Realist and constructivist semantics for modal modification: the case of alleged assassins

Bjørn Jespersen[1⋆] and Giuseppe Primiero[2⋆⋆]

[1] Department of Computer Science, Technical University of Ostrava;
Institute of Philosophy, Department of Logic,
Czech Academy of Sciences, Prague (Czech Republic)
bjornjespersen@gmail.com
[2] Centre for Logic and Philosophy of Science,
Ghent University (Belgium)
giuseppe.primiero@ugent.be

Abstract. Modal modification oscillates between being subsective and privative. Natural-language examples would be *Alleged*, *Reputed*, *Possible*, *Hoped-for*. This paper presents two kinds of procedural semantics regulating modal modification. One is a modal variation of the constructivist approach underlying Martin-Löf's Constructive Type Theory (CTT), the other is Tichý's Transparent Intensional Logic (TIL). CTT and TIL identify meanings with procedures, both consider functions basic, both embody a typed universe and both work with an interpreted syntax. While CTT and TIL differ over the semantics and logic of the oscillation between subsection and privation, they both see a strong similarity between modal modification and non-factive attitudes.

1 Introduction & Overview

Already the ancient Macedonians were well aware of the phenomenon of property modification. Thus, as Aristotle pointed out, a dead man is not a man. Nor does it follow that someone who is good and who is a cobbler is a good cobbler.[3]

Model-theoretic research into the formal, including logical, properties of natural language adjectives such as 'happy', 'big', 'skillful', 'forged', 'alleged', etc., conducted in the 1970s noted that some adjectives stand for *properties* and others for *modifiers*. It has become standard at least since then to operate with four different kinds of modifier:[4] *subsective*, *privative*, *intersective*, and *modal* (a.k.a.

⋆ Visiting Fellow, Project GACR 401/10/0792: "Temporal Aspects of Knowledge and Information".

⋆⋆ Postdoctoral Fellow of the Research Foundation - Flanders (FWO). Affiliated Researcher IEG, Oxford University.

[3] *De interpretatione*, c. xi: *"when in the adjunct there is some opposite which involves a contradiction, the predication of the simple term is impossible. Thus it is not right to call a dead man a man."*.

[4] See [9], [6].

intensional). Montague, at [12, pp. 211-14], makes *inter alia* two negative observations. One is that some adjectives denote *non-intersective* modifiers. The other is that some adjectives denote *non-subsective* modifiers.

Let a be an individual, F a property, M a function from properties to properties, $[MF]$ the property resulting from applying M to F, and $[[MF]a]$ the predication of $[MF]$ of a. Then the modifier M is a non-intersective if M fails to obey this rule of inference:

$$\frac{[[MF]a]}{Ma \wedge Fa}$$

Montague cites 'big' as an example: it is logically false (invalid) that every big flea is big (or a big entity).[5] As indeed it is, since we do not want to validate a transition from relative bigness to absolute bigness. But the above rule of inference is already formally flawed, in that M cannot be detached from the context $[MF]$ to figure as a property predicable of 'a'. M can, however, be pseudo-detached, in the sense defined and justified in [6], [3, §4.4], in accordance with the rule of pseudo-detachment, which allows the modifier M to be replaced by the property M^*, thus validating the inference

$$\frac{[[MF]a]}{M^*a \wedge Fa}$$

If M is the modifier Big and F the property $Flea$, the conclusion says that a is big, i.e. big with respect to some property or other, and that a is a flea. The crux of pseudo-detachment consists in existential quantification over properties: if a is a big flea then, trivially, there is a property p such that a is a big p; and 'big p' is abbreviated as 'big^*'. It is readily seen that pseudo-detachment is required to validate the first conjunct of intersective modification. Absent the rule, the conjunct that a is happy cannot be inferred from a being a happy logician, whenever "a is a happy logician" is not a truncated version of "a is happy and a is a logician". We shall not deal further with non-intersective modification here.

M is a non-subsective modifier if M fails to obey this rule of inference, which is syntactically an elimination rule for M:[6]

$$\frac{[[MF]a]}{Fa}$$

[5] See [12, p.211].

[6] If all modifiers turn out to be subsective, then they will all be right upward monotonic. But they are not, at least not in the straightforward sense in which happy children are children and forged banknotes are not banknotes. However, [13] from a model-theoretic perspecive sets out to 'reduce' every kind of modification to subsection. From the proof-theoretic perspective of CTT, intersection and privation are both cases of subsection. The modal extension to CTT presented in this paper to formalize modal modification, deviates from this pattern, also opening to non-monotonic derivability. On modification and monotonicity, see [23].

Typically, non-subsective modifiers come in two flavours: *modal* and *privative*. The logical difference between the two is whether it is *possible* or *necessary*, respectively, that a who or which is an $[MF]$ not be an F.

M is a privative modifier if M obeys this rule:

$$\frac{[[MF]a]}{\neg Fa}$$

Modal modification is not a primitive but derived kind of modification, being defined in terms of subsection, privation and disjunction introduction.[7] As a first attempt, M is a modal modifier if M obeys the following rule of inference:[8]

$$\frac{[[MF]a]}{Fa \vee \neg Fa}$$

However, the conclusion trivially applies to all other modifiers as well, since it is a classical tautology. E.g. if a is a skillful surgeon then either a is a surgeon or a is not a surgeon. It is just that the left disjunct is always true. Not so if M is modal. Then sometimes the left disjunct will be true and sometimes the right one. This is, of course, because modal modifiers oscillate between subsection and privation. Hence, the set-theoretic translation (in standard notation) of modal modification is in terms of the union of any set of Fs, $|F|$, and its complement, $\overline{|F|}$:

$$\{x \mid [M_{modal}F]x\} \in |F| \cup \overline{|F|}$$

The oscillation between subsection and privation points to a strong similarity between modal modifiers and non-factive attitudes.[9] Informally stated:

$$\frac{b \text{ knows whether } a \text{ is an assassin}}{\text{Possibly, } a \text{ is an assassin} \wedge \text{ possibly, } a \text{ is not an assassin}}$$

If you know that a is a skillful surgeon then a is a surgeon. If you know that b is a forged banknote then b is not a banknote. If you know that c is a skillful surgeon then c is skillful (i.e. skillful at something) and c is a surgeon. If you know that d is an alleged assassin then d is alleged to be something (which is a non-trivial property), but it is possible that d be an assassin and it is possible that d not be an assassin. So you know too little to know whether d is an assassin. What you do know is that there is a fact of the matter: either d is an assassin, or d is not an assassin. It is just that modal modification leaves it open which side the truth comes down on. It cannot be assumed, however, that there will never be

[7] The other derived kind of modification is intersection, defined in terms of subsection, pseudo-detachment and conjunction introduction.

[8] As for 'modal modifier', this is the term used by, e.g. Partee; see [13, p. 7]. Such modifiers are also known as 'intensional', see [1, p. 17] and [19, p. 276, n. 14].

[9] A deeper study of modal modifiers might reveal that many of them are attitudinal in nature, as exemplified by *hoped-for result* or *estimated value*. Such a study must await another occasion, however.

a third option. For instance, if the property is not *being an assassin* but *having stopped smoking*, then you do not know that either e stopped smoking or e did not stop smoking, for neither is true in case e never smoked in the first place. Yet it may be true that e has allegedly stopped smoking (or, is an alleged ex-smoker). This point, which bears on truth-value gaps, partiality and unfulfilled presuppositions, will re-emerge below. However, if we disregard, for now, the third option, it holds that modal modification is tucked into the conceptual space left between determinateness (cf. factive attitudes and the other modifiers) and epistemic possibility.

The premise of the rule of modal modification is true only if there is a fact of the matter (it is true that a is an assassin, or it is true that a is not an assassin); or, from an epistemic perspective, if b satisfies the necessary and sufficient conditions for asserting that it is true of a that (s)he is an assassin. But the conclusion leaves it open which of the two options obtains, hence either of Fa and $\neg Fa$ is epistemically possible. From a fact-oriented viewpoint, when learning (i.e. getting to know) that a is an alleged assassin, the obvious next step is to find out whether a is, in fact, an assassin. From an epistemic viewpoint, when learning that b knows whether a is an assassin, the obvious next step is to ask b under what conditions (s)he can claim that a is, in fact, an assassin. Attributee b knows which of the two options is true, but the knowing-whether attribution does not reveal which of the two b knows. We know that b knows but not what b knows.

In this paper we are going to extend to modal modification the two kinds of procedural semantics - one constructivist, the other Platonist - we already applied to privative modification and presented in great technical detail in [18].[10] Modal modifiers are characterized by two interlocking features:

1. oscillation between being subsective and being privative: if a is an alleged assassin, then either a is an assassin or a is not an assassin (with provisos for partiality);
2. a strong similarity to non-factive attitudes: if b knows whether/believes that a is an assassin then either a is an assassin or a is not an assassin (again with provisos for partiality).

The relevant constructivist, or proof-theoretic, semantics is Martin-Löf's Constructive Type Theory. CTT accommodates the two features in terms of laying down the epistemic conditions under which an agent is entitled to judge that a is an alleged assassin. The leading idea is to operate with possibility judgments induced from a truth dependent on (open) assumptions, as envisaged in [18, p.270], since this is the only constructivist way to say that it is *not* known whether a is an assassin. To obtain a typed language for open conditions, appropriate variations

[10] The vast majority of research into modification is conducted within linguistics. We wish to emphasize that our contribution is, instead, logical and semantic in nature. The logic of modal modifiers, in particular, remains a dimly lit corner of modification theory. We also wish to stress the experimental nature of this paper, thanks to its pioneering effort.

on the standard language will be needed, and a modal extension is provided to speak about such conditions in the language. A possibility judgement so obtained satisfies the basic requirement that knowing whether a is an assassin depends on whether the appropriate conditions are indeed satisfied: the judgement itself says that there are contexts in which the conditions can be satisfied, but it does not certify that there is no context in which the conditions are not refuted. This translates in an epistemically acceptable way the intuitionistically inadmissible conclusion of the rejected rule of modal modification.

The relevant Platonist semantics is Tichý's Transparent Intensional Logic. TIL literally builds upon Montague's model-theoretic possible-world semantics by erecting upon it a typed superstructure of logical objects in the form of hyper-intensionally individuated procedures detailing how to intellectually arrive at, or *construct*, an object of a particular type from one or more objects of one or more types. E.g. the meaning of "a is an alleged assassin" is a compound procedure producing a possible-world proposition, which is an empirical truth-condition. The components of that procedure are sub-procedures constructing a modifier, a property, an individual, worlds, times, and a truth-value (though emphatically not the actual truth-value of the sentence). The operations of functional application and abstraction tie together the objects so constructed. The leading idea with respect to modal modifiers consists in construing the conclusion of the rule of inference defining them as a procedure constructing a trichotomy of the sphere of world/time pairs (empirical parameters). There are those pairs at which those who have allegedly stopped smoking have stopped smoking; those pairs at which those who have allegedly stopped smoking have not stopped smoking; and those pairs at which those who have allegedly stopped smoking have neither stopped smoking nor failed to stop smoking (because they were non-smokers all along). (And then there are those pairs at which those who have stopped smoking are not alleged to have done so, because no allegations have been made about their change of habit.) When there are no presuppositions involved, as in the case of alleged assassins, the trichotomy is replaced by a dichotomy. Thus, for instance, the open question is which set of world/time pairs the proposition that a is an alleged assassin belongs to: the set where a is an assassin, or the set where a is not an assassin. Which one it is cannot be established logically but only empirically. What is eventually established is, in logical terms, whether a particular instance of *Alleged* is, on a particular occasion, subsective or privative. When the modified property is not $Assassin$, say, but $Having\ stopped\ smoking$, and the premise that a has allegedly stopped smoking is neither true nor false, but lacks a truth-value, the question of whether *Alleged* is subsective or privative does not arise, because *Alleged* is neither.

What makes the comparison between TIL and CTT illuminating is that, although they do so in different ways, both theories deploy

- the lambda calculus
- a typed universe
- an interpreted logical syntax

– a notion of meaning as a construction: a proof procedure for a proposition (CTT); a procedure for producing a lower-order entity, e.g. a possible-world proposition or a modified property (TIL).

Apart from the usual differences between proof-theoretic and (a successor theory of) model-theoretic semantics, the two most relevant differences between TIL and CTT is that TIL agrees with Montague that a modifier takes a property to a property, whereas CTT holds that a modifier takes a set to a set,[11] and that CTT construes privation as a special case of subsection pivoted on typed empty sets, whereas TIL construes subsection and privation as two basic kinds of modification that are mutually exclusive mirror images of one another.

The rest of this paper is organized as follows. In section 2 the explanation of modal modification in the TIL setting will be presented; in section 3 the appropriate counterpart in the language of CTT is introduced; section 4 will compare the results and discuss analogies.

2 TIL on Modal Modification

The core of TIL is made up of its notion of *construction* and its *ramified type hierarchy*, which incorporates a simple type theory regulating first-order objects (roughly, the objects of possible-world semantics, like individual concepts, relations-in-intension, etc., and their extensions). Constructions are higher-order logical objects arranged in the ramified type hierarchy and are Frege-Church-style higher-order modes of presentation of lower-order objects, whether these be lower-order constructions or first-order objects, which are functions (including 0-ary functions like individuals and truth-values) with neither domain nor range in constructions.[12] To analyze a piece of language is to cite the construction that is its meaning.

Where M is a modifier, the predication of a modified property of a has the (canonical) form

$$\lambda w \lambda t[[MF]_{wt}\ a]$$

The lambda term '$\lambda w \lambda t[[MF]_{wt}\ a]$' denotes a propositional construction that constructs an empirical truth-condition (a possible-world proposition), which in set-theoretic terms is the set of world/time pairs $\langle w, t\rangle$ at which the condition is satisfied (i.e. the satisfaction class of the possible-world proposition). $[MF]$ undergoes extensionalization, as per $[MF]_{wt}$, because $[MF]$ is a property (a function from worlds to a function from times to sets of individuals) and so cannot be directly predicated of an individual. What can be predicated is an

[11] This is in accordance with the general top-down approach of TIL and bottom-up approach of CTT. Cf. [15, Ch. 1] and [3, Ch. 1].
[12] See [3, §1.3.2], for definitions.

extensionalized property, which is a set (a characteristic function).[13] The proceedings can be made transparent by means of the simple type theory. Let the *basic types* be

- o truth-values $= \{1, 0\}$;
- ι individuals (constant domain);
- τ reals doubling as times (continuum);
- ω possible worlds (logical space).

Then, for arbitrary types α, β, the set $(\alpha\beta_1 \ldots \beta_m)$ of all m-ary partial functions from $\beta_1 \times \ldots \times \beta_m$ into α is a *functional* type over $\{o, \iota, \tau, \omega\}$. In particular, any functional type of type $(\alpha\omega)$, abbreviated 'α_ω', is a possible-world *intension.* In TIL the most common intensions are of type $((\alpha\tau)\omega)$, abbreviated as '$\alpha_{\tau\omega}$, which types functions from worlds to functions from times to α. So, for instance, the *proposition* that a is an F is a function from worlds to a partial function from times to truth-values, while the *property* of being an assassin is an intension of type $(o\iota)_{\tau\omega}$: a function from worlds to a function from times to a partial function from individuals to truth-values (characteristic functions of sets). Abbreviate '$(o\iota)_{\tau\omega}$' as 'π'. Then a *modifier* is of type $(\pi\pi)$.

The oscillation between subsection and privation translates into the oscillation between the properties F and $\lambda w \lambda t[\lambda x \neg[F_{wt}\ x]]$. Possibly, some alleged assassins are assassins and the rest are not. So we wish to express that it is contingent whether an alleged assassin is an assassin. The following will not do as an analysis, since the conclusion also trivially applies to all the other modifiers:

$$\frac{[[Alleged\ F]_{wt}\ a]}{\exists \lambda f[f_{wt}\ a \wedge [f = F \vee f = \lambda w \lambda t[\lambda x \neg[F_{wt}\ x]]]]}$$

$\exists$ is a function taking sets (in this case sets of properties) to truth-values, and f ranges over properties. The conclusion is true for any valuation of $\langle w, t\rangle$ at which the premise is true.[14] The reason is because the conclusion means that a has some property f at $\langle w, t\rangle$ and that property is either being an F or not being an F (or: *being a* $non - F$). Besides, if we change the example from *being an alleged assassin* to *allegedly having stopped smoking*, the conclusion fails to accommodate the third option, that the proposition that a has stopped smoking lacks a truth-value.[15]

But this rule captures what is unique about modal modifiers:

$$\frac{[[Alleged\ F]_{wt}\ a]}{[\exists \lambda w[\exists \lambda t[True_{wt} \lambda w \lambda t[F_{wt} a]]] \wedge \exists \lambda w'[\exists \lambda t' \neg[True_{w't'} \lambda w' \lambda t'[F_{w't'}\ a]]]]}$$

[13] See [5] for further discussion and technical details concerning extensionalization and predication.

[14] This argument, like any other valid argument, is a procedure (TIL construction) containing *analytic information* about how to obtain one truth from another. For the notion of analytic information, see [2].

[15] The conclusion is inspired by the TIL analysis of *knowing whether*. See [3, p.455, def. 5.1].

The conclusion states the analytical truth that there are those $\langle w, t\rangle$ pairs at which it is true that a is an assassin, and there are those $\langle w', t'\rangle$ pairs at which it is not true that a is an assassin. One might object that the conclusion is trivially true of any individual a, barring all-out individual essentialism to the effect that an assassin is necessarily an assassin, for, as a matter of logical possibility, each of us has a go at being an assassin at some, though not all, world/time pairs, and vice versa. The objection is absolutely correct. It highlights, in fact, the paucity of information characteristic of modal modifiers, just like *knowing whether* is much less informative than *knowing that.* However, no other modifier obeys a comparable rule, since no other modifier oscillates. For instance, at no world/time pair at which a is a skillful surgeon does it follow that a is not a surgeon; at no world/time pair is a forged banknote a banknote; etc.

When you learn that a is an alleged assassin at $\langle w, t\rangle$, your obvious next move will be to empirically establish, formally put, whether $Assassin_{wt}$ intersects with $[Alleged\ Assassin]_{wt}$ and a is in that set. That is, you wish to establish the truth-value of the allegation, which, formally put, is tantamount to establishing whether a particular instance of *Alleged* is, on a particular occasion, subsective or privative.[16]

Partiality ('gaps') is a complication pertaining to non-factive attitudes that also affects modal modification. One may know whether or believe that the King of Canada is an assassin, even when there is no King of Canada of whom it would be either true or false that he was an assassin. However, if it is true that the King is an alleged assassin then the existential presupposition that the King should exist has already been satisfied, because otherwise there would be no appropriate bearer of the property of being an alleged assassin. So it follows that it is either true or false that the King is an assassin.[17] But, if we again replace *being an assassin* by *having stopped smoking*, then it may be true that the King has allegedly stopped smoking while it is simultaneously neither true nor false that the King stopped smoking, namely in case the King was never a smoker. We do not want an argument to take us from a truth to a truth-value gap, which explains the propositional property $True$ occurring in the conclusion of the above rule.[18] Let p be a variable ranging over possible-world propositions. Then $[True_{wt}\ p]$ constructs the truth-value 1 at $\langle w, t\rangle$ iff p_{wt} constructs 1, otherwise $[True_{wt}\ p]$ constructs 0. Likewise, $[False_{wt}\ p]$ constructs 1 iff $[\neg p_{wt}]$ constructs 1, otherwise 0; and $[Undefined_{wt}\ p]$ constructs 1 iff $[\neg[True_{wt}\ p] \wedge \neg[False_{wt}\ p]]$ constructs 1, otherwise 0. Hence, $[Undefined_{wt}\ p] = [\neg[True_{wt}\ p] \wedge \neg[False_{wt}\ p]]$. The point of the second conjunct of the conclusion of the rule is to soak up two kinds of cases: it being false at $\langle w', t'\rangle$ that a is an $F_{w't'}$ and it being neither true nor false at $\langle w'', t''\rangle$ that a is an $F_{w''t''}$.

[16] For an up-to-date discussion of subsection versus privation in a realist semantics, see [7].
[17] For the logic of *knowing whether*, see [3, §5.1.4].
[18] See [3, §1.5.2.1].

3 CTT on Modal Modification

To allow for a system that induces contingent truth, a different design for constructions and refutations is needed than the standard strongly constructive semantics underlying Martin-Löf's Type Theory. Defining truth by constructors remains useful to preserve the distinction between different *degrees* of truth. To this end, one can distinguish between a refuted truth-claim and a missing construction.[19] Formally, we shall constrain a fragment of the language to a finitary set of formulae with only direct verification processes admissible: here the meaning of a valid judgement 'F *true*' justifies the further conclusion that no construction for '$\neg F$ *true*' is possible. The language is then extended so as to accommodate variables intended as place-holders for (yet) missing but admissible truths. The notion of admissible truth literally satisfies the logical concept of an assumption, computationally corresponding to a term which might not have an appropriate β-redux. The expression of satisfaction of conditions for truth *within* the language is performed by an extension to a modal fragment.

3.1 A Modal Type Theory for Contingent Truth

We rely on the previous work [17] for the complete presentation of this system and shall consider in particular the modal fragment. Let us present just the alphabet and a few explanations:

Types := $F\ type; F\ type_{inf}$;
Propositions := $F; F_1 \wedge F_2; F_1 \vee F_2; F_1 \rightarrow F_2; (\exists a_i : F_i)F; (\forall a_i : F_i)F; F_1 \supset F_2$;
Proof terms := $a : F; (a, b); a(b); \lambda(a(b)); <a, b>$;
Proof variables := $x : F; (x(b)); (x(b))(a)$;
Contexts := $\Gamma, x : F; \Gamma, a : F; \Box\Gamma; \Diamond\Gamma$;
Judgments := $F\ true; F\ true^*; \Gamma \vdash F\ true; \Diamond(F\ true); \Box(F\ true)$.

The language contains one basic sort *type* for categorical constructive judgments with corresponding term constructors $a, b, \ldots$; and one sort $type_{inf}$ (*information type*) for judgments in a context of refutable conditions, with corresponding variable constructors $x_1, x_2, \ldots$. Judgements of the first sort induce a constructive notion of truth (*true*), the second ones a weaker predicate of contingent truth (*true**). Identity of terms holds within *type*, and its constructors

[19] A standard explanation of constructive refutation is given by the meaning of intuitionistic negation as the condition that there is no construction for an absurdity; indirect proofs as *reductio ad absurdum* are standardly not admitted, whereas the usual intuitionistic absurdity rule interprets the classical *ex falso quodlibet*. See e.g. [22, p. 40]. The foundational work [10] translates classical mathematics into intuitionistic mathematics by reducing formulae of the former to intuitionistic ones as long as they are double-negated, and as such these are called 'pseudo-truths', the implication from $\neg\neg Fa$ to Fa being valid in the domain of the finitary. A stronger result can be obtained by defining pseudo-truths as entirely separated constructors, inducing a polymorphic language.

are composed standardly by way of listing, application, abstraction and pairing to define connectives and quantifiers: $\wedge, \vee, \rightarrow, \forall, \exists$. In particular: $\rightarrow$ is material implication obtained by application $a(b)$ of the construction a of the antecedent to the construction b of the consequent, which can be seen as a λ-term presented *together with* one of its α-terms. $\forall$ abstracts from enumerable sets of equivalent constructions; $\exists$ is justified by paired constructions. Admissibility of the $type_{inf}$ sort is defined in two steps: first, the construction $a : F$ establishes the inadmissibility of $\neg Fa$ in further contexts; secondly, a missing construction for $(F \rightarrow \bot)$ allows assumption formation $x : F$. Occurrences of variables in context are unique; abstraction and conversion on variables define respectively function formation and reduction to *type*. Material implication is therefore distinct from function formation $\supset$, the latter being given by abstraction on the admissible construction for the antecedent. Proof terms can occur both inside and outside of a context, proof variables hold only in contexts. Types are typically propositions (or equivalently sets) and judgments are built by declaration of *true* or *true** of propositions.[20] We omit for brevity also the identity rules that define Reflexivity, Symmetry and Transitivity on types as well as the structural rules, see [17].

Conversion rules are defined over terms of the $type_{inf}$ fragment, β for reduction of $type_{inf}$ terms to corresponding *type* terms (evaluation) and α-term equality:

$$\frac{F_1\ type_{inf} \qquad b : F_2[x : F_1] \qquad a : F_1}{(x(b))(a) = b[a/x] : F_2\ type[a/x]}\ \beta\text{-conversion}$$

$$\frac{\lambda((a_{1-i}(b))F_1, F_2) \qquad (b)[a_i := a]}{(a_i(b)) : F_1 \rightarrow F_2}\ \alpha\text{-conversion}$$

Two start rules are definable:

$$\frac{}{\Gamma, a : F, \Delta \vdash F\ true}\ \text{Premise Rule}$$

$$\frac{}{\Gamma, x : F, \Delta \vdash F\ true^*}\ \text{Hypothesis Rule}$$

The premise rule is meant to introduce explicitly verified contents; the hypothesis rule reflects the introduction of contents that are only assumed to be true; both rules can have $\Gamma, \Delta = \{\emptyset\}$. Correspondingly, the *true* predicate can be understood as validity (that is truth in every situation) and it corresponds to truth by verification, whereas the predicate *true** corresponds to validity in a context of assumptions.

[20] To preserve a consistent notation we have maintained the identity of literal F to indicate a type in the formula $a : F$ and in the formula $F\ true$. Nonetheless, it should be clear by our explanation that in the former formula the type F stands for a property that is predicated of an individual a, whereas in the second it expresses the proposition saying that property F is predicated of an individual.

Modalities are defined by internalizing the appropriate formation rules from categorical and contingent judgements.[21] Provided that the conditions for having the right to express a judgement are satisfied, the notion of judgemental necessity $\Box(F\ true)$ corresponds to that of an *apodictic judgement*: what is known to be so and cannot be known to be otherwise. The constructive interpretation identifies provability, truth and knowledge. The basic condition for the truth of Fa is the construction a that makes it true $(a : F)$; when F presupposes further propositions to be known, these represent the context in which F is known to be true, $\Gamma = (F_1\ true, \ldots, F_n\ true)$. Conditions in Γ can be seen as contextual or background knowledge. Hence, $\Box(F\ true)$ is knowledge for which no further contextual conditions are needed $(\Gamma = \emptyset)$.

The corresponding interpretation of a judgemental possibility operator starts from the propositional equivalence $\Box F \leftrightarrow \neg\Diamond\neg F$, which underlies the following formulated in [21]: $\Diamond(F\ true) \Leftrightarrow \neg\Box(\neg F\ true)$. If conditions needed for knowledge of $F\ true$ can be satisfied only with Γ empty, then this formula reduces to the conditions for $\Box(F\ true)$; otherwise, it means that truth is preserved under certain knowledge states in which appropriate conditions $\Gamma = (F_1\ true, \ldots, F_n\ true), n \geq 1$ are formulated. The latter amounts in our language to $x_i : F_i$ as a condition for $F\ true$.

$$\frac{a : F}{\Box(F\ true)}\ \Box\text{-Formation} \qquad \frac{x : F}{\Diamond(F\ true)}\ \Diamond\text{-Formation}$$

The inference to truth of contextual judgements requires generalization to contextual formulae:

Definition 1 (Necessitation Context). *For any context Γ, $\Box\Gamma$ is given by $\bigcup\{\Box F_i\ true \mid \text{ for all } F_i \in \Gamma\}$.*

Definition 2 (Normal Context). *For any context Γ, $\Diamond\Gamma$ is given by $\bigcup\{\circ F_i\ true \mid \circ = \{\Box, \Diamond\}$ and $\Diamond F_i$ true for at least one $F_i \in \Gamma\}$.*

Then a judgement valid under assumptions becomes a possibility judgement if its context remains normal, that is, at least one piece of its propositional contents is $true^*$.[22] For the use of contingent truths as key for modelling modal modification, we are here interested in the rule that characterizes the use of Normal Contexts. Local validity is defined by introduction and elimation rules for the $\Diamond$-operator:[23]

$$\frac{\Gamma, x_i : F_i \vdash F\ true^*}{\Box\Gamma, \Diamond(F_i\ true) \vdash \Diamond(F\ true)}\ \text{I-}\Diamond$$

[21] For more on this notion of epistemic modalities, see [16].

[22] *Necessitation* and *Normal Context* are equivalent to *Global* and *Local Context* from the literature on modal logic. Cf. [4].

[23] Structural Rules as Weakening, Contraction, Exchange hold in the form of theorems; also the Substitution of truth predicates and terms can be proved, plus the local inversion of these modal rules with the appropriate $\Box$-counterparts, corresponding to their soundness and completeness, see [17].

$$\frac{\Gamma, \Delta \vdash F_i\ true^* \qquad \Box\Gamma, \Diamond(F_i\ true) \vdash \Diamond(F\ true)}{\Gamma, \Delta \vdash F\ true^*}\ \text{E-}\Diamond$$

Notice that our $\Diamond$ rules prevent ill-behaving inferences, such as inferring that a has allegedly stopped smoking from the fact that a does not smoke – a proposition which does not have any relevant refutable assumption – or from *not* knowing that a has stopped smoking – which inference would need a constructor relying on *actually refuted* conditions for the property *being a smoker*. If conditions are actually refuted, then they are not expressed as merely *refutable*: whereas the latter state is properly expressed by our $\Diamond$ operator, the former induces a reduction to *type* (and so to $\Box$).

3.2 Modal Modification by Contingent Truth

The basic idea informing our simulation of modal modification within the procedural semantics underlying our Modal Type Theory is to interpret it as a function producing a non-terminating set of necessary conditions, so as to be able to assert a contingent truth. Our judgemental $\Diamond$ operator and its introduction and elimination rules regulate precisely the epistemic conditions for asserting a contingent truth.

Let us start by reconsidering an example: "John is an *alleged* assassin". We start with a an individual constructor in the kind *type*. F is the property (assassin) predicated of a, as Fa: in our language this corresponds to a judgement of the form $a{:}F$.

> $[Fa]$: "(It is true that) John is an assassin",

with the truth predicate hidden by the formalism. It is crucial for our construction to unveil the nature of such a truth predicate, i.e. to establish which rule it follows. When M is a modal modifier, $M[Fa]$ is an abbreviation of

> $[MF]a$: "[(It is true that) John is an alleged assassin]".

Our claim is that the modal modifier M over F applied to a can be simulated by analytically defining the conditions under which the truth of Fa is asserted in terms of our I-$\Diamond$ rule. The meaning of $[MF]a$ is in turn equivalent to $\Diamond(F\ true)$. $\Diamond(F\ true)$ expresses the unreducibility of F to *type*, so that one cannot judge either Fa or $\neg Fa$ to be true. Because on this reading the modifier operates on a judgement, our previous abbreviation of $[MF]a$ turns out to be equivalent to the following:

> $[MF]a$: "Allegedly[(It is true that) John is an assassin]".

We first exploit the nature of the predication as dependently defined. Besides very basic (and entirely uninformative) judgements, every assertion depends on the (eventually implicit) conditions under which it holds. The most obvious conditions can be made evident in terms of an analytic deconstruction of the predicate, as e.g. by saying that "John is an assassin, provided it is true that he killed a human being on purpose". Besides this analytical form, a judgement can be turned into a (truly) dependent one by referring to conditions dictated by the perspective from which judging occurs.[24] As an example, let us admit that the proposition "John is an assassin" is expressed in the perspective of a legal system where to be condemned as an assassin requires that the individual has been found guilty in the lowest to the highest courts. Then, an obvious formulation of our dependency relation would be of the form: "John is an assassin, provided it is true that he has been found guilty of killing a man by each of the required courts". The list of conditions can be further modified, by adding for example "under no mitigating circumstances".[25]

The validity of the dependent judgement corresponds to the reduction of the construction to the *type* fragment and hence of its assertion conditions to $\Box\Gamma$, so as to finally execute an inference $\Box\Gamma \vdash \Box(F\ true)$. To do so, we require that none of the conditions under which $F\ true$ holds can be falsifiable. The construction $\Diamond(F\ true)$ expresses instead the validity of $type_{inf}$ for at least one condition which does not reduce.

The formulation $[MF]a$ predicating of an individual a the modified property (MF) is constructively expressed as a function M such that for every element x in the set F taken as argument, it returns a function $M(x)$, formally $M(x)[x\!:\!F]$, which expresses that for at least one $F\ set[x_i\!:\!F_i]$ it holds that $F_i\ type_{inf}$.[26] To preserve the functional aspect of M in the constructive notation, we will refer to $M(x)[x\!:\!F]$ as the type satisfied by some $f\!:\!F$ modified by having a judgement of the form $f\!:\!F$, for which at least one $f_i\!:\!F_i$ cannot be shown to reduce:

$$\frac{F\ type[\Gamma] \qquad F_i\ type_{inf} \in \Gamma \qquad M(x)[x\!:\!F]}{\Box\Gamma, (x_i\!:\!F_i)f\!:\!F \vdash F\ type[(x_i(f))(f_i)\!:\!F_i]}\ \text{Modal Modification}$$

This rule can be read as follows: let there be an object type F that is satisfied provided that all the object types in $\Gamma = \{F_1, \ldots, F_n\}$ have appropriate type constructors; let it morever be the case that for $F_i \in \Gamma$ a constructor is admissible but the reduction is not actually obtained, so that $F_i\ type_{inf}$ holds; then it is the case that, provided all the constructors in Γ apart from F_i are satisfied, a modifier M holds for F such that F is an object type if and only if F_i has an appropriate β-redux, and it is not if F_i does not reduce. This rule is nothing else than an analytic definition of $type_{inf}$, inducing immediately the judgement

[24] On the perspectivist epistemology underlying Martin-Löf's Type Theory, see e.g. [20].

[25] The perspective can be easily changed so that also conditions change: for example, for someone who considers hunting an act of violence, the following might hold: "John is an assassin, given he killed a beast". Our concern is here only to assess the role of dependent judging in the formulation of a construction for modal modifiers, hence we shall constrain ourselves to the more evident and less problematic formulation of such meanings.

[26] Because M does get predicated of a, provided F has already been predicated of a, CTT has no need for a counterpart of pseudo-detachment.

$\Diamond(F\ true)$. Two remarks are in place to understand why such a construction qualifies as one for a modal modifier:

1. the modal operators express the separation between terminating and non-terminating terms, a property which is not available in the standard format of CTT; by presenting the constructor $\Box\Gamma, (x_i : F_i) f : F \vdash F\ type[(x_i(f))(f_i) : F_i]$, we refer to a term f that is modified by the missing reduction for a term f_i on which it depends;
2. given the admissibility of f_i, this construction simulates a derivability relation that satisfies the *tertium non datur* of the non-modal alternatives $Fa \vee \neg Fa$, though the language does not allow its formal derivability from $\Diamond(F\ true)$. The appropriate way of expressing the meaning of a modal modifier as 'alleged' in a sentence as "(It is true that) John is an alleged assassin" is to say: "(It is true that) John is an assassin if x is true" or "(It is false that) John is an assassin if x is not true".

4 Discussion and Conclusion

Above we presented a realist and a constructivist semantics for modal modifiers. Our shared point of departure was their strong similarity to non-factive attitudes, not least *knowing whether*. The crux of the realist semantics was that if a modally-modified property F is true of a then there are world/time pairs (empirical parameters) at which it is true that a is an F and other such pairs at which it is not true that a is an F. This is how the alethic contingency of an alleged assassin, say, (not) being an assassin is captured. This contingency is in contrast to the behaviour of the other modifiers; for instance, it is not contingent that if a is a skillful surgeon then a is a surgeon. The contingency is due to the oscillation between subsection and privation that is unique to modal modifiers. The crux of the constructivist semantics is that the judgement that a has a modally-modified property F is admissible to make, provided conditions for judging that a is an F are known to be satisfiable, though not every such condition is asserted as unrefutable. This is how the epistemic possibility, couched in terms of the contingency of constructivist truth, that an alleged assassin may not be an assassin is captured.

This analysis reveals a major rift between the two semantic theories. On one hand, once TIL types M as a property modifier, M must take properties as arguments and remains inapplicable to, say, individuals or propositions. This explains the introduction of pseudo-detachment to validate the first conjunct of the conclusion of the rule of intersective modification, where a property rather than a modifier is required. On the other hand, a modal modifier M is interpreted in CTT as a judgemental modal operator and it inherits this structure from the basic assumptions that expressions of the language are of the form "$F\ true$". This means that the modifier shifts from generating a modified property qualifying an individual to an operator qualifying a judgement, which induces the equivalence between the two formats "b is an alleged F" and "Allegedly, b is an F". It is disputable whether this equivalence between modal property modifiers and modal propositional modifiers holds in TIL. Whether it does remains at this point an open research question. If it does then a propositional modifier like *Allegedly* would embody empirical (i.e. non-analytic) modality in the sense laid out in [11].

In [18], the treatment of privation for both TIL and CTT is based on the underlying condition that there must be at least one substantial property shared both by F and $[MF]$, when M is a privative modifier: forged banknotes and banknotes have to show some substantial similiarities, though the former are of necessity not banknotes; of all the non-banknotes in the universe, forged banknotes resemble banknotes more than do the rest.

This is true, in a sense, also for modal modification according to the quasi-CTT semantics presented in this paper. The set of conditions that need to be satisfied in order to have the right to judge that a is an assassin and those needed to have the right to judge that a is an alleged assassin are literally the same. Only, in the former case, those conditions need to be completely and entirely fulfilled, whereas in the latter case one only requires that they *can* be fulfilled, although no guarantee is given that they may not be refuted, so they are as yet unfulfilled. So, an assassin and an alleged assassin who happens not to be an assassin are presented according to the same list of conditions, only in the first case these are satisfied, in the second case there is at least one which remains unfulfilled.

By contrast, TIL leaves room for wild allegations. An assassin and an alleged assassin who happens not to be an assassin may be quite dissimilar. For TIL there are no constraints on the privative variant of modal modification to the effect that an $[MF]$ and an F must be relevantly similar. TIL maintains only very general constraints, e.g. that an alleged assassin must be an agent capable of intentional action. People can be alleged assassins, beasts and inanimate objects cannot.[27] TIL is focussed solely on the logic of "*Alleged F*", which TIL takes as its point of departure, whereas CTT takes a step back and looks at the admissibility of generating "*Alleged F*". This explains why CTT can accommodate partiality only in the form of contextual indefineteness, whereas TIL is able to embrace partiality entirely, by allowing the alethic possibility that it be neither true nor false of somebody who allegedly stopped smoking that they stopped smoking. TIL embraces instances of "a is an alleged F" that are true, while "a is an F" is at the same time without a truth-value, as exemplified by our King of Canada who never smoked but is alleged to having stopped smoking. The modal semantics for CTT introduced here behaves differently, as it only allows to express lack of a

[27] However, exactly because modal modifiers flit between subsection and privation, laying down at least some of the non-trivial properties that must be instantiated by somebody who has the property of being an alleged assassin, or being presumed innocent, etc., may turn out to be a pointless undertaking. This explains why, unlike [18, §4.2], this paper does not introduce requisites (relations-in-extension between properties, of type $(o\pi\pi)$) of the property *being an alleged assassin* or any other modally-modified property.

truth-value with respect to the context in which the act of making a judgement is actually performed.

Modal modification is in need of a name that is just as explanatory as 'privative', 'subsective' or 'intersective'. Maybe the broad labels 'modal', 'intensional' and 'equivocating' for modifiers that are torn between subsection and privation are indicative of the elusive nature of such modifiers. Yet there is a direct sense in which modal modifiers deserve to be called just that – 'modal'. Thus, in CTT a modal M is analogous to the possibility operator (whether construed alethically or epistemically) of modal logic: M is applicable to Fa, just like $\Diamond$ is applicable to Fa.[28] In either case the result of the application is the possibility that a be an F, without either presupposing or entailing the truth of the proposition that Fa. Likewise, in TIL the truth of a being an $[MF]$ entails that there is a world/time pair at which it is true that Fa and another world/time pair at which it is not true that Fa. This is analogous to how modal logic defines $\Diamond Fa$ and $\Diamond\neg Fa$, which is, respectively, $\exists w[F_w\ a]$ and $\exists w'\neg[F_{w'}\ a]$.

Acknowledgements

Bjørn Jespersen wishes to thank Marie Duží for helpful comments and remarks. The two authors both of this paper and of [18] contributed equally to both papers.

References

1. Cresswell, M. Prepositions and points of view. *Linguistics and Philosophy*, vol. 2, pp. 1-41, 1978.
2. Duží, M. The paradox of inference and the non-triviality of analytic information. *Journal of Philosophical Logic*, vol. 39, pp.473-510, 2010.
3. Duží, M., Jespersen, B., Materna, P. *Procedural Semantics for Hyperintensional Logic. Foundations and Applications of Transparent Intensional Logic.* Logic, Epistemology and the Unity of Science, vol.17. Springer, Berlin (2010)
4. Fitting, M. Basic modal logic. In volume 4 of *Handbook of Logic in Artificial Intelligence and Logic Programming*, pp.368-449. Oxford University Press, 1994.
5. Jespersen, B. Predication and extensionalization. *Journal of Philosophical Logic*, vol.37, pp. 479-99, 2008.
6. Jespersen, B. Property modification and pseudo-detachment. In submission.
7. Jespersen, B. and M. Carrara. Two conceptions of technical malfunction. *Theoria*, forthcoming.
8. Kamp, H. Two Theories of Adjectives. In Keenan, E. (ed.). *Formal Semantics of Natural Language*, pp.123-155. Cambridge: Cambridge University Press, 1975.
9. Kamp, H. and B. Partee. Prototype theory and compositionality. *Cognition*, 57:129–191, 1995.
10. Kolmogorov, A. On the principle of excluded middle. In J. van Heijenoort, editor, *From Frege to Gödel: a Source Book in Mathematical Logic 1879-1931*, pages 414–437. Harvard University Press, 1967.

[28] Notice that this treatment of a *judgemental* modal operator significantly differs from a propositional definition, as the one presented in [14].

11. Materna, P. Ordinary modalities. *Logique & Analyse*, vols. 189-192, pp. 57-70.
12. Montague, R.. English as a formal language. In B. Visentini et al. (eds.), *Linguaggi nella societá e nella tecnica*. Milan, 189-224, 1970. Reprinted in R.H. Thomasson (ed.), *Formal Philosophy*. New Haven, London: Yale University Press, 1974.
13. Partee, B. Privative adjectives: subsective plus coercion. In R. Bäuerle, U. Reyle, and Thomas E. Zimmermann (eds.), *Presuppositions and Discourse*, Amsterdam: Elsevier, 2001. Available at `http://people.umass.edu/partee/docs/ParteeInPressKampFest.pdf`.
14. Pfenning, F. and R. Davies. A judgemental reconstruction of modal logic. *Mathematical Structures in Computer Science*, 11:511–540, 2001.
15. Primiero, G. *Information and Knowledge. A Constructive Type-Theoretical Approach*. Logic, Epistemology and the Unity of Science, vol.10. Springer, Berlin, 2008.
16. Primiero, G. Epistemic modalities. in G. Primiero, S. Rahman (eds.), *Acts of Knowledge - History, Philosophy and Logic*, pp.207-232. College Publications, London, 2009,
17. Primiero, G. Constructive contextual modal judgements for reasoning from open assumptions. In F. Ferreira, H. Guerra, E. Mayordomo, J. Rasga (eds.), *Proceedings of the Computability in Europe 2010 Conference - Proofs, Programs, Processes*, pp. 336–345. Centre for Applied Mathematics and Information Technology, Department of Mathematics, University of Azores, 2010.
18. Primiero, G. and B. Jespersen. Two kinds of procedural semantics for privative modification. *Lecture Notes in Artificial Intelligence*, vol. 6284, pp. 252-271, 2010.
19. Rotstein, C. and Y. Winter. Total adjectives vs. partial adjectives: scale structure and higher-order modifiers. *Natural Language Semantics*, 12:259–88, 2004.
20. Schaar v.d., M. The cognitive act and the first-person perspective: an epistemology for constructive type theory. *Synthese*, 2009. DOI:10.1007/s11229-009-9708-4.
21. Sundholm, B.G. Mind your P's and Q's. On the proper interpretation of modal logic. In T. Childers, O. Majer (eds.), *The Logica 2002 Yearbook*, pp. 101–111. Filosofia, Czech Academy of Science, 2003.
22. Troelstra, A.S. and D. van Dalen. *Constructivism in Mathematics: An Introduction*, vols. I,II. North-Holland, Amsterdam, 1988.
23. Winter, Y. Cross-Categorical Restrictions on Measure Phrase Modification. *Linguistics and Philosophy*, vol 28, pp.233-267, 2005.