A Sufficient Condition for Embedding Logics in Classical Logic*

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Abstract

It is shown that a set of logics, including many fragments of **CL**, (Classical Logic) can be embedded within **CL**. A logic belongs to the set iff it has a certain type of semantics, called nice semantics. The set includes many logics presented in the literature. The embedding turns finite premise sets into finite premise sets. So all embedded logics are semi-recursive and the partial decision methods for **CL** can be applied to them.

1 Aim of this Paper

A logic **L** is semi-recursive iff there is a Turing machine T with the following property: when given the input (A, Γ), T stops after finitely many steps with the answer YES iff $\Gamma \vdash_{\mathbf{L}} A$. The machine T, its tape, and the sequence of states of T and its tape can be described in **CL** (Classical Logic)—see for example [5]. This description can be seen as a kind of embedding of **L** in **CL** and every semi-recursive logic can be embedded in **CL** in this sense. Usually, however, "embedding" refers to more direct forms of embedding and this paper is about these.

It was shown in [4] that a set of propositional logics which are paraconsistent (A and $\neg A$ can be jointly true) or paracomplete (A and $\neg A$ can be jointly false) can be faithfully embedded within **CL**. In the present paper we generalize this result not only to the predicative version of those logics, but also to a large set of fragments of **CL** that allow for gluts and/or gaps with respect to other logical symbols than negation—that both A and $\neg A$ are true is a negation glut; that $A \wedge B$ is false while A and B are true is a conjunction gap. The embedding result is further generalized to all logics that have (what we shall call) a nice semantics. We moreover show that, where **L** has a nice semantics, A is a formula, and Γ is a recursive set of formulas, there exists a formula A' and a recursive Γ' such that

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$\Gamma \vdash_{\mathbf{L}} A$ iff $\Gamma' \vdash_{\mathbf{CL}} A'$. Moreover, if Γ is finite, then so is Γ'. One of the effects is that partial decision methods can be borrowed from **CL**, which entails that the consequence relation of a logic with a nice semantics is semi-recursive. We shall present an assignment that can serve as a common basis for the two-valued semantics of all such logics.

The embedding result depends essentially on the existence of a nice semantics for the embedded logic. This semantics will be deterministic and two-valued; the semantic metalanguage will always be classical. Before getting there, we need a few technicalities.

2 Preliminaries

Let $\mathcal{L}_s$ be the language of **CL** with the logical symbols $\neg$, $\wedge$, $\vee$, $\supset$, $\equiv$, $\forall$, $\exists$, and $=$ (but without function symbols); $\mathcal{C}$ is the set of (letters for) individual constants, $\mathcal{V}$ the set of individual variables, and $\mathcal{P}^r$ the set of predicates of rank $r \geq 0$—predicates of rank 0 will function as sentential letters. Officially, the members of $\mathcal{P}^r$ will be $P^r, Q^r, R^r, P^r_1, \ldots$, but we shall often write the superscripts invisibly, relying on the usual convention that we write only well-formed strings. Let $\mathcal{F}_s$ and $\mathcal{W}_s$ denote respectively the set of formulas and the set of closed formulas of $\mathcal{L}_s$.

By $\mathcal{L}$ we shall refer to any language that has the same non-logical symbols as $\mathcal{L}_s$ and an arbitrary set of logical symbols. In some contexts $\mathcal{L}$ will be a variable for such languages, in others it will refer to a specific such language. Let $\mathcal{F}$ and $\mathcal{W}$ denote respectively the set of formulas and the set of closed formulas of $\mathcal{L}$.

The easiest way to present the embedding is to consider a language $\mathcal{L}_\sharp$, which extends $\mathcal{L}_s$. We first introduce some functions that have $\mathcal{F}$ as their domain. Let $f(A)$ be the string obtained by replacing in $A \in \mathcal{F}$ every occurrence of an individual constant and every free occurrence of an individual variable by a centred dot. Thus $f(\exists y(Pay \supset Qbx)) = f(\exists y(Pxy \supset Qxx)) = \exists y(P \cdot y \supset Q \cdot\cdot)$. Let $h(A)$ be the number of centred dots that occur in $f(A)$—for example $h(\exists y(Pay \supset Qbx)) = 3$. Let $g(A)$ be the (possibly empty) string obtained by deleting from A all symbols except for occurrences of individual constants and free occurrences of individual variables. Thus $g(\exists y(Pay \supset Qbx)) = abx$, and $g(\exists y(Pxy \supset Qxx)) = xxx$. Finally, let the functions $g_i(A)$ denote the ith item in $g(A)$, $g_i(A)$ being undefined for $i \notin \{1, ..., h(A)\}$. For example, $g_2(\exists y(Pay \supset Qbx)) = b$ and $g_4(\exists y(Pay \supset Qbx))$ is undefined.

The language $\mathcal{L}_\sharp$ is obtained from $\mathcal{L}_s$ by adding (i) a new binary predicate I and (ii) a set of new predicates containing, for every $A \in \mathcal{W}$, a predicate $P^{h(A)}_{f(A)}$. Thus P^0_P and $P^0_{\forall x(Px \supset Qx)}$ are new predicates of rank 0, $P^1_{P\cdot}$ and $P^1_{\forall x(P\cdot \supset Qx)}$ are new predicates of rank 1, etc. Let, for every $r \in \mathbb{N}$, $\mathcal{P}^r_\sharp$ be the set of all predicates of rank r of $\mathcal{L}_\sharp$. Let $\mathcal{F}_\sharp$ and $\mathcal{W}_\sharp$ denote respectively the set of formulas and the set of closed formulas of $\mathcal{L}_\sharp$.

In order to simplify the characterization of the semantic systems, we introduce pseudo-languages. Let $\mathcal{O}$ be a set of *pseudo-constants*; $\mathcal{O}$ should have at least the cardinality of your largest set—the domain of a model is a set and a member of $\mathcal{O}$ should be mapped by the assignment v on every element of $\mathcal{O}$. The *pseudo-language* $^+\mathcal{L}$ is obtained from $\mathcal{L}$ by replacing $\mathcal{C}$ by $\mathcal{C} \cup \mathcal{O}$. Let $^+\mathcal{F}$ and $^+\mathcal{W}$ denote respectively the set of formulas and the set of closed formulas of $^+\mathcal{L}$. In a similar way one defines the pseudo-languages $^+\mathcal{L}_s$ and $^+\mathcal{L}_\sharp$ from $\mathcal{L}_s$

and $\mathcal{L}_\sharp$ respectively. Their sets of formulas are respectively ${}^+\mathcal{F}_s$ and ${}^+\mathcal{F}_\sharp$, their sets of closed formulas respectively ${}^+\mathcal{W}_s$ and ${}^+\mathcal{W}_\sharp$.

Extend the functions f, g, h, and g_i to the pseudo-languages ${}^+\mathcal{L}$, ${}^+\mathcal{L}_s$, and ${}^+\mathcal{L}_\sharp$ by letting them refer to $\mathcal{C} \cup \mathcal{O} \cup \mathcal{V}$ instead of to $\mathcal{C} \cup \mathcal{V}$. Let $\mathcal{Z}^0 = \{f(A) \mid A \in {}^+\mathcal{W}; h(A) = 0\} \cup {}^+\mathcal{W}$ and, for all $r > 0$, $\mathcal{Z}^r = \{f(A) \mid A \in {}^+\mathcal{W}; h(A) = r\}$. The sets $\mathcal{Z}^r_s$ and $\mathcal{Z}^r_\sharp$ are defined similarly (for all $r \geq 0$), replacing ${}^+\mathcal{W}$ by the suitable set of closed pseudo-formulas. Also extend f, g, h, and the g_i to the metalanguage in the standard way.

In the semantic systems, the assignment function v will assign a set of $h(A)$-tuples of members of the domain to every $f(A)$ for which A is a closed pseudo-formula of the language. So $v(f(P^2ab)) = v(P^2\cdot\cdot)$ is a set of couples. If v were to assign a value to P^2, one would obviously require that $v(P^2) = v(P^2\cdot\cdot)$. For this reason we identify, for every $\pi^r \in \mathcal{P}^r$, $v(\pi^r)$ with $v(\pi^r \cdot \ldots \cdot)$ (in which $\cdot \ldots \cdot$ denotes r centred dots). As an effect, $\mathcal{P}^r \subset \mathcal{Z}^r$. Moreover, a 0-tuple will be identified with $\emptyset$—see, for example, clause C2.1 in Section 4. So, if $h(\neg A) = 0$, $\langle v(g_1(\neg A)), \ldots, v(g_{h(\neg A)}(\neg A))\rangle$ is a 0-tuple, and hence is identified with $\emptyset$—see, for example, clause C2.$3^{o=}$ in Section 4.

Let, for every $r > 0$, $D^{(r)}$ denote the r-th Cartesian product of D and let $D^{(0)} = \{\emptyset\}$.

Let $\mathbb{P} \subset \mathcal{F}$ be the set of formulas that do not contain any logical symbols (not even identity), and let $\mathbb{P}^= = \mathbb{P} \cup \{\alpha = \beta \mid \alpha, \beta \in \mathcal{C} \cup \mathcal{V}\}$. Let ${}^+\mathbb{P}$ and ${}^+\mathbb{P}^=$ be defined analogously in terms of ${}^+\mathcal{F}$ and $\mathcal{C} \cup \mathcal{O} \cup \mathcal{V}$. Finally, let ${}^m\mathbb{P}$ be the set of metalinguistic formulas that do not contain any logical symbols and ${}^m\mathbb{P}^=$ the set of metalinguistic formulas that do not contain any logical symbols different from identity.

The further use of symbols will be self-explanatory, except (perhaps) for the following. ${}^m\mathcal{F}$ will denote the set of metalinguistic formulas (which contain only metavariables and logical symbols of the object language) and ${}^m\mathcal{W}$ the set of closed metalinguistic formulas. We shall use the following metametalinguistic variables: A and B as variables for metalinguistic formulas, P^r as a variable for metavariables for predicates of rank r, a, b, $\mathsf{a}_1, \ldots,$ as variables for metavariables for individual constants and individual pseudo-constants, and x as a variable for metavariables for individual variables. The symbols $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{A}'$, ... will be used as variables for metalinguistic statements that occur in semantic clauses (we shall call these statements *semantic statements*).

3 Nice semantics

All semantic systems will have the same type of models—$\mathcal{L}$ is a variable in the following definition.

Definition 1 *A model M (for the language ${}^+\mathcal{L}$ and hence for $\mathcal{L}$) is a couple $\langle D, v\rangle$ in which D is a non-empty set and the assignment v is as follows:*

C1.1 $v: \mathcal{C} \cup \mathcal{O} \to D$ *(where $D = \{v(\alpha) \mid \alpha \in \mathcal{C} \cup \mathcal{O}\}$)*
C1.2 $v: \mathcal{Z}^r \to \wp(D^{(r)})$ *(for every $r \in \mathbb{N}$)*

Let $\mathcal{M}$ be the set comprising the metavariables for non-logical symbols and the metavariables for formulas. Let $\overline{\mathsf{A}}$ be the set of members of $\mathcal{M}$ that occur in A. Let m be an *instantiation function* iff m maps every member of $\mathcal{M}$ on a symbol or formula from the object language for which it is a variable. The formula $m(\mathsf{A})$ is obtained by replacing every metavariable $\mu \in \mathcal{M}$ by $m(\mu)$. Let $i(\mathsf{A})$ be the set of all $A \in {}^{+}\mathcal{W}$ such that $m(\mathsf{A}) = A$ for an instantiation function m. A *logical form* ψ will be identified with a couple $\langle \mathsf{A}, \{\mathsf{B}_1, \ldots, \mathsf{B}_n\}\rangle$ ($n \geq 0$) and a formula A will be said to have the form $\psi = \langle \mathsf{A}, \{\mathsf{B}_1, \ldots, \mathsf{B}_n\}\rangle$ iff $A \in i(\mathsf{A}) - (i(\mathsf{B}_1) \cup \ldots \cup i(\mathsf{B}_n))$. If $n = 0$, we shall also say that A *has the form of* A.

We shall distinguish between two kinds of nice semantics: those for logics that follow the RoI schema and those for logics that do not. A logic **L** follows the RoI schema iff it validates the rule of replacement of identicals: $\alpha = \beta, A(\alpha) \vdash_{\mathbf{L}} A(\beta)$ for all $\alpha, \beta \in \mathcal{C}$.

Definition 2 *By* semantic elements *we shall mean the expressions that occur in quotation marks in (i)–(vi):*

(i) "$v_M(\mathsf{B}) = 1$", *with* $\mathsf{B} \in {}^{m}\mathcal{W}$,
(ii) "$\langle v(\mathsf{a}_1), \ldots, v(\mathsf{a}_r)\rangle \in v(\mathsf{P}^r)$",
(iii) "$v(\mathsf{a}) = v(\mathsf{b})$",
(iv) "$0 = 0$",
(v) "$\langle v(g_1(\mathsf{B})), \ldots, v(g_{h(\mathsf{B})}(\mathsf{B}))\rangle \in v(f(\mathsf{B}))$" *with* $\mathsf{B} \in {}^{m}\mathcal{W}$ *and* B *not of the form* $\mathsf{a} = \mathsf{b}$,
(vi) "$v(\mathsf{B}) = \{\emptyset\}$" *with* $\mathsf{B} \in {}^{m}\mathcal{W}$.

The semantic elements from (i)–(v) are RoI-semantic elements*, those from (i)–(iv) and (vi) are* non-RoI-semantic elements*; those from (ii)–(vi) are* semantic base elements.[1]

Definition 3 *A* RoI-valuation-defining clause *has officially the following structure:*

$$[\text{Where } \mathsf{A} \text{ has the form } \psi,]\ v_M(\mathsf{A}) = 1 \text{ iff } \mathfrak{A}.$$

provided (i) A *is the first element of* ψ*, (ii)* $\mathfrak{A}$ *is a finite semantic statement made up by parentheses, occurrences of "not", "or", "and", "for all* $\mathsf{a} \in \mathcal{C} \cup \mathcal{O}$*", "for at least one* $\mathsf{a} \in \mathcal{C} \cup \mathcal{O}$*", and one or more RoI-semantic elements, and (iii) every metavariable that occurs in* $\mathfrak{A}$ *either occurs in* A *or is bound by (or occurs in) a metaquantifier of the form "for all* $\mathsf{a} \in \mathcal{C} \cup \mathcal{O}$*" or "for at least one* $\mathsf{a} \in \mathcal{C} \cup \mathcal{O}$*".*

A non-RoI-valuation-defining clause *is defined similarly in terms of non-RoI-semantic elements.*

In the examples of semantic systems presented in subsequent sections, we shall often specify the form of A in a shorter way, leaving it to the reader to rephrase the clause in official structure. If $\psi = \langle \mathsf{A}, \emptyset\rangle$, the part between square brackets will be dropped altogether. Another example is clause C2.3$^{c\neg\wedge 1}$ of the $\mathbf{C_1}$-semantics. It reads "where $B \neq \neg A$, $v_M(\neg(A \wedge B)) = 1$ iff …", whereas its official structure is "where $\neg(A \wedge B)$ has the form $\langle\{\neg(A \wedge B)\}, \{\neg(A \wedge \neg A)\}\rangle$, $v_M(\neg(A \wedge B)) = 1$ iff …".

Where $\mathfrak{A}$ is a semantic statement and m is an instantiation function, define $m^*(\mathfrak{A})$ as the result of replacing every metalinguistic variable μ that occurs in $\mathfrak{A}$, by $m^*(\mu)$, which is defined as follows: (i) if μ occurs in a quantifier "for all $\mu \in \mathcal{C} \cup \mathcal{O}$" or "for at least one $\mu \in \mathcal{C} \cup \mathcal{O}$" or is bound by such a quantifier, then

[1] Semantic base elements do not refer to the (pre-)valuation value of other formulas.

$m^*(\mu) = \mu$, otherwise (ii) $m^*(\mu) = m(\mu)$. $m^*(\mathfrak{A})$ will be called an *instance* of $\mathfrak{A}$.

Definition 4 *An* instance *of the valuation-defining clause*

$$\text{[Where } \mathsf{A} \text{ has the form } \psi\text{,] } v_M(\mathsf{A}) = 1 \text{ iff } \mathfrak{A}.$$

is a statement

$$v_M(m(\mathsf{A})) = 1 \text{ iff } m^*(\mathfrak{A}).$$

provided m is an instantiation function and $m(\mathsf{A})$ has the form ψ.

Let $^{++}\mathcal{W}$ comprise all members of $^{+}\mathcal{W}$ together with the formulas that result from replacing in a member of $^{+}\mathcal{W}$ one or more members of $\mathcal{C} \cup \mathcal{O}$ by metavariables for individual constants. Let the *form* of the result be identical to the form of the formula from which it is obtained.

Definition 5 *A recursive set Ψ is a* complete set of logical forms *for $^{+}\mathcal{L}$ iff $\bigcup\{A \mid A \in {}^{++}\mathcal{W}$ has the form $\psi; \psi \in \Psi\} = {}^{++}\mathcal{W}$ and no formula of a form $\psi_1 \in \Psi$ has also a different form $\psi_2 \in \Psi$.*

Note that, in the following definition, α is an arbitrary variable for individual constants and a is an arbitrary metametavariable for those.

Definition 6 *A* regular complexity function *for $^{+}\mathcal{L}$ is a function $c\colon {}^{++}\mathcal{W} \to \mathbb{N}$ such that, if $B(\xi) \in {}^{+}\mathcal{F}$, then $c(B(\mathsf{a})) = c(B(\alpha))$.*

We shall say that a semantics is *complex* iff a pre-valuation function v_M is defined in terms of the assignment function v and the valuation function V_M is defined by $V_M(A) = v_M(\phi(A))$, in which $\phi\colon \mathcal{W} \to \mathcal{W}$ is a computable function. A semantics is *simple* if the valuation value of a formula coincides with its pre-valuation value. For the sake of uniformity, we shall then say that $V_M(A) = v_M(A)$. A special common case is that equivalence classes are defined by a (recursive) partition of all closed formulas and that all members of an equivalence class receive the same valuation value V_M in a model M. To realize this, let $s[\![A]\!]$ select an element from the equivalence class $[\![A]\!]$ and define $V_M(A) = v_M(s[\![A]\!])$.

Definition 7 *A semantics for a logic* **L** *with language $\mathcal{L}$ is* nice *iff (i) it has the models presented at the beginning of this section, (ii) there is a complete set of logical forms Ψ for $^{+}\mathcal{L}$ such that, for every $\psi \in \Psi$, the semantics has a unique valuation-defining clause*

$$\text{[Where } \mathsf{A} \text{ has the form } \psi\text{,] } v_M(\mathsf{A}) = 1 \text{ iff } \mathfrak{A}. \tag{1}$$

(iii) all the semantics' valuation-defining clauses are RoI or all are non-RoI, and (iv) there is a regular complexity function c such that it holds for every instance of every valuation-defining clause (1) of the semantics,

$$v_M(m(\mathsf{A})) = 1 \text{ iff } m^*(\mathfrak{A}).$$

that $c(B) < c(m(\mathsf{A}))$ whenever $B \in^{++} \mathcal{W}$ occurs in $m^*(\mathfrak{A})$.

For all logics with a nice semantics, truth in a model, semantic consequence, and validity are defined as usual—we shall sometimes write $M \Vdash A$ to express that M verifies A. As usual, "model" is used in this paragraph (and in similar passages later on) as comprising a model $M = \langle D, v\rangle$ in the strict sense *plus* the definition of the valuation function for the specific logic, which is here **CL**.

A *transparent semantic statement* is compounded from instances of semantic base elements by the connectives "(... and ...)", "(... or ...)", and "not ..." and by restricted quantifiers of the form "for all $\alpha \in \mathcal{C} \cup \mathcal{O}$" and "for at least one $\alpha \in \mathcal{C} \cup \mathcal{O}$". A *reduction statement* is a statement of the form "$v_M(A) = 1$ iff $\mathfrak{A}$", in which $\mathfrak{A}$ is a transparent semantic statement.[2]

Lemma 1 *In a nice semantics for* **L**, *a reduction statement "$v_M(A) = 1$ iff $\mathfrak{A}$" holds for every $A \in {}^{+}\mathcal{W}$, and there is an algorithm for constructing it.*

Proof. Let the semantics be nice in view of the complete set of logical forms Ψ and let c be a regular complexity function suitable for the nice semantics. We prove by an induction on $d(A) = c(A) - \min\{c(B) \mid B \in {}^{+}\mathcal{W}\}$ that a reduction statement "$v_M(A) = 1$ iff $\mathfrak{A}$" holds for every $A \in {}^{++}\mathcal{W}$. We also show the way in which the reduction statement is constructed.

For the basis, let $d(A) = 0$ and let $\psi \in \Psi$ be the form of A. As there is no $B \in {}^{++}\mathcal{W}$ for which $c(B) < c(A)$, the clause for ψ cannot contain semantic non-base elements in view of Definition 7. So the instance "$v_M(A) = 1$ iff $\mathfrak{A}$" of this clause is a reduction statement.

For the induction step, suppose that there is a reduction statement "$v_M(B) = 1$ iff $\mathfrak{B}$" for all $B \in {}^{++}\mathcal{W}$ for which $d(B) < n$. Consider an A for which $d(A) = n$ and let $\psi \in \Psi$ be the form of A. Consider the instance "$v_M(A) = 1$ iff $\mathfrak{A}'$" of the clause for ψ. In view of Definition 7, $d(B) < n$ for every $v_M(B) = 1$ that occurs in $\mathfrak{A}'$. So, for every such $v_M(B) = 1$, there is a reduction statement "$v_M(B) = 1$ iff $\mathfrak{B}$" in view of the induction hypothesis. Replacing in "$v_M(A) = 1$ iff $\mathfrak{A}'$", for every $B \in {}^{++}\mathcal{W}$, every $v_M(B) = 1$ by $\mathfrak{B}$ one obtains a reduction statement "$v_M(A) = 1$ iff $\mathfrak{A}$". ■

Corollary 1 *In every nice semantics, $v_M(A)$ is, for every A, a function of the model M.*

Corollary 2 *In a nice semantics for* **L**, *whether simple or complex, a reduction statement "$V_M(A) = 1$ iff $\mathfrak{A}$" holds for every closed formula A, and there is an algorithm for constructing it.*

Obviously, there may be several $\mathfrak{A}$ for which "$V_M(A) = 1$ iff $\mathfrak{A}$" is a reduction statement. If that is so, we shall take one such $\mathfrak{A}$ to be *selected*—$\mathfrak{A}$ will then be called the selected transparent statement.

4 Classical Logic and Its Basic Fragments

We begin with a nice semantics for **CL**. Its models are as defined at the outset of the previous section (here for the language ${}^{+}\mathcal{L}_s$). The valuation function $v_M\colon {}^{+}\mathcal{W} \to \{0,1\}$, determined by M, is defined by:

[2] Non-logical symbols of the object language occur in this occurrence of $\mathfrak{A}$ (and in future occurrences of similar expressions) and this was not the case for former occurrences. The context disambiguates everywhere.

C2.1	$v_M(\pi^r\alpha_1\ldots\alpha_r) = 1$ iff $\langle v(\alpha_1),\ldots,v(\alpha_r)\rangle \in v(\pi^r) \quad (r \geq 0)$[3]
C2.2	$v_M(\alpha = \beta) = 1$ iff $v(\alpha) = v(\beta)$
C2.3	$v_M(\neg A) = 1$ iff $v_M(A) = 0$
C2.4	$v_M(A \supset B) = 1$ iff $v_M(A) = 0$ or $v_M(B) = 1$
C2.5	$v_M(A \wedge B) = 1$ iff $v_M(A) = 1$ and $v_M(B) = 1$
C2.6	$v_M(A \vee B) = 1$ iff $v_M(A) = 1$ or $v_M(B) = 1$
C2.7	$v_M(A \equiv B) = 1$ iff $v_M(A) = v_M(B)$
C2.8	$v_M(\forall\xi A(\xi)) = 1$ iff $v_M(A(\alpha)) = 1$ for all $\alpha \in \mathcal{C} \cup \mathcal{O}$
C2.9	$v_M(\exists\xi A(\xi)) = 1$ iff $v_M(A(\alpha)) = 1$ for at least one $\alpha \in \mathcal{C} \cup \mathcal{O}$

For all $A \in \mathcal{W}$, $M \Vdash A$ iff $v_M(A) = 1$.

In order to extend the semantics to $\mathcal{L}_\sharp$, replace $\mathcal{Z}^r$ by $\mathcal{Z}^r_\sharp$ and ${}^+\mathcal{W}$ by ${}^+\mathcal{W}_\sharp$ in the definition of the assignment. This version will be used for the embedding.

Each of C2.1–9 specifies the valuation values of all formulas of a certain logical form. Let us call these nine logical forms the *simple logical forms*.

The basic fragments are obtained by removing one or both directions of the equivalences in the clauses C2.1–9. Thus, by removing "if $v_M(\neg A) = 1$, then $v_M(A) = 0$" some models will display negation gluts, by removing "$v_M(\neg A) = 1$ if $v_M(A) = 0$" some models will display negation gaps, and by removing both, some models will display both negation gluts and negation gaps.[4] Similarly, by removing "If $v_M(A \wedge B) = 1$, then $v_M(A) = 1$ and $v_M(B) = 1$" some models will display conjunction gluts and by removing "$v_M(\exists\xi A(\xi)) = 1$ if $v_M(A(\alpha)) = 1$ for at least one $\alpha \in \mathcal{C} \cup \mathcal{O}$" some models will display existential gaps. A semantics that allows for predicative gluts or gaps is obtained by removing one or both directions of C2.1.

The resulting semantic systems are indeterministic: the valuation values of formulas are not functions of the assignment values of their components. We shall devise equivalent nice (and hence deterministic) semantics, but first point to another peculiarity.

If some models of a logic **L** display gluts or gaps, RoI does not hold in **L**. In view of C2.1 and C2.2, $v_M(a = b) = 1$ warrants that $v_M(Pa) = v_M(Pb)$. But if there is, for example, a negation glut or gap, $v_M(a = b) = 1$ does not warrant that $v_M(\neg Pa) = v_M(\neg Pb)$. For some purposes, however, one will want to combine gluts or gaps with RoI. It is indeed possible to do so, as we now shall show.

An obvious example concerns gluts and gaps for negation. The six basic fragments handle negation gluts, negation gaps, or both negation gluts and gaps respectively. RoI does not hold in the first three logics, but holds in the last three (that have identity in the superscript). The nice semantics of the six logics is obtained from the above **CL**-semantics for $\mathcal{L}$ by replacing C2.3 according to the following table:

CL	**CLoN**	**CLuN**	**CLaN**	**CLoN**$^=$	**CLuN**$^=$	**CLaN**$^=$
C2.3	C2.3^o	C2.3^u	C2.3^a	C2.3$^{o=}$	C2.3$^{u=}$	C2.3$^{a=}$

The replacing clauses are:

C2.3^o $\quad v_M(\neg A) = 1$ iff $v(\neg A) = \{\emptyset\}$

[3] As stipulated in Section 2, $\langle v(\alpha_1),\ldots,v(\alpha_r)\rangle = \emptyset$ if $r = 0$. So $v_M(\pi^0) = 1$ iff $v(\pi^0) = \{\emptyset\}$.

[4] The resulting logics are called **CLuN** (for example in [3]), **CLaN** and **CLoN** respectively—they are like **CL** except in that they allow for, respectively, gl*u*ts, g*a*ps, and b*o*th gluts and gaps with respect to *n*egation.

C2.3^u $v_M(\neg A) = 1$ iff $v_M(A) = 0$ or $v(\neg A) = \{\emptyset\}$
C2.3^a $v_M(\neg A) = 1$ iff $v_M(A) = 0$ and $v(\neg A) = \{\emptyset\}$
C2.3$^{o=}$ $v_M(\neg A) = 1$ iff $\langle v(g_1(\neg A)), \ldots, v(g_{h(\neg A)}(\neg A))\rangle \in v(f(\neg A))$
C2.3$^{u=}$ $v_M(\neg A) = 1$ iff $v_M(A) = 0$ or $\langle v(g_1(\neg A)), \ldots, v(g_{h(\neg A)}(\neg A))\rangle \in v(f(\neg A))$
C2.3$^{a=}$ $v_M(\neg A) = 1$ iff $v_M(A) = 0$ and $\langle v(g_1(\neg A)), \ldots, v(g_{h(\neg A)}(\neg A))\rangle \in v(f(\neg A))$

Other gluts and gaps are handled similarly. Suppose that one wants to allow for gluts or gaps with respect to some logical symbol. In the above **CL**-semantics, the symbol is characterized by a simple form A and the clause for it reads "$v_M(\mathsf{A}) = 1$ iff Z" for some Z. In this clause, one replaces the expression "Z" by "Z or Y" to allow for gluts, by "Z and Y" to allow for gaps, and by "Y" to allow for both. In these expressions, Y is either $v(\mathsf{A}) = \{\emptyset\}$, in which case RoI is invalidated, or $\langle v(g_1(\mathsf{A})), \ldots, v(g_{h(\mathsf{A})}(\mathsf{A}))\rangle \in v(f(\mathsf{A}))$ in which case RoI is validated. Consider the clause for the universal quantifier as an example. Only gluts are allowed by the clause

$$v_M(\forall\xi A(\xi)) = 1 \text{ iff } v_M(A(\alpha)) = 1 \text{ for all } \alpha \in \mathcal{C} \cup \mathcal{O} \text{ or } v(\forall\xi A(\xi)) = \{\emptyset\}\,,$$

which invalidates RoI. Both gluts and gaps are allowed by the clause

$$v_M(\forall\xi A(\xi)) = 1 \text{ iff } \langle v(g_1(\forall\xi A(\xi))), \ldots, v(g_{h(\forall\xi A(\xi))}(\forall\xi A(\xi)))\rangle \in v(f(\forall\xi A(\xi)))\,,$$

which makes sure that RoI is validated.

Some special cases deserve a comment. The first case concerns predicative gluts or gaps. Consider the RoI variant of what the clause for predicative gluts would be:

$$v_M(\pi^r\alpha_1\ldots\alpha_r) = 1 \text{ iff } \langle v(\alpha_1), \ldots, v(\alpha_r)\rangle \in v(\pi^r) \text{ or } \langle v(g_1(\pi^r\alpha_1\ldots\alpha_r)), \ldots, v(g_{h(\pi^r\alpha_1\ldots\alpha_r)}(\pi^r\alpha_1\ldots\alpha_r))\rangle \in v(f(\pi^r\alpha_1\ldots\alpha_r))$$

which is equivalent to

$$v_M(\pi^r\alpha_1\ldots\alpha_r) = 1 \text{ iff } \langle v(\alpha_1), \ldots, v(\alpha_r)\rangle \in v(\pi^r) \text{ or } \langle v(\alpha_1), \ldots, v(\alpha_r)\rangle \in v(f(\pi^r\alpha_1\ldots\alpha_r)).$$

In Section 2 we have identified $v(\pi^r)$ with $v(f(\pi^r\alpha_1\ldots\alpha_r))$, but suppose we did not do so. Where $M = \langle D, v\rangle$, there obviously is a model $M' = \langle D, v'\rangle$ that is exactly as M except that $v'(\pi^r) = v(\pi^r) \cup v(f(\pi^r\alpha_1\ldots\alpha_r))$ and in which there are no predicative gluts in that the corresponding clause there reads:

$$v_{M'}(\pi^r\alpha_1\ldots\alpha_r) = 1 \text{ iff } \langle v'(\alpha_1), \ldots, v'(\alpha_r)\rangle \in v'(\pi^r)\,.$$

It is easily seen that $v_{M'}(A) = v_M(A)$ for all $A \in {}^{+}\mathcal{W}$.

So the semantics is equivalent to (defines the same consequence relation as) a simpler semantics. As this simpler semantics does not introduce predicative gluts, it follows at once that the original semantics does not introduce any identity gluts that show at the level of the consequence relation. By the same reasoning, one immediately sees that predicative gaps, either by themselves or combined with predicative gluts, are a useless complication if the semantics follows the RoI schema. So there is no harm in identifying $v(\pi^r)$ with $v(f(\pi^r\alpha_1\ldots\alpha_r))$ as we did.

If the logic does not follow the RoI schema, predicative gluts and gaps do have effect. Consider a semantics that is exactly like that for **CL** except that clause C2.1 is modified in order to allow for gluts and/or gaps. It is easily seen that RoI is not valid on this semantics.

The second special case is identity. As we are not interested here in the study of the basic logics themselves, two comments are sufficient. First, the RoI variant of the clause for identity gluts, which reads

$$v_M(\alpha = \beta) = 1 \text{ iff } v(\alpha) = v(\beta) \text{ or } \langle v(\alpha), v(\beta)\rangle \in v(\cdot = \cdot)\,,$$

obviously does not warrant the validity of RoI. Indeed, it allows for models in which $v(a) \neq v(b)$, $\langle v(a), v(b)\rangle \in v(\cdot = \cdot)$, $v(a) \in v(P)$, $v(b) \notin v(P)$, and hence $v_M(a = b) = v_M(Pa) = 1$ and $v_M(Pb) = 0$. Similarly for the RoI variant of the clause that allows for both identity gluts and gaps. The resulting logics have a semantics that follows the non-RoI schema.

We shall show that all basic fragments of **CL** can be embedded in **CL**. The same holds for certain extensions and fragments of them, which we discuss in the next section.

5 Other Logics that Have a Nice Semantics

An extension of a logic **L** may be defined in terms of axiom schemata. If one adds to the semantics of **L** a clause $v_M(\mathsf{A}) = 1$ for every new axiom schema A, the result will not be sensible because the new clauses may (and for some models will) contradict one of the original clauses for **L**. This, however may sometimes be repaired by first considering the original clause as a default (which is overruled by the new clauses) and next turning the semantics into a consistent and recursive one.

As a simple example, consider the extension of **CLoN** with the axiom schema $\neg\neg A \supset A$. The new semantic clause is $v_M(\neg\neg A \supset A) = 1$, which is contextually equivalent to "$v_M(\neg\neg A) = 0$ if $v_M(A) = 0$." It readily turns out that C2.3^o should be replaced by C2.3$'$ and C2.3$''$:

C2.3$'$ if A is not of the form $\neg B$, then $v_M(\neg A) = 1$ iff $v(\neg A) = \{\emptyset\}$
C2.3$''$ $v_M(\neg\neg A) = 1$ iff $v_M(A) = 1$ and $v(\neg\neg A) = \{\emptyset\}$

5.1 Some Maximal Fragments of CL

Two sets of logics between **CL** and those listed in the table in Section 4 will be considered.[5]

The first six will be called Schütte logics because their propositional fragments were first presented in [12]—their names are formed by appending a "s" to the systems they extend. The nice semantics for these systems is obtained from the **CL**-semantics of Section 4 by adding C2.3$^{\neg\neg}$–C2.3$^{\neg\exists}$ and by replacing C2.3 according to the following table:

CL	**CLoNs**	**CLuNs**	**CLaNs**	**CLoNs**$^=$	**CLuNs**$^=$	**CLaNs**$^=$
C2.3	C2.3^{op}	C2.3^{up}	C2.3^{ap}	C2.3$^{o=p}$	C2.3$^{u=p}$	C2.3$^{a=p}$

[5] All logics considered in this section have a characteristic three-valued semantics and their propositional fragments are maximally paraconsistent—see [2].

Here are the clauses:

C2.3^{op}	If $A \in {}^{+}\mathbb{P}^{=}$, $v_M(\neg A) = 1$ iff $v(\neg A) = \{\emptyset\}$
C2.3^{up}	If $A \in {}^{+}\mathbb{P}^{=}$, $v_M(\neg A) = 1$ iff $v_M(A) = 0$ or $v(\neg A) = \{\emptyset\}$
C2.3^{ap}	If $A \in {}^{+}\mathbb{P}^{=}$, $v_M(\neg A) = 1$ iff $v_M(A) = 0$ and $v(\neg A) = \{\emptyset\}$
C2.3$^{o=p}$	If $A \in {}^{+}\mathbb{P}^{=}$, $v_M(\neg A) = 1$ iff $\langle v(g_1(\neg A)), \ldots, v(g_{h(\neg A)}(\neg A))\rangle \in v(f(\neg A))$
C2.3$^{u=p}$	If $A \in {}^{+}\mathbb{P}^{=}$, $v_M(\neg A) = 1$ iff $v_M(A) = 0$ or $\langle v(g_1(\neg A)), \ldots, v(g_{h(\neg A)}(\neg A))\rangle \in v(f(\neg A))$
C2.3$^{a=p}$	If $A \in {}^{+}\mathbb{P}^{=}$, $v_M(\neg A) = 1$ iff $v_M(A) = 0$ and $\langle v(g_1(\neg A)), \ldots, v(g_{h(\neg A)}(\neg A))\rangle \in v(f(\neg A))$
C2.3$^{\neg\neg}$	$v_M(\neg\neg A) = v_M(A)$
C2.3$^{\neg\supset}$	$v_M(\neg(A \supset B)) = v_M(A \wedge \neg B)$
C2.3$^{\neg\vee}$	$v_M(\neg(A \wedge B)) = v_M(\neg A \vee \neg B)$
C2.3$^{\neg\wedge}$	$v_M(\neg(A \vee B)) = v_M(\neg A \wedge \neg B)$
C2.3$^{\neg\equiv}$	$v_M(\neg(A \equiv B)) = v_M((A \vee B) \wedge (\neg A \vee \neg B))$
C2.3$^{\neg\forall}$	$v_M(\neg\forall\xi A(\xi)) = v_M(\exists\xi\neg A(\xi))$
C2.3$^{\neg\exists}$	$v_M(\neg\exists\xi A(\xi)) = v_M(\forall\xi\neg A(\xi))$

While these six systems 'drive' negations of complex formulas 'inwards,' we now consider six systems in which negations behave classically in front of complex formulas. The logics arec called **CLoNv**, **CLuNv**, **CLaNv**, **CLoNv**$^{=}$, **CLuNv**$^{=}$, and **CLaNv**$^{=}$—the "v" refers to Arruda's so-called Vasil'ev system from [1], which is the propositional fragment of **CLuNv** and **CLuNv**$^{=}$.

The semantics of these logics is the same as that of the corresponding Schütte logic, except that C2.3^{v} is added instead of C2.3$^{\neg\neg}$–C2.3$^{\neg\exists}$:

C2.3^{v} where $A \in {}^{+}\mathcal{W} - {}^{+}\mathbb{P}^{=}$, $v_M(\neg A) = 1$ iff $v_M(A) = 0$,

5.2 Linguistic Extensions and Fragments

Several logics are fragments of the aforementioned ones, obtained by removing some logical symbols from the language. Their semantics is obtained by selecting the relevant valuation clauses from the logics of which they are fragments. Examples are **LP** from [10] (obtained from **CLuNs**$^{=}$ by removing $\supset$ from the language), the predicative version of $\mathbf{SK_3}$ from [8], etc.

Other logics are obtained from aforementioned fragments of **CL** by adding logical symbols that are definable in **CL**. Typical examples are logics extended with the missing classical connectives. Thus, if a logic handles negation gluts or gaps, the language may be extended with classical negation, say $\sim$. If it handles conjunction gaps or gluts, the language may be extended with classical conjunction, say $\sqcap$. The advantage of this linguistic extension is that it often greatly simplifies the metatheory.

The easiest way to handle linguistic extensions is to extend the language $\mathcal{L}$ and the pseudo-language ${}^{+}\mathcal{L}$ with the new symbol, and to extend the **CL**-semantics with an appropriate clause for the new symbol. In the case of added classical symbols, this clause will duplicate that for the original symbol (except for the single occurrence of the new symbol itself).

5.3 Other Roads to Gluts and Gaps

Many more logics than the ones described in this paper have a nice semantics and can be embedded in **CL** by the method described below.

Consider the result of replacing, in the **CL**-semantics from Section 4, C2.1 and C2.2 by

C2.1^o $v_M(\pi^r\alpha_1 \ldots \alpha_r) = \{\emptyset\}$ iff $v(\pi^r\alpha_1 \ldots \alpha_r) = 1 \quad (r \geq 0)$
C2.2^o $v_M(\alpha = \beta) = 1$ iff $v(\alpha = \beta) = \{\emptyset\}$

and C2.3 by C2.3$^{\neg\neg}$–C2.3$^{\neg\exists}$ together with

C2.3$^{\neg p}$ $v_M(\neg\pi^r\alpha_1 \ldots \alpha_r) = 1$ iff $\langle v(\alpha_1), \ldots, v(\alpha_r)\rangle \notin v(\pi^r) \quad (r \geq 0)$
C2.3$^{\neg=}$ $v_M(\neg\alpha = \beta) = 1$ iff $v(\alpha) \neq v(\beta)$

Suppose moreover that classical negation, $\sim$, is added to the language and correctly defined within the semantics—see the previous subsection.

The resulting logic allows for predicative gluts and gaps, for identity gluts and gaps, but also for negation gluts and gaps. The logic is not equivalent to any of the logics considered before, even if these are extended with classical negation. Indeed, unlike all previously considered logics, the present logic validates "If $\Gamma \vdash \sim\neg a = b$, then $\Gamma \vdash \neg A(a) \equiv \neg A(b)$."

Our proofs in Section 6 can handle gluts and gaps for different simple forms, provided all of them are RoI variants—we then say that the logic *follows the* RoI *schema*—or all of them are non-RoI variants—we then say that the logic *follows the non*-RoI *schema*.

5.4 A nice semantics for $\mathbf{C_1}$

A set of logics that have a nice semantics are the well-known $\mathbf{C}_n$-systems ($n \in \mathbb{N}$) from [6], further studied in [7] and many other papers—$\mathbf{C}_0$ is **CL**.

The nice semantics for $\mathbf{C_1}$ is like the one for **CL**, apart from the fact that v_M is not equal to V_M for $\mathbf{C_1}$ and that clause C2.3 must be replaced by clauses C2.3$^{c\neg p}$ to C2.3$^{c\neg\exists}$.

C3 $V_M(A) = v_M(T(A))$, where $T(A)$ is the result of first deleting all the vacuous quantifiers in A and then uniformly replacing all variables by the first variables of the alphabet in alphabetic order.
C2.3$^{c\neg p}$ Where $A \in \mathbb{P}^=$: $v_M(\neg A) = 1$ iff $v(A) = 0$ or $v(\neg A) = \{\emptyset\}$
C2.3$^{c\neg\neg}$ $v_M(\neg\neg A) = 1$ iff $v_M(\neg A) = 0$ or ($v_M(\neg A) = v_M(A) = 1$ and $v(\neg\neg A) = \{\emptyset\}$)
C2.3$^{c\neg\supset}$ $v_M(\neg(A \supset B)) = 1$ iff $v_M(A \supset B) = 0$ or (($v_M(\neg A) = v_M(A) = 1$ or $v_M(\neg B) = v_M(B) = 1$) and $v(\neg(A \supset B)) = \{\emptyset\}$)
C2.3$^{c\neg\vee}$ $v_M(\neg(A \vee B)) = 1$ iff $v_M(A \vee B) = 0$ or (($v_M(\neg B) = v_M(B) = 1$ or $v_M(\neg C) = v_M(C) = 1$) and $v(\neg(A \vee B)) = \{\emptyset\}$)
C2.3$^{c\neg\wedge 1}$ where $B \neq \neg A$: $v_M(\neg(A \wedge B)) = 1$ iff $v_M(A \wedge B) = 0$ or (($v_M(\neg A) = v_M(A) = 1$ or $v_M(\neg B) = v_M(B) = 1$) and $v(\neg(A \wedge B)) = \{\emptyset\}$)
C2.3$^{c\neg\wedge 2}$ $v_M(\neg(A \wedge \neg A)) = 1$ iff $v_M(\neg A) \neq v_M(A)$
C2.3$^{c\neg\forall}$ $v_M(\neg\forall\alpha A(\alpha)) = 1$ iff $v_M(\forall\alpha A(\alpha)) = 0$ or ($v_M(\neg A(\beta)) = v_M(A(\beta)) = 1$ for at least one $\beta \in \mathcal{C} \cup \mathcal{O}$ and $v(\neg\forall\alpha A(\alpha)) = \{\emptyset\}$)
C2.3$^{c\neg\exists}$ $v_M(\neg\exists\alpha A(\alpha)) = 1$ iff $v_M(\exists\alpha A(\alpha)) = 0$ or ($v_M(\neg A(\beta)) = v_M(A(\beta)) = 1$ for at least one $\beta \in \mathcal{C} \cup \mathcal{O}$ and $v(\neg\exists\alpha A(\alpha)) = \{\emptyset\}$)

Adjusting the semantics to any logic $\mathbf{C}_n$ ($n \in \mathbb{N}$) is straightforward.

5.5 A nice semantics for AN

The logic **AN** was presented in [9] by means of an elegant three-valued semantics. Its peculiarity is that paraconsistency is realized by weakening disjunction. **AN** validates all 'analysing rules' at the expense of giving up some 'constructive rules'.

The nice semantics for **AN** is like the one for **CL**, except that v_M is not equal to V_M, that the clauses for negation and disjunction are replaced as shown below, and that the clauses for implication and equivalence are removed—they are useless in view if C3.

C3 $V_M(A) = v_M(B)$, where B is the prenex conjunctive normal form of A—see, for example [5].
C2.3^{up} If $A \in {}^{+}\mathbb{P}^{=}$, $v_M(\neg A) = 1$ iff $v_M(A) = 0$ or $v(\neg A) = \{\emptyset\}$
C2.3$^{\neg\neg}$ $v_M(\neg\neg A) = v_M(A)$
C2.3$^{\neg\vee}$ $v_M(\neg(A \wedge B)) = v_M(\neg A \vee \neg B)$
C2.3$^{\neg\wedge}$ $v_M(\neg(A \vee B)) = v_M(\neg A \wedge \neg B)$
C2.3$^{\neg\forall}$ $v_M(\neg\forall\xi A(\xi)) = v_M(\exists\xi\neg A(\xi))$
C2.3$^{\neg\exists}$ $v_M(\neg\exists\xi A(\xi)) = v_M(\forall\xi\neg A(\xi))$
C2.6$^{a\vee}$ $v_M(A \vee B) = 1$ iff ($v_M(A) = 1$ and $v_M(\neg A) = 0$) or ($v_M(B) = 1$ and $v_M(\neg B) = 0$) or ($v_M(A) = v_M(B) = 1$)

5.6 A nice semantics for Łukasiewicz's m-valued logic Ł$_m$

The logical symbols of the language of Ł$_m$ are $\supset$, $\neg$ and $\forall$. A m-valued semantics (with values $1, 2, \ldots, m$, of which 1 is the only designated value) for these symbols is the following: $v_L(A \supset B) = max(1, 1 + v_L(B) - v_L(A))$, $v_L(\neg A) = m - v_L(A) + 1$ and $v_L(\forall\xi A(\xi)) = max\{v_L(A(\alpha)) \mid \alpha \in \mathcal{C} \cup \mathcal{O}\}$.

The nice semantics is most easily phrased in terms of defined symbols. As was established in [11], it is possible to define a set of symbols $\{I_k \mid k \in \{1, \ldots, m\}\}$ within Ł$_m$ such that (i) $v_L(I_k(A)) = 1$ iff $v_L(A) = k$ and (ii) $v_L(I_k(A)) = m$ otherwise. One most easily proceeds as follows.

D1 $A\&B =_{df} \neg(A \supset \neg B)$
D2 $A \wedge B =_{df} A\&(A \supset B)$
D3 $A \vee B =_{df} ((A \supset B) \supset B)\&((B \supset A) \supset A)$
D4 $A^i =_{df} \overbrace{A\&A\&\ldots\&A}^{i \text{ times}}$

Let $f_m(k)$ denote the least integer $n \geq \frac{m-k}{k-1}$.

D5 Define $I_k(A)$ recursively by:
(i) $I_1(A) = A^{m-1}$,
(ii) if $k \leq max(1, m-((k-1) \times f_m(k)))$, then $I_k(A) = ((\neg A^{f_m(k)} \vee A) \supset (\neg A^{f_m(k)} \wedge A))^{m-1}$, and
(iii) if $k > max(1, m-((k-1) \times f_m(k)))$, then $I_k(A) = I_{f_m(k)}(\neg(A^{f_m(k)}))$.
D6 $J_k(A) = I_1(A) \vee I_2(A) \vee \ldots \vee I_k(A)$

The nice semantics, which is two-valued, looks as follows (where $v_M(I_k(A)) = 1$ iff $v_L(A) = k$).

C2.1^1 $v_M(I_1(\pi^r\alpha_1 \ldots \alpha_r)) = 1$ iff $\langle v(\alpha_1), \ldots, v(\alpha_r)\rangle \in v(\pi^r)$ $(r \geq 0)$

C2.1^k where $k \in \{2, \ldots, m-1\}$, $v_M(I_k(\pi^r\alpha_1 \ldots \alpha_r)) = 1$ iff $\langle v(\alpha_1), \ldots, v(\alpha_r)\rangle \in v(f(I_k(\pi^r\alpha_1 \ldots \alpha_r)))$ and for every $i < k$, $v_M(I_i(\pi^r\alpha_1 \ldots \alpha_r)) = 0$ $(r \geq 0)$

C2.1^m $v_M(I_m(\pi^r\alpha_1 \ldots \alpha_r)) = 1$ iff for every $i < m$, $v_M(I_i(\pi^r\alpha_1 \ldots \alpha_r)) = 0$ $(r \geq 0)$

C2.3$^{Ł\neg}$ $v_M(I_k(\neg A) = 1$ iff $v_M(I_{m+1-k}(A)) = 1$

C2.4$^{Ł\supset}$ $v_M(I_k(A \supset B)) = 1$ iff $v_M(I_i(A)) = 1$ and $v_M(I_j(B)) = 1$ where i and j are such that $k = max(1, 1 + j - i)$

C2.8$^{Ł\forall}$ $v_M(I_k(\forall\alpha A(\alpha))) = 1$ iff for all $\beta \in \mathcal{C} \cup \mathcal{O}$, $v_M(J_k(A(\beta))) = 1$ and for at least one $\beta \in \mathcal{C} \cup \mathcal{O}$, $v_M(J_{k+1}(A(\beta))) = 0$

C2.10 where A is not of the form $I_k(B)$, $v_M(A) = v_M(I_1(A))$

Some of the above clauses look rather like schemata for clauses because m is unspecified and k is a variable. Once a specific logic $Ł_m$ is chosen, C2.1^k may be turned into $m - 2$ clauses—similarly for the other clause schemata. The quantifiers over natural numbers are then turned into finite disjunctions or conjunctions of semantic statements. So once a specific logic $Ł_m$ is chosen, one obtains a nice semantics for it.

A suitable complexity function c for this nice semantics is defined by the following clauses: if $A \in \mathbb{P}$, then $c(A) = 1$; $c(I_k(A)) = c(A) + k - 2$;[6] if $\neg A$ does not have the form $I_k(A)$, $c(\neg A) = c(A) + m + 1$; if $A \supset B$ does not have the form $I_k(A)$, $c(A \supset B) = c(A) + c(B) + (2 \times (m + 1))$; finally $c(\forall\alpha A(\alpha)) = (3 \times m \times c(A(\beta))) + (15 \times (m + 1)^2)$ for all $\beta \in \mathcal{C}$.

6 The Embedding

Let **L** be a logic that has an adequate nice semantics. In order to show that **L** can be embedded in **CL**, we shall first turn the **L**-semantics into a translation function tr which maps formulas (and sets of formulas) from $\mathcal{W}$ to formulas (and sets of formulas) from $\mathcal{W}_\sharp$, thus taking care of the embedding. We shall distinguish between two cases according as **L** follows the RoI schema or not. The second case is slightly more complicated.

6.1 Logics Following the RoI Schema

Let **L** be a logic that follows the RoI schema and has a nice semantics—so without gluts or gaps for either predicates or identity. We shall prove that, where $\Gamma \subset \mathcal{W}$ and $A \in \mathcal{W}$, $\Gamma \vdash_{\mathbf{L}} A$ iff $\mathrm{tr}(\Gamma) \vdash_{\mathbf{CL}} \mathrm{tr}(A)$, in which $\mathrm{tr}(\Gamma)$ is a finite set whenever Γ is finite.

Definition 8 *Where* **L** *is a logic that follows the RoI schema and has a nice semantics, the translation function* $\mathrm{tr}\colon \mathcal{W} \to \mathcal{W}_\sharp$ *for* **L** *is defined as:*

$$\mathrm{tr}(A) = \mathrm{TRoI}(\mathfrak{A}),$$

where $\mathfrak{A}$ *is the selected transparent semantic statement for which "*$V_M(A) = 1$ *iff* $\mathfrak{A}$*" is a reduction statement*[7] *and* TRoI *is the function from transparent*

[6]Note that $I_k(A)$ has the form $\neg B$ if $m > 2$ and has the form $\neg B$ or $B \supset C$ in case $m = 2$.
[7]See Corollary 2 and the subsequent paragraph.

(i)	$\text{TRoI}((\mathfrak{A})) = (\text{TRoI}(\mathfrak{A}))$
(ii)	$\text{TRoI}(\text{not } \mathfrak{A}) = \neg\text{TRoI}(\mathfrak{A})$
(iii)	$\text{TRoI}(\mathfrak{A} \text{ and } \mathfrak{B}) = \text{TRoI}(\mathfrak{B}) \wedge \text{TRoI}(\mathfrak{B})$
(iv)	$\text{TRoI}(\mathfrak{A} \text{ or } \mathfrak{B}) = \text{TRoI}(\mathfrak{A}) \vee \text{TRoI}(\mathfrak{B})$
(v)	$\text{TRoI}(\mathfrak{A}(\mathsf{a}) \text{ for all } \mathsf{a} \in \mathcal{C} \cup \mathcal{O}) = \forall\xi\, \text{TRoI}(\mathfrak{A}(\xi))$, where ξ is a variable that does not occur in $\mathfrak{A}(\mathsf{a})$
(vi)	$\text{TRoI}(\mathfrak{A}(\mathsf{a}) \text{ for at least one } \mathsf{a} \in \mathcal{C} \cup \mathcal{O}) = \exists\xi\, \text{TRoI}(\mathfrak{A}(\xi))$, where ξ is a variable that does not occur in $\mathfrak{A}(\mathsf{a})$
(vii)	$\text{TRoI}(\langle v(\alpha_1), \ldots, v(\alpha_r)\rangle \in v(\pi^r)) = \pi^r\alpha_1 \ldots \alpha_r$
(viii)	$\text{TRoI}(v(\alpha) = v(\beta)) = \alpha = \beta$
(ix)	if $A \notin {}^{+}\mathbb{P}^{=}$, $\text{TRoI}(\langle v(g_1(A)), \ldots, v(g_{h(A)}(A))\rangle \in v(f(A))) = P^{h(A)}_{f(A)} g(A)$
(x)	$\text{TRoI}(0 = 0) = (P^0 \vee \neg P^0)$

Table 1: RoI schema: from the semantics to tr

semantic statements to $\mathcal{W}_\sharp$-formulas that is recursively defined in Table 1. The translation function is extended to sets by $\text{tr}(\Gamma) = \{\text{tr}(A) \mid A \in \Gamma\}$.

Together with definition 7, the definition of the translation function warrants that tr is a total function.

Let us at once consider a complex example, viz. the translation function for **CLuNs**$^{=}$. The translation function is recursively defined as follows:[8]

T1	$\text{tr}(\pi^r\alpha_1 \ldots \alpha_r) = \pi^r\alpha_1 \ldots \alpha_r \quad (r \geq 0)$
T2	$\text{tr}(\alpha = \beta) = \alpha = \beta$
T3	$\text{tr}(A \supset B) = \text{tr}(A) \supset \text{tr}(B)$
T4	$\text{tr}(A \wedge B) = \text{tr}(A) \wedge \text{tr}(B)$
T5	$\text{tr}(A \vee B) = \text{tr}(A) \vee \text{tr}(B)$
T6	$\text{tr}(A \equiv B) = \text{tr}(A) \equiv \text{tr}(B)$
T7	$\text{tr}(\forall\xi A) = \forall\xi\, \text{tr}(A)$
T8	$\text{tr}(\exists\xi A) = \exists\xi\, \text{tr}(A)$
$\text{T9}^{u=p}$	If $A \in {}^{+}\mathbb{P}^{=}$, $\text{tr}(\neg A) = \neg\text{tr}(A) \vee P^{h(\neg A)}_{f(\neg A)} g(\neg A)$
$\text{T9}^{s\neg\neg}$	$\text{tr}(\neg\neg A) = \text{tr}(A)$
$\text{T9}^{s\neg\supset}$	$\text{tr}(\neg(A \supset B)) = \text{tr}(A) \wedge \text{tr}(\neg B)$
$\text{T9}^{s\neg\wedge}$	$\text{tr}(\neg(A \wedge B)) = \text{tr}(\neg A) \vee \text{tr}(\neg B)$
$\text{T9}^{s\neg\vee}$	$\text{tr}(\neg(A \vee B) = \text{tr}(\neg A) \wedge \text{tr}(\neg B)$
$\text{T9}^{s\neg\equiv}$	$\text{tr}(\neg(A \equiv B)) = (\text{tr}(A) \vee \text{tr}(B)) \wedge (\text{tr}(\neg A) \vee \text{tr}(\neg B))$
$\text{T9}^{s\neg\forall}$	$\text{tr}(\neg\forall\xi A) = \exists\xi\, \text{tr}(\neg A)$
$\text{T9}^{s\neg\exists}$	$\text{tr}(\neg\exists\xi A) = \forall\xi\, \text{tr}(\neg A)$

Definition 9 *Where $M = \langle D, v\rangle$ is a* **L***-model for ${}^{+}\mathcal{L}$ and $M' = \langle D, v'\rangle$ is a* **CL***-model for ${}^{+}\mathcal{L}_\sharp$, let RMM' iff the following conditions are fulfilled:*
R1 If $\alpha \in \mathcal{C} \cup \mathcal{O}$, then $v'(\alpha) = v(\alpha)$.
R2 If $A \in {}^{+}\mathbb{P}$, then $v'(A) = v(A)$.

[8]The translation function as presented here is slightly more restricted than the one obtained by Definition 8. The difference concerns only the clauses for the quantifiers, for which Definition 8 may lead to a relettering of the obtained formula.

R3 If $A \notin {}^{+}\mathbb{P}^{=}$, then $v'(P^{h(A)}_{f(A)}) = v(f(A))$.

Lemma 2 *(i) For every* **L***-model $M = \langle D, v\rangle$ for ${}^{+}\mathcal{L}$ there is a* **CL***-model $M' = \langle D, v'\rangle$ for ${}^{+}\mathcal{L}_{\sharp}$ such that RMM' and (ii) for every* **CL***-model $M' = \langle D, v'\rangle$ for ${}^{+}\mathcal{L}_{\sharp}$ there is a* **L***-model $M = \langle D, v\rangle$ for ${}^{+}\mathcal{L}$ such that RMM'.*

Proof. Immediate in view of Definition 9. ■

Lemma 3 *If $M = \langle D, v\rangle$ is a* **L***-model for ${}^{+}\mathcal{L}$, $M' = \langle D, v'\rangle$ is a* **CL***-model for ${}^{+}\mathcal{L}_{\sharp}$, RMM', and $\mathfrak{A}$ is an instance of a semantic base element, then $\mathfrak{A}$ holds true in M iff $v_{M'}(\mathrm{TRoI}(\mathfrak{A})) = 1$.*

Proof. Suppose that the antecedent is true. There are four cases.

Case 1: $\mathfrak{A}$ has the form $\langle v(\alpha_1), \ldots, v(\alpha_r)\rangle \in v(\pi^r)$, whence $\mathrm{TRoI}(\mathfrak{A})$ is $\pi^r\alpha_1 \ldots \alpha_r$. The consequent of the lemma follows in view of R1, R2, and C2.1.

Case 2: $\mathfrak{A}$ has the form $v(\alpha) = v(\beta)$, whence $\mathrm{TRoI}(\mathfrak{A})$ is $\alpha = \beta$. The consequent of the lemma follows in view of R1 and C2.2.

Case 3: $\mathfrak{A}$ has the form $\langle v(g_1(A)), \ldots, v(g_{h(A)}(A))\rangle \in v(f(A))$ and $A \notin {}^{+}\mathbb{P}^{=}$, whence $\mathrm{TRoI}(\mathfrak{A})$ is $P^{h(A)}_{f(A)}g(A)$. The consequent of the lemma follows in view of R1, R3, and C2.1.

Case 4: $\mathfrak{A}$ has the form $0 = 0$, whence $\mathrm{TRoI}(\mathfrak{A})$ is $P^0 \vee \neg P^0$. The consequent of the lemma follows in view of C2.6 and C2.3. ■

Lemma 4 If *$M = \langle D, v\rangle$ is a* **L***-model for ${}^{+}\mathcal{L}$, $M' = \langle D, v'\rangle$ is a* **CL***-model for ${}^{+}\mathcal{L}_{\sharp}$, RMM', and $\mathfrak{A}$ is a transparent semantic statement,* then *$\mathfrak{A}$ holds true in M iff $v_{M'}(\mathrm{TRoI}(\mathfrak{A})) = 1$.*

Proof. Suppose $M = \langle D, v\rangle$ is a **L**-model for ${}^{+}\mathcal{L}$, $M' = \langle D, v'\rangle$ a **CL**-model for ${}^{+}\mathcal{L}_{\sharp}$ and RMM'. We prove that

$$\mathfrak{A} \text{ holds true in } M \text{ iff } v_{M'}(\mathrm{TRoI}(\mathfrak{A})) = 1 \tag{2}$$

for every transparent semantic statement $\mathfrak{A}$, by means of an induction on the complexity of transparent semantic statements[9].

Base case: $\mathfrak{A}$ is an instance of a semantic base element. So (2) follows by Lemma 3.

For the induction step, suppose that, for every transparent semantic statement $\mathfrak{B}$ that is less complex than $\mathfrak{A}$, $\mathfrak{B}$ holds true in M iff $v_{M'}(\mathrm{TRoI}(\mathfrak{B})) = 1$. There are five cases.

Case 1–3. $\mathfrak{A}$ is of the form "not $\mathfrak{B}$", "$\mathfrak{B}_1$ and $\mathfrak{B}_2$", or "$\mathfrak{B}_1$ or $\mathfrak{B}_2$", whence respectively $\mathrm{TRoI}(\mathfrak{A}) = \neg\mathrm{TRoI}(\mathfrak{B})$, $\mathrm{TRoI}(\mathfrak{A}) = \mathrm{TRoI}(\mathfrak{B}_1) \wedge \mathrm{TRoI}(\mathfrak{B}_2)$ and $\mathrm{TRoI}(\mathfrak{A}) = \mathrm{TRoI}(\mathfrak{B}_1) \vee \mathrm{TRoI}(\mathfrak{B}_2)$. (2) follows in view of C2.3, C2.5, C2.6 and the induction hypothesis.

Case 4 and 5. $\mathfrak{A}$ is of the form "$\mathfrak{B}(\mathsf{a})$ for at least one $\mathsf{a} \in \mathcal{C} \cup \mathcal{O}$" or "$\mathfrak{B}(\mathsf{a})$ for every $\mathsf{a} \in \mathcal{C} \cup \mathcal{O}$", whence respectively $\mathrm{TRoI}(\mathfrak{A}) = \exists\xi\mathrm{TRoI}(\mathfrak{B}(\xi))$ and $\mathrm{TRoI}(\mathfrak{A}) = \forall\xi\mathrm{TRoI}(\mathfrak{B}(\xi))$. In view of C2.9 and C2.8, $v_{M'}(\exists\xi\mathrm{TRoI}(\mathfrak{B}(\xi))) = 1$ iff $v_{M'}(\mathrm{TRoI}(\mathfrak{B}(\alpha))) = 1$ for at least one $\alpha \in \mathcal{C} \cup \mathcal{O}$ and $v_{M'}(\forall\xi\mathrm{TRoI}(\mathfrak{B}(\xi))) = 1$ iff $v_{M'}(\mathrm{TRoI}(\mathfrak{B}(\alpha))) = 1$ for all $\alpha \in \mathcal{C} \cup \mathcal{O}$. The induction hypothesis entails

[9] The complexity of a transparent semantic statement, which should not be confused with the complexity function of a nice semantics, is the number of connectives and quantifiers that occur (in English) in the statement.

that for all $\alpha \in \mathcal{C} \cup \mathcal{O}$, $\mathfrak{B}(\alpha)$ holds true in M iff $v_{M'}(\text{TRoI}(\mathfrak{B}(\alpha))) = 1$. Hence, $v_{M'}(\exists\xi\text{TRoI}(\mathfrak{B}(\xi))) = 1$ iff $\mathfrak{B}(\alpha)$ holds true in M for at least one $\alpha \in \mathcal{C} \cup \mathcal{O}$ and $v_{M'}(\forall\xi\text{TRoI}(\mathfrak{B}(\xi))) = 1$ iff $\mathfrak{B}(\alpha)$ holds true in M for all $\alpha \in \mathcal{C} \cup \mathcal{O}$. So we have established (2). ■

Lemma 5 *If* tr *is the translation function for* **L**, $M = \langle D, v\rangle$ *is a* **L**-*model for* ${}^{+}\mathcal{L}$, $M' = \langle D, v'\rangle$ *a* **CL**-*model for* ${}^{+}\mathcal{L}_\sharp$, *and* RMM', *then* $v_{M'}(\text{tr}(A)) = v_M(A)$.

Proof. Immediate in view of Definition 8, Lemma 4 and Lemma 3. ■

Theorem 1 *If* **L** *has a nice semantics that follows the RoI schema and* tr *is the translation function for* **L**, *then* $\Gamma \vDash_{\mathbf{L}} A$ *iff* $\text{tr}(\Gamma) \vDash_{\mathbf{CL}} \text{tr}(A)$.

Proof. By Lemmas 2 and 5, if a **L**-model M verifies Γ and falsifies A, then there is a **CL**-model M' that verifies $\text{tr}(\Gamma)$ and falsifies $\text{tr}(A)$, and vice versa. ■

If Γ is a finite set, then so is $\text{tr}(\Gamma)$.

6.2 Logics Following the Non-RoI Schema

Let $\Delta^{=} = \{\forall x Ixx, \forall x\forall y\forall z(Ixy \supset (Ixz \equiv Iyz))\} \cup \{\forall x\forall y(Ixy \supset (A(x) \equiv A(y))) \mid A(x) \in \mathbb{P}\}$.[10] The main general difference with the previous subsection is that, whenever **L** has a nice semantics, the translation function tr will be such that where $\Gamma \subset \mathcal{W}$ and $A \in \mathcal{W}$, $\Gamma \vdash_{\mathbf{L}} A$ iff $\text{tr}(\Gamma) \cup \Delta^{=} \vdash_{\mathbf{CL}} \text{tr}(A)$. $\Delta^{=}$ is an infinite set, but we shall also be able to show that, under the above conditions, $\Gamma \vdash_{\mathbf{L}} A$ iff $\text{tr}(\Gamma) \cup \Delta^{=}_{\Gamma\cup\{A\}} \vdash_{\mathbf{CL}} \text{tr}(A)$, in which $\text{tr}(\Gamma) \cup \Delta^{=}_{\Gamma\cup\{A\}}$ is a finite set whenever Γ is finite.

Definition 10 *Where* **L** *is a logic that follows the non-RoI schema and has a nice semantics, the translation function* $\text{tr}\colon \mathcal{W} \to \mathcal{W}_\sharp$ *for* **L** *is defined as:*

$$\text{tr}(A) = \text{TNRoI}(\mathfrak{A}),$$

where $\mathfrak{A}$ *is the selected transparent semantic statement for which "*$V_M(A) = 1$ *iff* $\mathfrak{A}$*" is a reduction statement and* TNRoI *is the function from transparent semantic statements to* $\mathcal{W}_\sharp$*-formulas that is recursively defined in Table 2. The translation function is extended to sets by* $\text{tr}(\Gamma) = \{\text{tr}(A) \mid A \in \Gamma\}$.

Let us at once consider a complex example, viz. the translation function for $\mathbf{C_1}$. The translation function is defined by:[11]

T1 $\text{tr}(A) = \text{tr}'(T(A))$, where $T(A)$ is the result of first deleting all the vacuous quantifiers in A and then uniformly replacing all variables by the first variables of the alphabet in alphabetic order.
T2 $\text{tr}'(\pi^r\alpha_1 \ldots \alpha_r) = \pi^r\alpha_1 \ldots \alpha_r \quad (r \geq 0)$
T3 $\text{tr}'(\alpha = \beta) = I\alpha\beta$
T4 $\text{tr}'(A \supset B) = \text{tr}'(A) \supset \text{tr}'(B)$
T5 $\text{tr}'(A \wedge B) = \text{tr}'(A) \wedge \text{tr}'(B)$

[10] $\Delta^{=}$ **CL**-entails $\forall x\forall y(Ixy \supset Iyx)$ as well as $\forall x\forall y\forall z(Ixy \supset (Iyz \supset Ixz))$.

[11] The translation function as presented here is slightly more restricted than the one obtained by Definition 10. The difference concerns the clauses for formulas that contain quantifiers, for which Definition 10 may lead to a relettering of (parts of) the obtained formula.

(i)	TNRoI(($\mathfrak{A}$)) = (TNRoI($\mathfrak{A}$))
(ii)	TNRoI(not $\mathfrak{A}$) = ¬TNRoI($\mathfrak{A}$)
(iii)	TNRoI($\mathfrak{A}$ and $\mathfrak{B}$) = TNRoI($\mathfrak{B}$) ∧ TNRoI($\mathfrak{B}$)
(iv)	TNRoI($\mathfrak{A}$ or $\mathfrak{B}$) = TNRoI($\mathfrak{A}$) ∨ TNRoI($\mathfrak{B}$)
(v)	TNRoI($\mathfrak{A}(\mathsf{a})$ for all $\mathsf{a} \in \mathcal{C} \cup \mathcal{O}$) = $\forall\xi$ TNRoI($\mathfrak{A}(\xi)$), where ξ is a variable that does not occur in $\mathfrak{A}(\mathsf{a})$
(vi)	TNRoI($\mathfrak{A}(\mathsf{a})$ for at least one $\mathsf{a} \in \mathcal{C} \cup \mathcal{O}$) = $\exists\xi$ TNRoI($\mathfrak{A}(\xi)$), where ξ is a variable that does not occur in $\mathfrak{A}(\mathsf{a})$
(vii)	TNRoI($\langle v(\alpha_1), \ldots, v(\alpha_r)\rangle \in v(\pi^r)$) = $\pi^r\alpha_1 \ldots \alpha_r$
(viii)	TNRoI($v(\alpha) = v(\beta)$) = $I\alpha\beta$
(ix)	TNRoI($v(A) = \{\emptyset\}$) = $P^{h(A)}_{f(A)} g(A)$
(x)	TNRoI($0 = 0$) = $(P^0 \vee \neg P^0)$

Table 2: Without RoI: from the semantics to tr

T6	$\mathrm{tr}'(A \vee B) = \mathrm{tr}'(A) \vee \mathrm{tr}'(B)$
T7	$\mathrm{tr}'(A \equiv B) = \mathrm{tr}'(A) \equiv \mathrm{tr}'(B)$
T8	$\mathrm{tr}'(\forall\xi A) = \forall\xi\, \mathrm{tr}'(A)$
T9	$\mathrm{tr}'(\exists\xi A) = \exists\xi\, \mathrm{tr}'(A)$
T10$^{u=p}$	If $A \in {}^{+}\mathbb{P}^{=}$, $\mathrm{tr}'(\neg A) = \neg\mathrm{tr}'(A) \vee P^{h(\neg A)}_{f(\neg A)} g(\neg A)$
T10$^{s\neg\neg}$	$\mathrm{tr}'(\neg\neg A) = \neg\mathrm{tr}'(\neg A) \vee (\mathrm{tr}'(\neg A) \wedge \mathrm{tr}'(A) \wedge P^{h(\neg\neg A)}_{f(\neg\neg A)} g(\neg\neg A))$
T10$^{s\neg\supset}$	$\mathrm{tr}'(\neg(A \supset B)) = \neg\mathrm{tr}'(A \supset B) \vee (((\mathrm{tr}'(\neg A) \wedge \mathrm{tr}'(A)) \vee (\mathrm{tr}'(\neg B) \wedge \mathrm{tr}'(B))) \wedge P^{h(\neg(A\supset B))}_{f(\neg(A\supset B))} g(\neg(A \supset B)))$
T10$^{s\neg\wedge 1}$	if B is not of the form $\neg A$: $\mathrm{tr}'(\neg(A \wedge B)) = \neg\mathrm{tr}'(A \wedge B) \vee (((\mathrm{tr}'(\neg A) \wedge \mathrm{tr}'(A)) \vee (\mathrm{tr}'(\neg B) \wedge \mathrm{tr}'(B))) \wedge P^{h(\neg(A\wedge B))}_{f(\neg(A\wedge B))} g(\neg(A \wedge B)))$
T10$^{s\neg\wedge 2}$	$\mathrm{tr}'(\neg(A \wedge \neg A)) = \neg(\mathrm{tr}'(A) \wedge \mathrm{tr}'(\neg A))$
T10$^{s\neg\vee}$	$\mathrm{tr}'(\neg(A \vee B)) = \neg\mathrm{tr}'(A \vee B) \vee (((\mathrm{tr}'(\neg A) \wedge \mathrm{tr}'(A)) \vee (\mathrm{tr}'(\neg B) \wedge \mathrm{tr}'(B))) \wedge P^{h(\neg(A\vee B))}_{f(\neg(A\vee B))} g(\neg(A \vee B)))$
T10$^{s\neg\equiv}$	$\mathrm{tr}'(\neg(A \equiv B)) = \neg\mathrm{tr}'(A \equiv B) \vee (((\mathrm{tr}'(\neg A) \wedge \mathrm{tr}'(A)) \vee (\mathrm{tr}'(\neg B) \wedge \mathrm{tr}'(B))) \wedge P^{h(\neg(A\equiv B))}_{f(\neg(A\equiv B))} g(\neg(A \equiv B)))$
T10$^{s\neg\forall}$	$\mathrm{tr}'(\neg\forall\xi A(\xi)) = \neg\mathrm{tr}'(\forall\xi A(\xi)) \vee (\exists\xi(\mathrm{tr}'(\neg A(\xi)) \wedge \mathrm{tr}'(A(\xi))) \wedge P^{h(\neg\forall\xi A(\xi))}_{f(\neg\forall\xi A(\xi))} g(\neg\forall\xi A(\xi)))$
T10$^{s\neg\exists}$	$\mathrm{tr}'(\neg\exists\xi A(\xi)) = \neg\mathrm{tr}'(\exists\xi A(\xi)) \vee (\exists\xi(\mathrm{tr}'(\neg A(\xi)) \wedge \mathrm{tr}'(A(\xi))) \wedge P^{h(\neg\exists\xi A(\xi))}_{f(\neg\exists\xi A(\xi))} g(\neg\exists\xi A(\xi)))$

Definition 11 *Where $M = \langle D, v\rangle$ is a* **L***-model for ${}^{+}\mathcal{L}$ and $M' = \langle D, v'\rangle$ a* **CL***-model for ${}^{+}\mathcal{L}_\sharp$, let SMM' iff the following conditions are fulfilled:*
S1 If $\alpha \in \mathcal{C} \cup \mathcal{O}$, then $v'(\alpha) = \alpha$.
S2 If $\pi^r \in \mathcal{P}^r$, then $v'(\pi^r) = \{\langle\alpha_1, \ldots, \alpha_r\rangle \mid \langle v(\alpha_1), \ldots, v(\alpha_r)\rangle \in v(\pi^r)\}$.
S3 $v'(I) = \{\langle\alpha, \beta\rangle \mid v(\alpha) = v(\beta)\}$.
S4 For all $A \in {}^{+}\mathcal{W}$, if $f(A) \neq I$, then $v'(P^{h(A)}_{f(A)}) = \{\langle\alpha_1, \ldots, \alpha_r\rangle \mid$ for some $B \in {}^{+}\mathcal{W}$, $f(B) = f(A)$, $v(B) = \{\emptyset\}$ and $g(B) = \alpha_1 \ldots \alpha_r\}$.

Two models (for the same language) are *equivalent* iff they verify the same set of formulas. Where $M = \langle D, v\rangle$ is a **CL**-model for ${}^{+}\mathcal{L}_\sharp$, a predicate $\pi^2 \in \mathcal{P}^2_\sharp$ will be called *an identity relation over ${}^{+}\mathcal{L}_\sharp$ in M* iff $v(\pi^2)$ is reflexive, symmetric and transitive and, for all $\rho^r \in \mathcal{P}^r$, if $\langle v(\alpha_1), \ldots, v(\alpha_i), \ldots, v(\alpha_r)\rangle \in v(\rho^r)$ and $\langle v(\alpha_i), v(\beta)\rangle \in v(\pi^2)$, then $\langle v(\alpha_1), \ldots, v(\beta), \ldots, v(\alpha_r)\rangle \in v(\rho^r)$.

Lemma 6 *(i) For every* **L***-model* $M = \langle D, v\rangle$ *for* ${}^+\mathcal{L}$ *there is a* **CL***-model* $M' = \langle \mathcal{C} \cup \mathcal{O}, v'\rangle$ *for* ${}^+\mathcal{L}_\sharp$ *such that* SMM'*, and (ii) for every* **CL***-model* $M'' = \langle D'', v''\rangle$ *for* ${}^+\mathcal{L}_\sharp$ *in which* I *is an identity relation over* ${}^+\mathcal{L}_\sharp$*, there is an equivalent* **CL***-model* $M' = \langle \mathcal{C} \cup \mathcal{O}, v'\rangle$ *for* ${}^+\mathcal{L}_\sharp$ *and there is a* **L***-model* $M = \langle D, v\rangle$ *for* ${}^+\mathcal{L}$ *such that* SMM'*.*

Proof. The proof of (i) is immediate in view of the definition of SMM'. For the proof of (ii), consider a **CL**-model $M'' = \langle D'', v''\rangle$ for ${}^+\mathcal{L}_\sharp$ in which I is an identity relation over ${}^+\mathcal{L}_\sharp$. Let $M' = \langle \mathcal{C} \cup \mathcal{O}, v'\rangle$ be a **CL**-model for ${}^+\mathcal{L}_\sharp$ in which v' fulfills the following conditions:

(a) Where $\alpha \in \mathcal{C} \cup \mathcal{O}$, $v'(\alpha) = \alpha$.
(b) Where $\pi^r \in \mathcal{P}^r_\sharp$, $v'(\pi^r) = \{\langle \alpha_1, \ldots, \alpha_r\rangle \mid \langle v''(\alpha_1), \ldots, v''(\alpha_r)\rangle \in v''(\pi^r)\}$.

We leave it to the reader to prove that M' is equivalent with M'' and that I is an identity relation over ${}^+\mathcal{L}_\sharp$ in M'.

Let, for all $\alpha \in \mathcal{C} \cup \mathcal{O}$, $[\![\alpha]\!] = \{\beta \in \mathcal{C} \cup \mathcal{O} \mid \langle \alpha, \beta\rangle \in v'(I)\}$. Define a **L**-model $M = \langle D, v\rangle$ in which $D = \{[\![\alpha]\!] \mid \alpha \in \mathcal{C} \cup \mathcal{O}\}$ and v fulfills the following conditions:

$v1$ Where $\alpha \in \mathcal{C} \cup \mathcal{O}$, $v(\alpha) = [\![\alpha]\!]$.
$v2$ Where $\pi^r \in \mathcal{P}^r$, $v(\pi^r) = \{\langle [\![\alpha_1]\!], \ldots, [\![\alpha_r]\!]\rangle \mid \langle \alpha_1, \ldots, \alpha_r\rangle \in v'(\pi^r)\}$.
$v3$ Where $A \in {}^+\mathcal{W}$ and $g(A) = \alpha_1 \ldots \alpha_{h(A)}$, $v(A) = \{\emptyset\}$ iff $\langle \alpha_1, \ldots, \alpha_{h(A)}\rangle \in v'(P^{h(A)}_{f(A)})$.

SMM' holds because M and M' are models of the right sorts, (a) warrants S1, S2 is warranted by $v1$ together with $v2$ and the fact that I is an identity relation over ${}^+\mathcal{L}_\sharp$ in M', $v3$ warrants S4, and, given the way in which D is defined, $v1$ warrants S3. ■

Lemma 7 *If* SMM'*, then* $M' \Vdash \Delta^=$*.*

Proof. Suppose that SMM'. S1, S3 and C2.1 jointly warrant that $v_{M'}(\forall x Ixx) = v_{M'}(\forall x \forall y \forall z(Ixy \supset (Ixz \equiv Iyz))) = 1$.

Suppose moreover that $v_{M'}(\forall x \forall y(Ixy \supset (A(x) \equiv A(y)))) = 0$ for some $A(x) \in \mathbb{P}$, whence $v_{M'}(\forall x \forall y(Ixy \supset (A(x) \supset A(y)))) = 0$ or $v_{M'}(\forall x \forall y(Ixy \supset (A(y) \supset A(x)))) = 0$. We only consider the first possibility. It follows that there are $\alpha, \beta \in \mathcal{C} \cup \mathcal{O}$ such that $v_{M'}(I\alpha\beta) = v_{M'}(A(\alpha)) = 1$ and $v_{M'}(A(\beta)) = 0$. We shall show that this is impossible.

As $v_{M'}(I\alpha\beta) = 1$ (for those α and β), $v(\alpha) = v(\beta)$ by S1, S3 and C2.1. As $A(x) \in \mathbb{P}$, and hence $A(\alpha) \in {}^+\mathbb{P}$, it follows that $A(\alpha)$ has the form $\pi^r \gamma_1 \ldots \gamma_r$ and that α is one of the γ_i $(1 \leq i \leq r)$. Let us represent this by $\pi^r \gamma_1 \ldots \alpha \ldots \gamma_r$. The following equivalences obtain:

	$v_{M'}(\pi^r \gamma_1 \ldots \alpha \ldots \gamma_r) = 1$
iff (by C2.1)	$\langle v'(\gamma_1), \ldots, v'(\alpha), \ldots, v'(\gamma_r)\rangle \in v'(\pi^r)$
iff (by S1)	$\langle \gamma_1, \ldots, \alpha, \ldots, \gamma_r\rangle \in v'(\pi^r)$
iff (by S2)	$\langle v(\gamma_1), \ldots, v(\alpha), \ldots, v(\gamma_r)\rangle \in v(\pi^r)$
iff (as $v(\alpha) = v(\beta)$)	$\langle v(\gamma_1), \ldots, v(\beta), \ldots, v(\gamma_r)\rangle \in v(\pi^r)$
iff (by S1 and S2)	$\langle v'(\gamma_1), \ldots, v'(\beta), \ldots, v'(\gamma_r)\rangle \in v'(\pi^r)$
iff (by C2.1)	$v_{M'}(\pi^r \gamma_1 \ldots \beta \ldots \gamma_r) = 1$.

As $\pi^r \gamma_1 \ldots \beta \ldots \gamma_r$ is $A(\beta)$, this contradicts $v_{M'}(A(\beta)) = 0$. ■

Lemma 8 *If $M = \langle D, v\rangle$ is a* **L***-model for* $^+\mathcal{L}$*,* $M' = \langle D, v'\rangle$ *is a* **CL***-model for* $^+\mathcal{L}_\sharp$*, SMM', and $\mathfrak{A}$ is an instance of a semantic base element, then $\mathfrak{A}$ holds true in M iff* $v_{M'}(\mathrm{TNRoI}(\mathfrak{A})) = 1$.

Proof. Suppose that the antecedent is true. There are four cases.

Case 1: $\mathfrak{A}$ has the form $\langle v(\alpha_1), \ldots, v(\alpha_r)\rangle \in v(\pi^r)$, whence $\mathrm{TNRoI}(\mathfrak{A})$ is $\pi^r\alpha_1 \ldots \alpha_r$. The consequent of the lemma follows in view of S1, S2, and C2.1.

Case 2: $\mathfrak{A}$ has the form $v(\alpha) = v(\beta)$, whence $\mathrm{TNRoI}(\mathfrak{A})$ is $I\alpha\beta$. The consequent of the lemma follows in view of S1, S3 and C2.2.

Case 3: $\mathfrak{A}$ has the form $v(A) = \{\emptyset\}$ whence $\mathrm{TNRoI}(\mathfrak{A})$ is $P^{h(A)}_{f(A)}g(A)$. The consequent of the lemma follows in view of S1, S4, and C2.1.

Case 4: $\mathfrak{A}$ has the form $0 = 0$, whence $\mathrm{TNRoI}(\mathfrak{A})$ is $P^0 \vee \neg P^0$. The consequent of the lemma follows in view of C2.6 and C2.3. ■

Lemma 9 *If $M = \langle D, v\rangle$ is a* **L***-model for* $^+\mathcal{L}$*,* $M' = \langle D, v'\rangle$ *is a* **CL***-model for* $^+\mathcal{L}_\sharp$*, SMM', and $\mathfrak{A}$ is a transparent semantic statement, then $\mathfrak{A}$ holds true in M iff* $v_{M'}(\mathrm{TNRoI}(\mathfrak{A})) = 1$.

Proof. The proof is identical to the proof of Lemma 4, apart from the following two aspects: the reference to Lemma 3 should be changed into a reference to Lemma 8 and every occurrence of TRoI should be replaced by TNRoI. ■

Lemma 10 *If* tr *is the translation function for* **L***, $M = \langle D, v\rangle$ is a* **L***-model for* $^+\mathcal{L}$*,* $M' = \langle D, v'\rangle$ *a* **CL***-model for* $^+\mathcal{L}_\sharp$*, and SMM', then* $v_{M'}(\mathrm{tr}(A)) = v_M(A)$.

Proof. Immediate in view of Definition 10, Lemma 9 and Lemma 8. ■

Theorem 2 *If* **L** *has a nice semantics that follows the non-RoI schema and* tr *is the translation function for* **L***:* $\Gamma \vDash_{\mathbf{L}} A$ *iff* $\mathrm{tr}(\Gamma) \cup \Delta^= \vDash_{\mathbf{CL}} \mathrm{tr}(A)$.

Proof. For the first direction, suppose that there is a **CL**-model $M'' = \langle D'', v''\rangle$ for $^+\mathcal{L}_\sharp$ such that $M'' \Vdash \mathrm{tr}(\Gamma) \cup \Delta^=$ and $M'' \nVdash \mathrm{tr}(A)$. As $M'' \Vdash \Delta^=$, I is an identity relation over $^+\mathcal{L}_\sharp$ in M''. Hence, by Lemma 6, there is an equivalent **CL**-model $M' = \langle \mathcal{C} \cup \mathcal{O}, v'\rangle$ for $^+\mathcal{L}_\sharp$ and there is a **L**-model M for $^+\mathcal{L}$ such that SMM'. In view of Lemma 10, $M \Vdash \Gamma$ and $M \nVdash A$.

For the second direction, suppose that there is a **L**-model M such that $M \Vdash \Gamma$ and $M \nVdash A$. By Lemma 6, there is a **CL**-model M' such that SMM'. $M' \Vdash \Delta^=$ in view of Lemma 7; $M' \Vdash \mathrm{tr}(\Gamma)$ and $M' \nVdash \mathrm{tr}(A)$ in view of Lemma 10. ■

Even if Γ is a finite set, $\mathrm{tr}(\Gamma) \cup \Delta^=$ is an infinite set, which is inconvenient from a computational point of view. Let $\mathcal{P}_{\Gamma\cup\{A\}}$ be the set of members of $\mathcal{P}$ that occur in Γ or in A, let $\mathsf{Pr}(\Gamma \cup \{A\}) = \{\pi^r x_1 \ldots x_r \mid \pi^r \in \mathcal{P}_{\Gamma\cup\{A\}}\}$, and let $\forall\!\!\forall A$ be the universal closure of A (A preceded by a universal quantifier over every variable free in A). Finally, let $\Delta^=_{\Gamma\cup\{A\}} = \{\forall x Ixx, \forall x\forall y\forall z(Ixy \supset (Ixz \equiv Iyz))\} \cup \{\forall\!\!\forall(Ixy \supset (B(x) \equiv B(y))) \mid B(x) \in \mathsf{Pr}(\Gamma \cup \{A\})\}$. Clearly $\Delta^=_{\Gamma\cup\{A\}}$ is finite whenever Γ is so.

Theorem 3 $\mathrm{tr}(\Gamma) \cup \Delta^= \vDash_{\mathbf{CL}} \mathrm{tr}(A)$ *iff* $\mathrm{tr}(\Gamma) \cup \Delta^=_{\Gamma\cup\{A\}} \vDash_{\mathbf{CL}} \mathrm{tr}(A)$.

Proof. As $\Delta^=_{\Gamma\cup\{A\}} \subseteq \Delta^=$, the right–left direction is obvious. For the left–right direction, suppose that $\mathrm{tr}(\Gamma) \cup \Delta^=_{\Gamma\cup\{A\}} \nvDash_{\mathbf{CL}} \mathrm{tr}(A)$. It follows that there is a **CL**-model $M = \langle D, v\rangle$ for $^+\mathcal{L}_\sharp$ that verifies $\mathrm{tr}(\Gamma) \cup \Delta^=_{\Gamma\cup\{A\}}$ and falsifies $\mathrm{tr}(A)$. Let $M' = \langle D, v'\rangle$ be exactly as M, except that $v'(\pi^r) = \emptyset$ for all $\pi^r \in \mathcal{P} - \mathcal{P}_{\Gamma\cup\{A\}}$. It follows that M' verifies $\mathrm{tr}(\Gamma) \cup \Delta^=$ and falsifies $\mathrm{tr}(A)$. ■

7 In Conclusion

Our distinction in terms of the RoI schema is useful when one devises a nice semantics for a given logic and moreover simplifies the proof in the previous section. However, it is not difficult to unify the matter. First, modify two clauses in the definition of the function TRoI:

(viii) $\text{TRoI}(v(\alpha) = v(\beta)) = I\alpha\beta$
(ix) if $A \notin {}^{+}\mathbb{P}^{=}$,
$\text{TRoI}(\langle v(g_1(A)), \ldots, v(g_{h(A)}(A))\rangle \in v(f(A))) = Q^{h(A)}_{f(A)}g(A)$

Extend $\mathcal{L}_\sharp$ with predicates $Q^{h(A)}_{f(A)}$ for $A \in \mathcal{W}$ and replace, in the definition of $\Delta^{=}$, $\mathbb{P}$ by $\mathbb{P} \cup \{Q^{h(A)}_{f(A)}g(A) \mid A \in \mathcal{W}\}$. It is easily seen that the embedding still goes through for logics that follow the RoI schema. Moreover, the functions TRoI and TNRoI do not conflict. So they may be replaced by a single function that takes care of the embedding of both kinds of logics (and also of logics that follow the RoI schema at some points and not at others).[12] With these changes, Theorem 1 becomes invalid but Theorem 2 holds for all logics that have a nice semantics. Obviously the proofs should be adjusted.

When one comes across a new logic **L**, and devises a nice semantics for it—or possibly finds **L** by devising a nice semantics—our result provides an embedding of **L** in **CL** and constructively warrants that **L** is a semi-recursive logic.

An interesting open problem concerns the delineation of the set of logics that have a nice semantics and the procedure to devise, where this is possible, a nice semantics for a given logic.

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