightarrow$ i + 1) Let $\Gamma' = Cn_{\text{SAL}^m_{(i+1)}}(\Gamma)$. By Definition 2, $M_{\text{AL}^m_{(i+1)}}(\Gamma') = \{ M \in M_{\text{LLL}}^+(\Gamma') \mid \text{there is no } M' \in M_{\text{LLL}}^+(\Gamma') \text{ such that } Ab(M') \cap \Omega_{(i+1)} \subset Ab(M) \cap \Omega_{(i+1)} \}$. By the induction hypotheses and Definition 5,

$$M_{\text{AL}^m_{(i+1)}}(\Gamma') = M_{\text{SAL}^m_{(i+1)}}(\Gamma) \quad (4)$$

By Theorem 12 and (4), $M_{\text{AL}^m_{(i+1)}}(\Gamma') = M_{\text{AL}^m_{(i+1)}}(\Gamma)$. By Theorem 6.2, we obtain that $\Phi^{(i+1)}(\Gamma') = \Phi^{(i+1)}(\Gamma)$, whence also

$$\mathcal{C}_c(\Gamma') = \mathcal{C}_c(\Gamma) \quad (5)$$

By the supposition and Lemma 9, $\mathcal{C}_c(\Gamma)$ has no infinite minimal choice sets. Hence in view of (5), $\mathcal{C}_c(\Gamma)$ has no infinite minimal choice sets. By Theorem 14.1 and Lemma 6,

$$M_{\text{LLL}}^+(Cn_{\text{AL}^m_{(i+1)}}(\Gamma')) = M_{\text{AL}^m_{(i+1)}}(\Gamma') \quad (6)$$

Hence in view of Definition 7, $M_{\text{LLL}}^+(Cn_{\text{SAL}^m_{(i+1)}}(\Gamma)) = M_{\text{AL}^m_{(i+1)}}(\Gamma')$. By (4), $M_{\text{LLL}}^+(Cn_{\text{SAL}^m_{(i+1)}}(\Gamma)) = M_{\text{SAL}^m_{(i+1)}}(\Gamma)$. $\blacksquare$

Lemma 12 Where $\Gamma \subseteq W$: if $M_{\text{LLL}}^+(Cn_{\text{SAL}^m}(\Gamma)) = M_{\text{SAL}^m}(\Gamma)$ then $\Gamma$ satisfies $\star_{\text{SAL}^m}$.

Proof. Suppose $\mathcal{C}_c(\Gamma)$ has an infinite minimal choice set. By Lemma 5, $M_{\text{LLL}}^+(Cn_{\text{SAL}^m}(\Gamma)) \neq M_{\text{AL}^m}(\Gamma)$. By the soundness of $\text{AL}^m$, $M_{\text{AL}^m}(\Gamma) \subseteq M_{\text{LLL}}^+(Cn_{\text{SAL}^m}(\Gamma))$. It follows that there is an $M \in M_{\text{LLL}}^+(Cn_{\text{SAL}^m}(\Gamma)) - M_{\text{AL}^m}(\Gamma)$. By Corollary 6 and the monotonicity of $\text{LLL}^+$, $M \in M_{\text{LLL}}^+(\Gamma)$. By Corollary 1, $M \not\in M_{\text{SAL}^m}(\Gamma)$. $\blacksquare$

Corollary 10 Where $\Gamma \subseteq W$: $\Gamma$ satisfies $\star_{\text{SAL}^m}$ iff $M_{\text{LLL}}^+(Cn_{\text{SAL}^m}(\Gamma)) = M_{\text{SAL}^m}(\Gamma)$.

In view of Theorem 20, we immediately obtain: $\text{SAL}^m$ is not complete for all premise sets – we refer to Appendix C.3 for a counterexample. Notably, this example also illustrates that in some cases, $\text{HAL}^m$ may yield more consequences than $\text{SAL}^m$. 

19$\text{SAL}^m$ is not complete for all premise sets – we refer to Appendix C.3 for a counterexample. Notably, this example also illustrates that in some cases, $\text{HAL}^m$ may yield more consequences than $\text{SAL}^m$. 

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Corollary 11  Where $\Gamma \subseteq W$: if $\Gamma$ satisfies $\bigstar_{SAL^m}$, then each of the following holds:

1. $A \in Cn_{SAL^m}(\Gamma)$ iff $\Gamma \models_{SAL^m} A$
2. $Cn_{SAL^m}(\Gamma) = Cn_{AL^m}(\Gamma)$

5.3 Restricted Completeness and Equivalence for HAL\textsuperscript{m}

The following (restricted) completeness result for HAL\textsuperscript{m} was proven in [20]: whenever $\Phi(\Gamma)$ is finite for a $\Gamma \subseteq W$, then $A \in Cn_{HAL^m}(\Gamma)$ iff $\Gamma \models_{HAL^m} A$.

In the following, we further generalize this result to all premise sets that satisfy the following criterion:

($\bigstar_{HAL^m}$) for every $i \in I$, $\Phi^i(\Gamma)$ has no infinite minimal choice sets.

Lemma 13  Where $\Gamma \subseteq W$: if $\Gamma$ satisfies $\bigstar_{HAL^m}$, then $M_{LLL^+}(Cn_{HAL^m}(\Gamma)) = M_{HAL^m}(\Gamma)$.

Proof. Suppose the antecedent holds. By Lemma 8, it follows that ($\dagger$) for every $i \in I$, $M_{LLL^+}(Cn_{HAL^m(\Gamma)}) = M_{AL^m(\Gamma)}$. By Definition 8, ($\dagger$) and Definition 6 consecutively, we have $M_{LLL^+}(Cn_{HAL^m(\Gamma)}) = \bigcap_{i \in I} M_{LLL^+}(Cn_{AL^m(\Gamma)}) = \bigcap_{i \in I} M_{AL^m(\Gamma)} = M_{HAL^m}(\Gamma)$. \hfill $\blacksquare$

Unlike for SAL\textsuperscript{m}, the right-left direction of the above lemma fails – we refer to Appendix C.4 for a counterexample. By Theorem 20, we immediately obtain:\textsuperscript{20}

Corollary 12  Where $\Gamma \subseteq W$: if $\Gamma$ satisfies $\bigstar_{HAL^m}$, then each of the following holds:

1. $A \in Cn_{SAL^m}(\Gamma)$ iff $\Gamma \models_{SAL^m} A$
2. $Cn_{SAL^m}(\Gamma) = Cn_{AL^m}(\Gamma)$

5.4 Some Weaker Completeness and Equivalence Criteria

In the preceding, we saw two sufficient syntactic criteria for the completeness and equivalence results for SAL\textsuperscript{m}, resp. HAL\textsuperscript{m}. As we will now show, several more straightforward criteria can be listed, each of which imply that either one or both of the conditions for equivalence are obeyed. Hence in concrete applications, there are various ways to establish that e.g. $Cn_{SAL^m}(\Gamma) = Cn_{HAL^m}(\Gamma)$, or that $\Gamma \models_{HAL^m} A$ iff $A \in Cn_{HAL^m}(\Gamma)$. The following is proven in Appendix B:

Theorem 21  Each of the following holds for every $\Gamma \subseteq W$:

1. $\Sigma(\Gamma)$ is finite iff every $\varphi \in \Phi(\Gamma)$ is finite.
2. If every $\varphi \in \Phi(\Gamma)$ is finite, then $\Phi(\Gamma)$ is finite.
3. If $\Phi(\Gamma)$ is finite, then for all $i \in I$, $\Phi^i(\Gamma)$ is finite.
4. If $\Phi^i(\Gamma)$ is finite for every $i \in I$, then $\Gamma$ satisfies $\bigstar_{HAL^m}$.
5. If $\Phi(\Gamma)$ is finite, then $\Phi^c(\Gamma)$ is finite.
6. If $\Phi^c(\Gamma)$ is finite, then for all $i \in I$, $\Phi^c(\Gamma)$ is finite.

\textsuperscript{20}As shown in Appendix C.2, unrestricted completeness fails for HAL\textsuperscript{m}. 20
7. If $\Phi^c(\Gamma)$ is finite, then $\Gamma$ satisfies $\star_{SAL_m}$. 
8. $\Gamma$ satisfies $\star_{SAL_m}$ iff for no $i \in I$, $\Phi^{c(i)}(\Gamma)$ has infinite minimal choice sets.

Figure 1: Syntactic criteria for completeness and equivalence

Figure 1 illustrates the relation between the criteria listed in Theorem 21 and the criteria $\star_{SAL_m}$ and $\star_{HAL_m}$. In Appendix C.4, we show by concrete examples that the converses of items 2-7 in Theorem 21 fail.

6 Some Additional Results

Some properties which were hitherto not proven for $SAL_m$ and $HAL_m$, follow almost immediately from the soundness results together with Corollaries 11 and 12. These properties are:

**Cumulative Transitivity:** where $\Gamma' \subseteq Cn_L(\Gamma) : Cn_L(\Gamma \cup \Gamma') \subseteq Cn_L(\Gamma)$

**Fixed Point:** $Cn_L(\Gamma) = Cn_L(Cn_L(\Gamma))$

**The Deduction Theorem:** If $B \in Cn_L(\Gamma \cup \{A\})$, then $A \vdash B \in Cn_L(\Gamma)$

Each of these was proven to hold for $L = AL_m$ in [24]. We will now show that a restricted version of them can be easily derived for $SAL_m$ and $HAL_m$, in view of the soundness and equivalence results from the two preceding sections. In the remainder, let $PAL \in \{HAL_m, SAL_m\}$. The following Corollary summarizes Corollaries 6 and 7:

**Corollary 13** Where $\Gamma \subseteq W$: $Cn_{PAL}(\Gamma) \subseteq Cn_{AL_m}(\Gamma)$.

**Theorem 22** Where $\Gamma \subseteq W$ and $\Gamma$ satisfies $\star_{PAL}$: if $\Gamma' \subseteq Cn_{PAL}(\Gamma)$, then $Cn_{PAL}(\Gamma \cup \Gamma') \subseteq Cn_{PAL}(\Gamma)$. (Restricted Cumulative Transitivity)

**Proof.** Suppose the antecedent holds. By the soundness of $PAL$ and Corollary 1, $\Gamma' \subseteq \{A \mid \Gamma \models_{PAL} A\} = \{A \mid \Gamma \models_{AL_m} A\}$. Hence by Corollary 1 and Lemma...
\[ M_{\text{PAL}}(\Gamma) = M_{\text{AL}}(\Gamma) = M_{\text{PAL}}(\Gamma \cup \Gamma') = M_{\text{PAL}}(\Gamma') \]  

(7)

Suppose that \( A \in C_{n_{\text{PAL}}} \Gamma_{\cup \Gamma'} \). By the soundness of \( \text{PAL} \), \( A \) is true in every \( M \in M_{\text{PAL}}(\Gamma \cup \Gamma') \). Hence by (7), \( A \) is true in every \( M \in M_{\text{PAL}}(\Gamma) \). Since \( \Gamma \) obeys \( \text{PAL} \), it follows that \( A \in C_{n_{\text{PAL}}} \Gamma \).

**Theorem 23** Where \( \Gamma \subseteq W \) and \( \Gamma \) satisfies \( \text{PAL} \): \( C_{n_{\text{PAL}}} \Gamma = C_{n_{\text{PAL}}} (C_{n_{\text{PAL}}} (\Gamma)) \).


(Restricted Fixed Point)

**Proof.** Suppose the antecedent holds. \( (C_{n_{\text{PAL}}} (\Gamma) \subseteq C_{n_{\text{PAL}}} (C_{n_{\text{PAL}}} (\Gamma))) \) Immediate in view of the reflexivity of \( \text{PAL} \).

\( (C_{n_{\text{PAL}}} (C_{n_{\text{PAL}}} (\Gamma)) \subseteq C_{n_{\text{PAL}}} (\Gamma)) \) By the reflexivity of \( \text{PAL} \), \( C_{n_{\text{PAL}}} (\Gamma) = \Gamma \cup C_{n_{\text{PAL}}} (\Gamma) \). But then \( C_{n_{\text{PAL}}} (C_{n_{\text{PAL}}} (\Gamma)) = C_{n_{\text{PAL}}} (\Gamma \cup C_{n_{\text{PAL}}} (\Gamma)) \), whence by the restricted Cumulative Transitivity of \( \text{PAL} \), \( C_{n_{\text{PAL}}} (C_{n_{\text{PAL}}} (\Gamma)) \subseteq C_{n_{\text{PAL}}} (\Gamma) \).

**Theorem 24** Where \( \Gamma \subseteq W \), \( \Gamma \) satisfies \( \text{PAL} \) and \( A \in W \): if \( B \in C_{n_{\text{AL}}} (\Gamma \cup \{ A \}) \), then \( A \supseteq B \in C_{n_{\text{PAL}}} (\Gamma) \). (Restricted Deduction Theorem)

**Proof.** Suppose the antecedent holds. By Corollary 13, \( B \in C_{n_{\text{AL}}} (\Gamma \cup \{ A \}) \).

Hence since the Deduction Theorem holds for \( \text{AL} \), \( A \supseteq B \in C_{n_{\text{AL}}} (\Gamma) \). By Corollary 11.2 (for \( \text{SAL} \)), resp. Corollary 12.2 (for \( \text{HAL} \)) and the supposition, \( A \supseteq B \in C_{n_{\text{PAL}}} (\Gamma) \).

7 In Conclusion

Let us briefly recapitulate the main results of this paper. We have shown that the semantic selections defined by \( \text{AL} \), \( \text{SAL} \) and \( \text{HAL} \) lead to an identical set of models, whence these logics define the same semantic consequence relation.

We have shown that \( \text{SAL} \) is sound with respect to its semantics, and that both \( \text{HAL} \) and \( \text{SAL} \) are weaker than \( \text{AL} \). Finally, we have established criteria for the completeness of \( \text{SAL} \), resp. \( \text{HAL} \), which immediately imply their equivalence to \( \text{AL} \). These facts allowed us to easily prove some additional properties of \( \text{SAL} \) and \( \text{HAL} \).

On the basis of these results, future research may further extend the metatheory of prioritized ALs, e.g. by showing whether one can prove a restricted cautious monotonicity theorem for \( \text{SAL} \) and \( \text{HAL} \) as well.\(^{21}\) Also, the current results may help in the establishment of a unified and generic proof theory for sequential superpositions and more specifically, logics in the \( \text{SAL} \)-format.

References


\(^{21}\)For readers not familiar with this property: \( L \) is cautiously monotonic if for every \( \Gamma' \subseteq C_{n_{L}} (\Gamma): C_{n_{L}} (\Gamma) \subseteq C_{n_{L}} (\Gamma' \cup \Gamma') \).


APPENDIX

A Some Syntactic Proofs

In this section, we define a generic proof theory for logics in $AL_{m}^{\prec}$-format – as before, $\prec$ is a metavariable for both $\subset$ and $\sqsubseteq$. We do so without further comments – for illustrations and philosophical motivations, we refer to [24, Section 2]. Subsequently, we prove two Theorems which call for syntactic meta-proofs and were therefore omitted from the main text. Both meta-proofs rely only on theorems stated in Section 2.

A.1 The Proof Theory of $AL_{m}^{\prec}$

Every $AL_{m}^{\prec}$-proof consists of lines that have four elements: a line number $i$, a formula $A$, a justification (consisting of a series of line numbers and a derivation rule) and a condition $\Delta \subseteq \Omega$. Where $\Gamma$ is the set of premises, the inference rules are given by:

**PREM** If $A \in \Gamma$:

\[
\begin{array}{c}
\vdots \\
A \\
\hline
\emptyset
\end{array}
\]

**RU** If $A_1, \ldots, A_n \vdash_{LLL\prec} B$:

\[
\begin{array}{c}
A_1 \ \Delta_1 \\
\vdots \\
A_n \ \Delta_n \\
\hline
B \ \Delta_1 \cup \ldots \cup \Delta_n
\end{array}
\]

**RC** If $A_1, \ldots, A_n \vdash_{LLL\prec} B \lor \mathsf{Dab}(\Theta)$:

\[
\begin{array}{c}
A_1 \ \Delta_1 \\
\vdots \\
A_n \ \Delta_n \\
\hline
B \ \Delta_1 \cup \ldots \cup \Delta_n \cup \Theta
\end{array}
\]
A stage of a proof is a sequence of lines, obtained by the application of the above rules. A proof is a sequence of stages. Every proof starts off with stage 1. Adding a line to a proof by applying one of the rules of inference brings the proof to a successor stage, which is the sequence of all lines written so far. An extension of a proof at stage $s$ is simply the same proof at a later stage $s'$.

A distinctive feature of adaptive proofs is the marking definition. At every stage of a proof, this definition determines for each line in the proof whether it is marked or not. If a line that has as its second element $A$ is marked at stage $s$, this indicates that according to our best insights at this stage, $A$ cannot be considered derivable. If the line is unmarked at stage $s$, we say that $A$ is derived at stage $s$ of the proof. To prepare for the marking definition, we need some more conventions.

Where $\emptyset \neq \Delta \subset \Omega$, $\text{Dab}(\Delta)$ is a Dab-formula at stage $s$ of a proof iff it is the second element of a line at stage $s$ with an empty condition. $\text{Dab}(\Delta)$ is a minimal Dab-formula at stage $s$ iff there is no other Dab-formula $\text{Dab}(\Delta')$ at stage $s$ for which $\Delta' \subset \Delta$. Where $\text{Dab}(\Delta_1), \text{Dab}(\Delta_2), \ldots$ are the minimal Dab-formulas at stage $s$ of a proof, let $\Sigma(\Gamma) = \{\Delta_1, \Delta_2, \ldots\}$. Let $\Phi^\Delta(\Gamma)$ be the set of $\prec$-minimal choice sets of $\Sigma(\Gamma)$.

**Definition 10** $\text{AL}^m$. Marking: a line $l$ with formula $A$ is marked at stage $s$ iff, where its condition is $\Delta$: (i) there is no $\varphi \in \Phi^\Delta(\Gamma)$ such that $\varphi \cap \Delta = \emptyset$, or (ii) for a $\varphi \in \Phi^\Delta(\Gamma)$, there is no line on which $A$ is derived on a condition $\Theta$ for which $\Theta \cap \varphi = \emptyset$.

Put differently: a line with formula $A$ is unmarked at stage $s$ iff its condition has an empty intersection with at least one $\varphi \in \Phi^\Delta(\Gamma)$, and for every $\psi \in \Phi^\Delta(\Gamma)$, there is a line on which $A$ is derived on a condition $\Delta$ such that $\Delta \cap \psi = \emptyset$. As a line may be marked at stage $s$, unmarked at a later stage $s'$ and marked again at a still later stage $s''$, we also define a stable notion of derivability.

**Definition 11** $A$ is finally derived from $\Gamma$ on line $l$ of a finite stage $s$ iff (i) $A$ is the second element of line $l$, (ii) line $l$ is unmarked at stage $s$, and (iii) every extension of the proof at stage $s$, in which line $l$ is marked may be further extended in such a way that line $l$ is unmarked again.

**Definition 12** $\Gamma \vdash_{\text{AL}^m} A$ iff $A$ is finally derived on a line of an $\text{AL}^m$-proof from $\Gamma$.

### A.2 A Specific Kind of Adequacy for $\text{AL}^m$

**Lemma 14** $\Phi(\Gamma) \neq \emptyset$.

*Proof. Case 1: $\Gamma$ has no $\text{LLL}^+$-models. By the completeness of $\text{LLL}^+$, $\Gamma \vdash_{\text{LLL}^+} A$ for every $A \in \Omega$, whence $\Sigma(\Gamma) = \{\{A\} \mid A \in \Omega\}$ and hence $\Phi(\Gamma) = \{\emptyset\} \neq \emptyset$.

Case 2: $\Gamma$ has $\text{LLL}^+$-models. By Theorem 2, $\Gamma$ has $\text{AL}^m$-models. By Theorem 6.2, $\Phi(\Gamma) = \{\text{Ab}(M) \mid M \in \text{M}_{\text{AL}^m}(\Gamma)\} \neq \emptyset$. ■

**Lemma 15** For every finite $\Delta \subset \Omega$: $\Gamma \models_{\text{LLL}^+} \text{Dab}(\Delta)$ iff $\Gamma \models_{\text{AL}^m} \text{Dab}(\Delta)$.\(^{22}\)

*Proof. $(\Rightarrow)$ Immediate in view of the fact that every $\text{AL}^m$-model of $\Gamma$ is an $\text{LLL}^+$-model of $\Gamma$ — see Definition 2. $(\Leftarrow)$ Suppose $\Gamma \models_{\text{AL}^m} \text{Dab}(\Delta)$. Let $M \in \text{M}_{\text{AL}^m}(\Gamma)$. If $M \in \text{M}_{\text{LLL}^+}(\Gamma)$, it follows by the supposition that $M \models \text{Dab}(\Delta)$. If $M \in \text{M}_{\text{LLL}^+}(\Gamma) - \text{M}_{\text{AL}^m}(\Gamma)$, then by Theorem 2, there is an $M' \in \text{M}_{\text{AL}^m}(\Gamma)$ such that $\text{Ab}(M') \subset \text{Ab}(M)$. In view of the supposition, $M' \models \text{Dab}(\Delta)$, whence $M' \models A$ for an $A \in \Delta$. It follows immediately that also $M \models A$, whence $M \models \text{Dab}(\Delta)$. ■

\(^{22}\)This lemma is a semantic variant of the one for Theorem 10 in [3], but generalized to every $\Gamma \subseteq \text{W}_+$. 

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Lemma 16 If $\Gamma \models_{\text{AL}_m} A$, then for every $\varphi \in \Phi^-(\Gamma)$, there is a $\Delta \subseteq \Omega - \varphi$ such that $\Gamma \vdash_{\text{LLL}^+} A \lor \text{Dab}(\Delta)$. 

Proof. Suppose $\Gamma \models_{\text{AL}_m} A$. Assume that there is a $\varphi \in \Phi^-(\Gamma)$ for which there is no $\Delta \subseteq \Omega - \varphi$ such that $\Gamma \models_{\text{LLL}^+} A \lor \text{Dab}(\Delta)$. This implies that $\Gamma$ is $\text{LLL}^+$-satisfiable. By the compactness of $\text{LLL}^+$ there is an $M \in \mathcal{M}_{\text{LLL}^+}(\Gamma)$ for which $M \not\models A$ and $M \models \text{Dab}(\Delta)$ for all $\Delta \subseteq \Omega - \varphi$. Hence, $M \not\models B$ for all $B \in \Omega - \varphi$. Hence, $\text{Ab}(M) \subseteq \varphi$. By Theorem 6.2 there is an $M' \in \mathcal{M}_{\text{AL}_m}(\Gamma)$ for which $\text{Ab}(M') = \varphi$. Since $M'$ is $\sim$-minimally abnormal, $\text{Ab}(M) = \text{Ab}(M')$. Hence, by Theorem 6.1, $M \in \mathcal{M}_{\text{AL}_m}(\Gamma)$. This is a contradiction since $M \not\models A$ and $\Gamma \models_{\text{AL}_m} A$. $\blacksquare$

Proof of Theorem 5. Note that if $\Gamma \vdash_{\text{AL}_m} A$, then $\Gamma \models_{\text{AL}_m} A$ by the (unrestricted) soundness of $\text{AL}_m$. For the other direction, suppose that $(\dagger) \Gamma = C_{\text{LLL}^+}(\Gamma)$ and $(\ddagger) \Gamma \models_{\text{AL}_m} A$. By Lemma 14, there is a $\varphi \in \Phi(\Gamma)$. By $(\dagger)$ and Lemma 16, there is a $\Delta \subseteq \Omega - \varphi$ such that $\Gamma \models_{\text{LLL}^+} A \lor \text{Dab}(\Delta)$. By the completeness of $\text{LLL}^+$, $\Gamma \vdash_{\text{LLL}^+} A \lor \text{Dab}(\Delta)$, whence by $(\dagger)$, $A \lor \text{Dab}(\Delta) \in \Gamma$. We start an $\text{AL}_m$-proof from $\Gamma$ as follows: (a) introduce the premise $A \lor \text{Dab}(\Delta)$ on line 1; (b) derive $A$ on line 2, using the rule RC, on the condition $\Delta$. Let $s_2$ be the stage consisting of these two lines.

Suppose line 2 is marked at stage $s_2$. This implies that $A \lor \text{Dab}(\Delta)$ is a Dab-formula, whence also $A$ is a Dab-formula. But then, by $(\dagger)$ and Lemma 15, $\Gamma \models_{\text{LLL}^+} A$. By the completeness of $\text{LLL}^+$, $\Gamma \vdash_{\text{LLL}^+} A$ whence by $(\dagger)$, $A \in \Gamma$. By the reflexivity of $\text{AL}_m$, $A \in C_{\text{AL}_m}(\Gamma)$.

Suppose line 2 is not marked at stage $s_2$. Suppose moreover that, in an extension of the proof, line 2 is marked. In view of the preceding, we may further extend the extended proof, such that (c) every minimal Dab-consequence of $\Gamma$ is derived in it, and (d) for every $\varphi' \in \Phi(\Gamma)$, $A$ is derived on a condition $\Delta'$ that has an empty intersection with $\varphi'$. Let $s$ be the stage of the second extension. In view of (c), $\Phi_s(\Gamma) = \Phi(\Gamma)$. Hence in view of (d), line 2 is unmarked at stage $s$. But then, by Definition 11, $A$ is finally derived at line 2, whence by Definition 12, $A \in C_{\text{AL}_m}(\Gamma)$. $\blacksquare$

A.3 Normal Premise Sets Revisited: $\text{AL}_m^+$

Lemma 17 If $\Gamma$ is normal up to level $i$, then there is no $\Delta \subseteq \Omega(i)$ such that $\Gamma \vdash_{\text{LLL}^+} \text{Dab}(\Delta)$. 

Proof. Suppose the antecedent holds, and let $\Delta$ be a finite subset of $\Gamma$. By the supposition, there is an $M \in \mathcal{M}_{\text{LLL}^+}(\Gamma \cup \Omega(i))$, whence $M \not\models \text{Dab}(\Delta)$. By the soundness of $\text{LLL}^+$, $\Gamma \not\vdash_{\text{LLL}^+} \text{Dab}(\Delta)$. $\blacksquare$

Lemma 18 If $\Gamma$ is normal up to level $i$, then at every stage $s$ of a proof from $\Gamma$, $\varphi \cap \Omega(i) = \emptyset$ for every $\varphi \in \Phi^+(\Gamma)$. 

Proof. Suppose the antecedent holds and let $s$ be a stage of a proof from $\Gamma$. Assume that for a $\varphi \in \Phi^+(\Gamma)$, $\varphi \cap \Omega(i) \neq \emptyset$. Let $\psi = \bigcup \{ \Theta - \Omega(i) \mid \Theta \in \Sigma^i(\Gamma) \}$. By the supposition and Lemma 17, every $\Theta \in \Sigma^i(\Gamma)$ is such that $\Theta - \Omega(i) \neq \emptyset$, whence $\psi$ is a choice set of $\Sigma^i(\Gamma)$. However, since $\psi \cap \Omega(i) = \emptyset$, it follows that $\psi \subseteq \varphi$ — a contradiction. $\blacksquare$

Theorem 25 If $\Gamma$ is normal up to level $i$, then $C_{\text{Dab}}(\Gamma) \subseteq C_{\text{AL}_m}(\Gamma)$.

Proof. Suppose the antecedent holds and $A \in C_{\text{Dab}}(\Gamma)$, whence $\Gamma \cup \Omega(i) \vdash_{\text{LLL}^+} A$. By the compactness of $\text{LLL}^+$, there are $B_1, \ldots, B_n \in \Gamma$ and there is a finite $\Delta \subseteq \Omega(i)$ such that $\{B_1, \ldots, B_n\} \cup \Delta^i \vdash_{\text{LLL}^+} A$. By the Deduction Theorem, $\{B_1, \ldots, B_n\} \vdash_{\text{LLL}^+} A \lor \text{Dab}(\Delta)$. 

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Let $p$ be an $AL^m$-proof from $\Gamma$, obtained by (i) introducing all the premises $B_i$ ($i \leq n$) and (ii) deriving $A$ on the condition $\Delta$ from these premises, by the rule RC. Let $l$ be the line on which $A$ is derived.

Assume that $l$ is marked. It follows that there is a $\phi \in \Phi^m(\Gamma)$ such that $\phi \cap \Delta \neq \emptyset$, whence $\phi \cap \Omega(i) \neq \emptyset$. By Lemma 18, $\Gamma$ is not normal up to level $i$ — a contradiction. By the same reasoning, it follows that in every extension of $p$, line $l$ remains unmarked. Hence $A$ is finally derived in $p$, whence $A \in Cn_{AL^m}(\Gamma)$. ■

B Criteria for Equivalence

For the proof of Theorem 21, we will rely on two facts and a lemma about minimal choice sets. The first fact was proven in [17] (Lemma 3.2.4), the second is an immediate consequence of Theorem 12 and Definition 5.

Fact 11 If every $\phi \in \Phi(\Gamma)$ is finite, then $\Phi(\Gamma)$ is finite.

Fact 12 $M_{AL^m}(\Gamma) = M_{SAL^m}(\Gamma) = \cap_{i \in I} M_{SAL^m}(\Gamma) = \cap_{i \in I} M_{AL^m}(\Gamma)$. 

Lemma 19 If $\Sigma$ is a finite set of sets, then $\Sigma$ has no infinite minimal choice sets.

Proof. Let $\Sigma = \{\Theta_i \mid i \leq n\}$ and let $\phi$ be an infinite choice set of $\Sigma$. For every $i \leq n$, let $A_i$ be an arbitrary element of $\phi \cap \Theta_i$, and let $\phi' = \{A_1, \ldots, A_n\}$. Note that since $\phi'$ is finite, $\phi' \subset \phi$. Since $\phi'$ is a choice set of $\Sigma$, $\phi$ is not a minimal choice set of $\Sigma$. ■

Proof of Theorem 21. Let $\Gamma \subseteq W$. Ad 1. ($\Rightarrow$) Immediate in view of Lemma 19. ($\Leftarrow$) Suppose that every $\phi \in \Phi(\Gamma)$ is finite. By Fact 11, $\Phi(\Gamma)$ is finite, whence also $\bigcup \Phi(\Gamma)$ and $\phi(\bigcup \Phi(\Gamma))$ are finite. As stated in [3, Theorem 11.5], $\bigcup \Sigma(\Gamma) = \bigcup \Phi(\Gamma)$. It follows that $\Sigma(\Gamma) \subseteq \phi(\bigcup \Phi(\Gamma))$, whence $\Sigma(\Gamma)$ is finite.

Ad 3. See Lemma 7 from [20].

Ad 4 and 7. Immediate in view of Lemma 19.

Ad 5. Immediate in view of Theorem 7.

Ad 8. ($\Rightarrow$) This is Lemma 9. ($\Leftarrow$) Suppose that for no $i \in I$, $\phi^{c,i}(\Gamma)$ has infinite minimal choice sets. Hence (1) for every $i \in I$, $M_{LLL^+}(Cn_{AL^m}(\Gamma)) = M_{AL^m}(\Gamma)$. By $(\cdot)$, Fact 5.4 and the monotonicity of $LLL^+$, for every $i \in I$, $M_{LLL^+}(Cn_{AL^m}(\Gamma)) \subseteq M_{AL^m}(\Gamma)$. By Fact 12, $M_{LLL^+}(Cn_{AL^m}(\Gamma)) \subseteq M_{AL^m}(\Gamma)$. Also, by the soundness of $AL^m$, $M_{LLL^+}(Cn_{AL^m}(\Gamma)) \supseteq M_{AL^m}(\Gamma)$. Hence $M_{LLL^+}(Cn_{AL^m}(\Gamma)) = M_{AL^m}(\Gamma)$. By Lemma 5, $\phi(\Gamma)$ has no infinite minimal choice sets. ■

C Some Negative Results

In this section, we define the prioritized adaptive logics $K_{2}^m$, $SK_{2}^m$ and $HK_{2}^m$, and use them to prove some negative results about the respective formats $AL^m$, $SAL^m$ and $HAL^m$. The concrete systems are merely presented for argumentative purposes — we refer to [24, Section 3] for more details and examples.

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23This is a well-established fact in the study of flat ALs. However, it is usually spelled out in terms of the set of unreliable formulas $U(\Gamma)$, which is identical to what we call $\bigcup \Sigma(\Gamma)$.
C.1 Some Particular Prioritized Adaptive Logics

Several ALs have been developed to explicate reasoning with prioritized belief bases—see [8], [27] and [26], and [23, Section 6]. The ALs that deal with such belief bases typically use a certain logical operator or a sequence of such operators to express that a belief has a certain degree of plausibility. We will define three such systems, one in each of the three generic formats, in order to prove the negative results we promised in Section 1.

We restrict the logic to the propositional level. We use the standard modal language $\mathcal{L}^M$ of Kripke’s minimal normal modal logic $K$. As usually, we define $\Box A = \neg \square \neg A$. Let $\mathcal{W}^M$ denote the set of modal wffs, and $\mathcal{W}^L$ the set of literals (sentential letters and their negations). To express the plausibility degree of a piece of information, sequences of diamonds are used: $\Diamond \Diamond \ldots \Diamond A$. The longer the sequence, the less plausible the information. A sequence of $i$ diamonds will be abbreviated by $\Diamond^i A$. Every set of abnormalities $\Omega^K_i$ ($i \in \mathbb{N}$) is defined as follows:

$$\Omega^K_1 = \{ ! A \mid A \in \mathcal{W}^L \}$$

Let in the remainder $\Omega^K_0 = \Omega^K_1 \cup \ldots \cup \Omega^K_i$ for all $i \in \mathbb{N}$. Every flat adaptive logic $K^{m}_{(i)}$ is defined by the triple $\langle K, \Omega(i), m \rangle$. We will often restrict ourselves to systems that only consider two degrees of plausibility, whence the number 2 in the name of the logics. By the triple characterization, we can readily define the following three systems:

1. the AL$m$-logic $K^{2m}$ defined by the triple $\langle K, \langle \Omega^K_1, \Omega^K_2 \rangle, m \rangle$
2. the superposition-logic $SK^{2m}$ defined by the sequence of logics $\langle K^{m}_{(1)}, K^{m}_{(2)} \rangle$
3. the hierarchic logic $HK^{2m}$ defined by the sequence of logics $\langle K^{m}_{(1)}, K^{m}_{(2)} \rangle$

Before we prove the negative results for $HK^{2m}$ and $SK^{2m}$, let us introduce some conventions that will lighten notation in the remainder. Where $i \in \mathbb{N}$, we say that $Dab(\Delta)$ is a Dab$_i$-consequence of $\Gamma$ iff $\Delta \subseteq \Omega(i)$. Where $Dab(\Delta_1), Dab(\Delta_2), \ldots$ are the minimal Dab$_i$-consequences of $\Gamma$, we use $\Phi^{(i)}(\Gamma)$ to refer to the set of $\subseteq$-minimal choice sets of $\Sigma^{(i)}(\Gamma) = \{ \Delta_1, \Delta_2, \ldots \}$. We use $\Phi^{(i)}(\Gamma)$ to refer to the $\subseteq$-minimal choice sets of $\Sigma^{(i)}(\Gamma)$, where the order $\subseteq$ is spelled out as in Definition 1, but replacing each $\Omega_i$ by $\Omega^K_i$. Finally, where $M \in \mathcal{M}_{K^{m}}$, let $Ab^{(i)}(M) = \{ A \in \Omega(i) \mid M \models A \}$. Slightly abusing notation, we write $M \models A$ to denote that $M \models A$ for every $A \in \Delta$. Where $\Delta \subseteq \mathcal{W}$, $\Delta^+ = \{ \vdash A \mid A \in \Delta \}$.

C.2 The Incompleteness Result for $HK^{2m}$

As a result of Corollary 12, whenever $\Gamma$ satisfies $\star_{HAL}^m$, then it holds that $A \in C\Gamma_{HK^{2m}}(\Gamma)$ iff $\Gamma \models_{HK^{2m}} A$. We will now give an example that shows that $HAL^m$ is not in general complete with respect to its semantics.

Let $\Gamma_1 = \Gamma_1^1 \cup \Gamma_1^2 \cup \Gamma_1^3 \cup \Gamma_1^4$, where

$$\begin{align*}
\Gamma_1^1 &= \{ !^i p \vee !^j q \mid i, j \in \mathbb{N}, i \geq j \} \\
\Gamma_1^2 &= \{ !^i q \vee !^j r \mid i, j \in \mathbb{N}, i \neq j \} \\
\Gamma_1^3 &= \{ !^i q \vee !^i r \mid i \in \mathbb{N} \} \\
\Gamma_1^4 &= \{ \bot \} 
\end{align*}$$

Lemma 20 $\Phi^{(2)}(\Gamma_1) = \{ !^i q \mid i \in \mathbb{N} \}$.

Proof. First of all, note that $\Phi^{(2)}(\Gamma_1) = \Upsilon_1 \cup \Upsilon_2$, where

$$\begin{align*}
\Upsilon_1 &= \{ !^i q \mid i \in \mathbb{N} \} \\
\Upsilon_2 &= \{ \bot \}
\end{align*}$$

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Υ₂ = {φk | k ∈ N} = {{i²φi, i²pj, i²r | i ∈ N \ {k}, j ≥ k | k ∈ N}
By Theorem 7, Φ⁻¹(Γ₁) ⊆ Φ⁽²⁾(Γ₁). Note that for every φ ∈ Υ₂, {i²φi | i ∈ N} ⊆ K⁻φ.
Hence Φ⁻¹(Γ₁) = {i²φi | i ∈ N}.

By Theorem 21.7, we can derive:

Corollary 14  ⁶Φ⁻¹(Γ₁) has no infinite minimal choice sets.

Lemma 21  Γ₁ ⊨HK₂⁺ s.
Proof. We prove that Γ₁ ⊨HK₂⁺ s – the rest is immediate in view of Corollary 1.
By Lemma 20 and Theorem 6, for every M ∈ M̂₂⁺(Γ₁), Ab(M) = {i²φi | i ∈ N}.
But then for every such M, M ⊭ s, whence in view of Γ₁, M ⊬ s. □

To show that s is not in the HK₂⁺-consequence set of Γ₁, we will need a slightly longer proof. Note that there is no Θ ⊂ Ω(Γ₁) such that Γ₁ ⊩ HK₂⁺ Dab(Θ). Hence Γ₁ is normal with respect to Ω(Γ₁).
By Theorem 11, we have:

Lemma 22  Cn(Γ₁) = CnHK₂⁺(Γ₁).

Lemma 23  s ∉ CnHK₂⁺(Γ₁).
Proof. Suppose s ∈ CnHK₂⁺(Γ₁). By Definition 8, CnHK₂⁺(Γ₁) ∪ CnHK₂⁺(Γ₁) ⊩HK⁺ s.
By Lemma 22, CnHK⁺(Γ₁) ∪ CnHK⁺(Γ₁) ⊩HK⁺ s. Since K⁺ is monotonic, transitive and reflexive, we can derive that Γ₁ ∪ Ω(Γ₁) ⮕ CnHK⁺(Γ₁) ⊩HK⁺ s.
Since K₂⁺(Γ₁) is reflexive, Ω(Γ₁) ⮕ CnHK⁺(Γ₁) ⊩HK⁺ s. By the compactness of K⁺, Ω(Γ₁) ⮕ CnHK⁺(Γ₁) ⊩HK⁺ s for a finite Θ ⊂ Ω(Γ₁). But then, by the deduction theorem, CnHK⁺(Γ₁) □HK⁺ s ⮕ Dab(Θ). By Theorem 3, Γ₁ □HK⁺ s ⮕ Dab(Θ).

Since Θ is finite, there is a k ∈ N such that, for every l ≥ k: !i²φi ∉ Θ. Let M ∈ M̂⁺(Γ₁) be such that each of the following holds:²⁴
(C₁)  Ab(Γ₁) = φk
(C₂)  M ⊭ s

By Theorem 6 and Lemma 20, M ∈ M̂⁺(Γ₁). By (C₁), M ⊭ Dab(Θ), whence by (C₂), also M ⊭ s ⮕ Dab(Θ). By the soundness of K₂⁺, Γ₁ ⊬HK⁺ s ⮕ Dab(Θ) — a contradiction. □

By Corollary 14, Lemma 21 and Lemma 23, we immediately have:

Theorem 26  There are Γ, A for which ⁶Φ⁻¹(Γ) has no infinite minimal choice sets and Γ ⊨HK₂⁺ A, but A ∉ CnHK₂⁺(Γ).

Likewise, by Lemma 21, Corollary 1 and Theorem 26, it follows that:

Theorem 27  There are Γ, A such that A ∈ CnHK₂⁺(Γ), whereas A ∉ CnHK₂⁺(Γ).

²⁴See Lemma 20 for the definition of φk.
C.3 Incomparability of $HK2^m_c$ and $SK2^m_c$

Lemma 24 $s \in Cn_{SK2^m_c}(\Gamma_1)$.

Proof. From Lemma 21 and Corollary 1, we can infer that $\Gamma_1 \models_{SK2^m_c} s$. By Corollaries 11.1 and 14, it follows that $s \in Cn_{SK2^m_c}(\Gamma_1)$.

Proof. Suppose that there is a $\Theta \subset \Omega_{(2)}(\Gamma_2)$, such that $Cn_{SK2^m_c}(\Gamma_2) \models_{K^+} r \in Dab(\Theta) = \emptyset$. Also, in view of $\Gamma_2$, let $\Gamma_2$ be such that $r \in Dab(\Theta) = \emptyset$. By Theorem 8, $r \in Cn_{\Omega_{(2)}^K}(\Gamma_2)$. Hence by Definition 8 and the reflexivity of $LLL^+$, $r \in Cn_{SK2^m_c}(\Gamma_2)$.

Proof. Let $\Theta \subset \Omega_{(2)}(\Gamma_2)$ be such that $Cn_{SK2^m_c}(\Gamma_2) \models_{K^+} r \in Dab(\Theta)$. By Theorem 8, $r \in Cn_{\Omega_{(2)}^K}(\Gamma_2)$. Hence by Definition 8 and the reflexivity of $LLL^+$, $r \in Cn_{SK2^m_c}(\Gamma_2)$. 

Lemma 25 $r \in Cn_{HK2^m_c}(\Gamma_2)$.

Proof. Note that $\Phi^{(2)}(\Gamma_2) = \Psi_1 \cup \Psi_2$, where

$\Psi_1 = \{\varphi_0\} = \{\{p_i \mid i \in N\}\}$
$\Psi_2 = \{\varphi_j \mid j \in N\} = \{\{p_i \mid i \in N \setminus \{j\}\}\} \cup \{\{q_i \mid k \geq j\}\} \cup \{\{s_i \mid j \in N\}\}$

In view of $\Gamma_2$, for every $\varphi_j \in \Psi_2$, there is a $\Theta_j = \{p_i, q_i\}$, such that $\Gamma_2 \models_{K^+} r \in Dab(\Theta_j)$ and $\Theta_j \cap \varphi_j = \emptyset$. Also, in view of $\Gamma_2$, $\Gamma_2 \models_{K^+} r \in Dab(\{s_i\})$, and $\{s_i\} \cap \varphi_0 = \emptyset$. By Theorem 8, $r \in Cn_{\Omega_{(2)}^K}(\Gamma_2)$. Hence by Definition 8 and the reflexivity of $LLL^+$, $r \in Cn_{SK2^m_c}(\Gamma_2)$.

Lemma 26 $\Gamma_2 \models_{SK2^m_c} r$.

Proof. From Lemma 21 and Corollary 1, we can infer that $\Gamma_1 \models_{SK2^m_c} s$. By Corollaries 11.1 and 14, it follows that $s \in Cn_{SK2^m_c}(\Gamma_1)$.

Proof. Suppose that there is a $\Theta \subset \Omega_{(2)}(\Gamma_2)$, such that $Cn_{SK2^m_c}(\Gamma_2) \models_{K^+} r \in Dab(\Theta)$. By Theorem 8, $r \in Cn_{\Omega_{(2)}^K}(\Gamma_2)$. Hence by Definition 8 and the reflexivity of $LLL^+$, $r \in Cn_{SK2^m_c}(\Gamma_2)$.

Proof. Let $\Theta \subset \Omega_{(2)}(\Gamma_2)$ be such that $Cn_{SK2^m_c}(\Gamma_2) \models_{K^+} r \in Dab(\Theta)$. By Theorem 8, $r \in Cn_{\Omega_{(2)}^K}(\Gamma_2)$. Hence by Definition 8 and the reflexivity of $LLL^+$, $r \in Cn_{SK2^m_c}(\Gamma_2)$.
(C₆) \( M \not\models A \) for every \( A \in \Omega^K_{\Theta} \setminus \{\langle i \rangle p_k \mid i \in \mathbb{N} \setminus \{k\}\} \)
(C₇) \( M \not\models A \) for every \( A \in \Omega^K_{\Theta} \setminus \{\langle i \rangle t_l, \langle i \rangle q_l, \langle i \rangle s_l \mid l \geq k\} \)

Note that by (C₆), \( M \not\models \Gamma^2 \); by (C₇), \( M \not\models \Gamma^2 \); by (C₆) and (C₇), \( M \not\models \Gamma^2 \); finally, by (C₆), \( M \not\models \Gamma^3 \). Suppose there is an \( M' \in M_{Km}(\Gamma^2) \) such that \( Ab^{(1)}(M') \subseteq Ab^{(1)}(M) \). In that case, \( M' \not\models \langle i \rangle p_k \) for an \( i \neq k \). But then, in view of \( \Gamma^1 \), \( M' \not\models \langle i \rangle p_k \), whence \( \langle i \rangle p_k \in Ab^{(1)}(M') \setminus Ab^{(1)}(M) \) — a contradiction. It follows that \( M \in M_{Km}(\Gamma^2) \).

Note that by (C₆) and (C₇), \( M \not\models Dab(\Lambda) \) for every \( \Lambda \subseteq \Omega^K_{\Theta} \setminus \{\langle i \rangle p_k \mid i \in \mathbb{N} \} \cup \{\langle i \rangle t_l, \langle i \rangle q_l, \langle i \rangle s_l \mid l \geq k\} \), whence by (8), \( M \not\models Dab(\Theta) \). Together with (C₆), this implies that \( M \not\models r \lor Dab(\Theta) \). By the completeness of \( K_{m}^{m}(\Gamma^2) \), \( \Gamma^2 \not\models Dab(\Theta) \) — a contradiction.

**Lemma 28** \( r \not\in Cn_{SK_{2}^{m}}(\Gamma^2) \).

**Proof.** First of all, note that \( \Phi^{(1)}(\Gamma^2) = \{\langle i \rangle p_k \mid i \in \mathbb{N} \setminus \{j\} \mid j \in \mathbb{J}\} \). In view of \( \Gamma^3 \), for every \( \varphi \in \Phi^{(1)}(\Gamma^2) \), \( \Gamma^2 \vdash \varphi \lor Dab(\Theta) \) for a \( \Theta \subseteq \Omega^K_{\Theta} \) such that \( \varphi \cap \Theta = \emptyset \). Hence by Theorem 8, \( \varphi \lor Dab(\Theta) \) holds for every \( \varphi \in \Phi^{(1)}(\Gamma^2) \).

By Lemma 27, there is no \( \Theta \subseteq \Omega^K_{\Theta} \) such that \( Cn_{SK_{2}^{m}}(\Gamma^2) \vdash \varphi \lor Dab(\Theta) \). By Theorem 8, \( r \not\in Cn_{SK_{2}^{m}}(\Gamma^2) \).

By Lemma 25 and Lemma 28, we immediately have:

**Theorem 29** There are \( \Gamma, A \) such that \( A \in Cn_{HK_{2}^{m}}(\Gamma) \), whereas \( A \not\in Cn_{SK_{2}^{m}}(\Gamma) \).

Since the \( SK_{2}^{m} \)-semantics and the \( HK_{2}^{m} \)-semantics are equivalent, and in view of Theorem 16, it follows immediately that \( SK_{2}^{m} \) is not complete with respect to its semantics:

**Theorem 30** There are \( \Gamma, A \) such that \( \Gamma \models_{SK_{2}^{m}} A \), whereas \( A \not\in Cn_{SK_{2}^{m}}(\Gamma) \).

Finally, by Corollary 1, Theorem 30 and the soundness and completeness of \( K_{2}^{m} \):

**Theorem 31** There are \( \Gamma, A \) such that \( A \in Cn_{K_{2}^{m}}(\Gamma) \), whereas \( A \not\in Cn_{SK_{2}^{m}}(\Gamma) \).

### C.4 The Conditions for Equivalence

In this section, we briefly show that the converses of items 2-7 of Theorem 21 fail, and that the right-left direction of Lemma 13 fails. We will not give full proofs for these claims, but simply list the counterexamples and some of their most salient properties.

Let us start with Theorem 21:

**Ad 2.** Let \( \Theta_1 = \{\langle i \rangle p_i \mid i \in \mathbb{N}\} \). Note that there is an infinite minimal choice set of \( \Sigma(\Theta_1) \), i.e. the set \( \varphi = \{\langle i \rangle p_i \mid i \in \mathbb{N}\} \). Still, \( \Phi(\Theta_1) = \{\langle i \rangle p_i \mid i \in \mathbb{N}\} \) is finite.

**Ad 3 and 6.** Let \( \Theta_2 = \{\langle i \rangle p_1, \langle i \rangle p_2 \mid i \in \mathbb{N}\} \). Let \( \Phi(\Theta_2) \) be the set of minimal choice sets with respect to the flat adaption logic \( K_{m}^{m} = (K, \bigcup_{i \in \mathbb{N}} \Omega^K_{\Theta}(\Theta), m) \) and let \( \Phi^{(c)}(\Theta_2) \) be the set of minimal choice sets with respect to the prioritized adaptive logic \( K_{m}^{m} = (K, \bigcup_{i \in \mathbb{N}} \Omega^K_{\Theta}(\Theta), m) \). Note that for every \( i \in \mathbb{N} \), \( \Sigma(\Theta_2) \) is finite, whence \( \Phi^{(1)}(\Theta_2) \) and \( \Phi^{(c)}(\Theta_2) \) have only finitely many minimal choice sets. However, \( \Phi(\Theta_2) \) is \( \Phi^{(c)}(\Theta_2) \) is infinite.

**Ad 4 and 7.** Let \( \Theta_3 = \{\langle i \rangle p_2, \langle i \rangle p_{2m+1} \mid n \in \mathbb{N}\} \). Note that \( \Phi^{(1)}(\Theta_3) = \Phi^{(c)}(\Theta_3) \) is infinite. Nevertheless, every minimal choice set of \( \Phi^{(1)}(\Theta_3) \) is \( \Phi^{(c)}(\Theta_3) \) is a couple: \( \Phi^{(1)}(\Theta_3) = \{\langle i \rangle p_2, \langle i \rangle p_3, \langle i \rangle p_4, \langle i \rangle p_5, \ldots\} \).
Ad 5. Let Θ₄ = {i^1_p_i ∨ i^2_q | i, j ∈ N, i ≠ j}. Let Ψ = {{i^1_p_i | i ∈ N − {k}} | k ∈ N}. Note that Φ(2)(Θ₄) = {{i^2_q}} ∪ Ψ, whereas ΦC(2)(Θ₄) = {{i^2_q}}.

The last example is also a counterexample for the right-left direction of Lemma 13. That is, although ℳΦ(2)(Θ₄) has an infinite minimal choice set (i.e. the set {i^1_p_i | i ∈ N}), it can be shown that ℳHK⁺(Θ₄) = ℳHK⁻(Θ₄).