

Type-theoretical Dynamics

Exploring Belief Revision in a constructive framework

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Abstract In the present paper a dynamics for type theory is introduced. The formalization provides epistemic explanations for the basic notions of belief state and belief set by referring to assertion conditions for type-theoretical judgements; it interprets expectations in terms of default assumptions for such a structure and it adapts the usual revision operations and the analogues of the Ramsey test. The model, restricted to operations of revision, merging and information preference, provides a constructive type-theoretical approach to epistemic dynamics.

1 Introduction

Various non classical models for dynamic reasoning have provided extremely fruitful results during the last two decades. The first aim of such frameworks is to interpret the basic standard properties of dynamic every-day reasoning: non-monotonicity, because conclusions earlier drawn on the basis of an insufficient set of informations can be rejected by new information obtained at a later stage; paraconsistency, because often the agent infers from contradictory informational contents; adaptive procedures, because reasoning is very often performed on the basis of an internal dynamics, which leads to the rejection of previously accepted consequences due to a better understanding or a modification of the starting set of premises. More in general, the standard reference for belief dynamics is to the AGM framework [1], and the large body of work that emerged from there.¹ In the present contribution, we

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¹ See [16] and [33] for two of the most comprehensive treatments.

explore the type-theoretical model of knowledge as a framework to formalize non-monotonic procedures, by means of a proper interpretation of the logical notions at the basis of its formalism. This also provides a conceptual ground for a formal treatment of errors.

The type-theoretical dynamics provides a model defining independent items of information as beliefs, along the usual epistemic definition of knowledge heritage of the constructive approach to logic. It provides the due operations on these states as the dynamic core of the entire knowledge process; it allows, moreover, an explanation of the procedures of information preference and it defines degrees of belief in terms of confidence and resistance to change. Our work proceeds as follows: we first introduce in section 2 the type-theoretical framework in the context of conditions for developing an epistemic dynamics, together with the appropriate epistemic explanations of the notions of belief, belief state and belief set, expectation; in section 3 we interpret the revision operator, the Ramsey test and the Minimal Change Principle; in section 4 an attempt is provided to visualize the due operations for belief merging; we conclude in section 5 with some further notions, useful in the context of extensions of the framework.

2 Conditions for type-theoretical Dynamics

The well-known idea at the basis of the classic AGM model² is to integrate new information within the starting knowledge frame of a rational agent, in order to consider which inconsistencies rise up, and how to deal with them. The basic way of treating with such a case is by accepting the *new information* and by changing the starting beliefs as little as possible (minimal change principle, MCP); the resulting belief state should be syntax independent. This model has been reformulated in different frameworks, each based on certain formal and conceptual constraints, and for each of them the meaning of “rational” is obviously depending on the type of logic used to represent the agent’s procedures. Each of these models needs to express the conditions to formulate belief revision and non-monotonic procedures in terms of epistemic notions.

In the case of a type-theoretical frame, such conditions are easily presented. They can be informally introduced as follows:

1. to provide an appropriate formulation of the conceptual conditions (or equivalently of an order of priority) among the elements of the theory;
2. to describe an epistemic model in which an intuitive interpretation of the basic notions of the dynamic of theories is provided, like e.g. for the notions of belief state and belief set, the revision operators and so on.

² Cf. [1].

The first task must be obtained in a comparison with the usual descriptions of logical relations for contents of theories, namely by referring to one of the following methods:

- *epistemic entrenchment*, originated with the AGM model, is an ordering among the sentences in a language in the form of a binary relation \leq , which tries to capture the importance of one of these sentences in face of a change³;
- the *system of spheres* treats with set of consistent complete theories and the order of relation among (parts of) them⁴;
- a *preorder on models* as a structure equivalent both to entrenchment and spheres⁵.

These relations formulate the order of priority holding between propositional contents or (parts of) theories. This allows for any procedure of revision to be applied on ordered contents, i.e. depending on the relation of priority among the contents contained in the starting knowledge state of the agent.

In the type-theoretical framework, the definition of the conditional order is quite important and at the same time intuitive: the first problem is to define such relation among the judgemental (rather than propositional) contents of the theory, after which it appears natural to define the nature of doxastic states and their informational content in a completely new way; moreover, it is on the basis of such conditional order that revision operators and the epistemic notions involved (i.e. our second task) can be formulated. We proceed in doing this in the remaining of this section.

Type Theory as a Theory of Beliefs

The type-theoretical approach to belief revision provides a quite developed framework to treat with beliefs, expectations and justifications.⁶ Let us re-

³ See [9].

⁴ See [12].

⁵ See [23].

⁶ The framework presented in [4] is the first type-theoretical approach to belief revision. They stress the possibility to express explicitly an agent's beliefs as well as her justifications for these beliefs, by means of constructions which act as first-class citizens in the theory. The connection between types, their justifications, and the notion of belief is therefore based on the idea that any propositional content which in the type-theoretical formalization comes equipped with its instantiation (proof, instance) is intended as the content of a belief, which therefore results also justified. In the following, I will distinguish between contents as beliefs and contents as instances of knowledge.

member that the formal expressions of the theory are the standard categorical judgements⁷

$$\begin{aligned} & a:A \\ & a = b:A \end{aligned}$$

and the dependent judgements:

$$\begin{aligned} & a:A(x_1:A_1, \dots, x_n:A_n) \\ & a = b:A(x_1:A_1, \dots, x_n:A_n). \end{aligned}$$

Dependent judgements are formulated under assumptions contained in contexts (within brackets) and the primary condition for assumptions within contexts consists in the predicability of the types involved, in the form:

$$A:\text{type}.$$

Our first aim is to explain this order of priority within judgments formation for Type Theory in relation to the models of theories for belief revision.

Notoriously, two models exist for belief revision theories: the first is represented by the “Foundations theories”, in which one needs to keep track of justifications for one’s beliefs, and beliefs are only accepted if justified; the second model is notoriously that of the “Coherence theories”, in which beliefs are accepted on the basis of their coherence to other beliefs.⁸ In the following, I shall maintain that the constructive type-theoretical approach to knowledge and belief falls among the Foundations Theories.

The first and major innovation provided by the type theory in the foundational perspective is obviously that it provides contents accompanied by appropriate justifications.⁹ This has been considered one of the main elements in determining the nature of our (basic) beliefs,¹⁰ and has direct consequences on the definition of our belief states (see following subsection). If one understands type-theoretical judgements simply as judgements expressing beliefs, one will obtain a theory of “justified beliefs”, because of the explicit formulation of constructions for types. On the other hand, a related problem, based also on the constructive foundation of the theory, is that it becomes extremely difficult to make sense of the nature of fallible theories: to accept the identity between judgemental forms and belief contents implies there are only *true beliefs*; no space is left for the formulation of *wrongly justified beliefs*, and *false beliefs* are interpreted as those propositional contents explicitly implying a contradiction: $a:A \rightarrow \perp$ (and they are therefore rejected). This reading

⁷ For all of these expressions, small letters represent instances or proofs, and capital letters represent types. The identity between instance and proof is based on the Curry-Howard isomorphism.

⁸ For this distinction, see especially [10, ch.4, pp. 47-65].

⁹ For a similar perspective in a different context, see [2].

¹⁰ Cf. [14, pp.22-23].

simply substitutes the usually understood notion of knowledge for the type theory by a notion of belief¹¹. In the following, we choose to follow a different route: a proper notion of belief will be formally introduced as distinguished from knowledge, the distinction being based on the two having a different epistemic status. Belief is considered to enjoy a weaker epistemic status than the strongly justified notion of knowledge. This explanation of belief contents has to be coherent to the epistemic definition of truth as presence of a proof, and it should express a content which can be submitted to revision and rejected when false. This will be obtained by interpreting the schema of conditions holding for the type-theoretical judgements, i.e. in terms of judgements not equipped with proper analytic constructions (proofs), or whose constructions are only assumed to be possessed¹². This seems to express a more intuitive notion of belief as what (by definition) is not true, rather *maintained to be true*. The interpretation of type-theoretical contexts as belief states provides a new element to the identity between beliefs as contents which need to be processed before their acceptance, a method that recalls the notion of *databases* in computer science and the idea of *acceptance system* from epistemology¹³.

The second issue that allows to treat our formalism among the Foundations Theories is the crucial role of priority relations in the building of contexts, which turns out to be another major issue in the description of belief generation and revision.¹⁴ The formal structure introduces type theory as based on a calculus of contexts, and each content in such structure is strictly dependent on the formulation of the previous ones in the same context; moreover, a formal dynamics among contexts has already been developed,¹⁵ and we are going to make extensive use of that too. The crucial step is to justify the interpretation of contexts as belief sets, which in turn let to interpret the

¹¹ This is the idea at the basis of the type-theoretical interpretation of [4].

¹² Let us consider e.g. the entry in the Oxford Dictionary (Paperback Edition, 2001):

belief, n. **1** a feeling that something exists or is true, especially one without a proof. **2** a firmly held opinion. **3 (belief in)** trust or confidence in. **4** religious faith.

¹³ Cf. [20]. The definition of belief set for the type-theoretical framework provided above contains therefore as its core an internal definition of the informational contents of belief states at the agent's disposal. In [27] this is formulated in terms of the epistemic distinction between the "information" that a rational agent accepts in her system (and another agent could reject), and her "knowledge", as the stronger epistemic status of a certain content supported by a proof (amounting to Lehrer's distinction between "acceptance condition" and "knowledge").

¹⁴ See [33, p.135]:

"In accordance with the foundationalist philosophy of belief change, what gets revised in the first place is not *theories*, but rather *prioritized belief bases*."

¹⁵ Cf. [31].

relation of extension between contexts as the connection between different consecutive belief sets, providing a model for the standard operations of a belief revision theory, to be extended to the operation of merging. To accomplish this aim also means to import all the structural properties of contexts in the analysis of belief states, hence also gaining the prioritized structure and the possibility of defining revision operations on such structured contents. The next step is therefore the analysis of a belief state as a dependent judgement, with a knowledge declaration valid on the basis of an appropriate belief set.

Belief States and Belief Set

The standard representation of doxastic states is sentential or propositional, i.e. beliefs are coded as formulas representing propositions. Following Martin-Lof's explanation of formulae in type theory, the propositional content A is embraced into the judgemental form A is true, obtained by abstraction on the formal expression $a:A$, saying that there exists a construction or proof a for A . This means that the formulation of the formal derivation (or construction) a justifies the truth of the concept or predicate expressed by the type A . This structure establishes therefore the truth of propositional contents on the basis of the proofs represented by the close derivations, and it defines in this way types as its formal objects¹⁶. Given this conceptual switch from simple propositions to the judgemental form predicating truth of the propositional contents, close derivations are the basic conditions for the latter, namely their *proof-conditions* (in the light of the Brouwer-Heyting-Kolmogorov interpretation of propositions). Judgements which provide proof-conditions for their propositional contents, must be in the following intended as pieces of knowledge.

In addition to their strict proof conditions, it is also required that further conditions for judgements be formulated: those are expressed by the related *assertion conditions*, intended as the basis needed in order to formulate some knowledge content¹⁷. These assertion conditions for the type-theoretical formalism are represented by the expressions contained in contexts:

$$\Gamma = (x_1 : A_1, \dots, x_n : A_n) : \text{context}.$$

A context collects assumptions and recall (implicitly) presuppositions for the knowledge content that is formulated. Contexts provide an informational content playing an essential role in the formulation of new judgements, and

¹⁶ See [21], [22].

¹⁷ The conceptual and formal frame of conditions for type-theoretical judgements is fully presented and investigated in [25].

they let to formulate the required conceptual order between expressions of the theory.

The relation of dependency holding among the content of such contexts and the judgements derived from them, is based on the transmission of knowability, rather than on truth-preservation as in classical models. The assumption on the knowability of certain propositional contents provide conditions for the knowability of further propositional contents. In this sense, it is the degree of knowledge and therefore the confidence in the truth involved by such contents that allows the interpretation of the notion of “belief” within the type-theoretical formalism. The basic role of the contextual contents is in fact to provide “independent items of information”, not necessarily derived by actual processes of inference, and representing the foundation to build further contents.¹⁸ In standard belief revision theories, these are called “default assumptions”. The role of assumptions as independent items of information is simple to describe:

- an assumption of the truth of a certain propositional content is called an alethic assumption and it is of the form $(x : A)$;
- an assumption on the knowledge of a close derivation making a certain content true is called an epistemic assumption and it is of the form $(a : A)$.¹⁹

Clearly, an alethic assumption can be considered an abstraction on an epistemic one, and viceversa, an epistemic assumption is the actual construction of an alethic one, thus being conceptually prior.²⁰ The formulation of different kinds of assumptions for type theory is essential for the epistemic counterparts of the notions of belief set, belief state and expectation. In the usual terminology of belief revision theories, the term belief set (B_{set}) refers to the agent’s set of beliefs closed under logical consequence, i.e. the set of actual beliefs and their consequences; the term belief state (B_{state}) refers instead only to the actual beliefs maintained by the agent in a certain knowledge process. To be more accurate: a belief base/state is a partial description of the world, which together with some inference operations generates a belief set. The beliefs in a base are therefore basic ones with an independent warrant, whereas those that follows from the base to obtain a state are the derived ones: the beliefs in the base are the foundation on which a belief set is built.²¹

In the type-theoretical model, contexts clearly represent beliefs states, that come additionally with appropriate justifications; the latter, in the alethic representation of assumptions, are abstracted. A B_{state} will therefore contain the agent’s beliefs, and it will be part of a B_{set} , the latter providing a piece of knowledge with the appropriate conditions to express it:

¹⁸ See e.g. [33, ch.1].

¹⁹ See [25]. In both cases the presence of brackets refers to the use of the expressions in a context.

²⁰ For the notion of abstraction involved by proof processes and variables, see [34] and [28].

²¹ Cf. [33, p.22].

$$\begin{array}{c}
 B_{state} \\
 \widehat{(\Gamma)} \\
 B_{set} = \\
 \Downarrow \\
 a:A
 \end{array}$$

This distinction is completely natural in the type theory, where a belief set can be based on different belief states (contexts), each providing a different collection of conditions on the knowledge expressed by the entire set. On the basis of the epistemic constructive interpretation here provided, the agent will be aware of further logical consequences only if equipped with the due constructions: this means that each B_{set} can be simply understood in terms of a knowledge state, containing a set of belief contents (assumptions and presuppositions) from which constructions are derived for new known propositional contents.

Beliefs and Expectations

On the basis of the above considered distinction between alethic and epistemic assumptions, the type-theoretical framework allows now to introduce a proper description for beliefs and expectations. These notions are usually explained²² in terms of the agent's attitude towards the contents: a full belief is defined as a propositional content actually held true; when the same content is submitted to revision, one says that the expectation is contradicted, for example by new observations. This description can now be accounted in our epistemic schema by referring to different states produced by assumptions. Whereas the notion of full belief is clearly the proper equivalent of a justified belief, i.e. of a known content, the relation between a belief and an expectation is completely based on the structure of dependent judgements.

A dependent judgement which is based on alethic assumptions $(x_1:A_1, \dots, x_n:A_n)$, has obviously a weaker status than a judgement depending on epistemic assumptions $(a_1:A_1, \dots, a_n:A_n)$, i.e. when the agent is actually able to show constructions that justifies her assumptions. Hence, the belief set based on alethic assumptions is a conditional relation among the supposed truth of certain propositions and the truth of a consequence; when the belief set is based instead on epistemic assumptions, it expresses directly the knowability of contents. Assuming to possess a close derivation for a certain propositional content implies therefore also that one assumes to be constrained by that derivation to a known content. This means that if the conditional relation really holds, the knowability of the conclusion requires the assumed knowability of the premises, and those known assumptions lead invariantly

²² See e.g. [11].

to that conclusion. Under these conditions, constructions should not be revised, up to the point an error is found in the construction itself, i.e. when a syntactical error is done. In other words, a dependent judgement holding under epistemic assumptions, really expresses an expectation (where by this word one also refers to something which is considered most likely to happen). A simple belief does not impose the same epistemic constraints.

The difference between these forms of assumptions is therefore essential also in relation to the kind of errors involved: an alethic assumption ($x:A$) can be rejected because either it is recognized to be meaningless, i.e. it fails to have a well-formed type-introduction $A: \text{type}$, or it assumes an old variable, i.e. the new assumption would correspond to an old construction. On the other hand, a revision performed on an epistemic assumption ($a:A$) refers essentially to the syntactic structure of its construction a . A revision of a possible construction within the type A , recalls (at least implicitly) a transformation into the alethic correlative ($x:A$). The correspondence between different forms of assumptions and the dynamic-theoretical distinction between beliefs and expectations has a second result, which is of the greatest importance: it allows to provide a correct type-theoretical interpretation of the Ramsey Test and therefore to explain the identity between revision and conditionals²³.

3 Belief Revision

The wide range of non-classical approaches formulated for the problem of non monotonic reasoning, and in particular for the treatment of belief revision processes has produced a number of formal tools to treat with the cases of incoherence and revision, see e.g. [7]. Localization, representation and treatment of inconsistent data is a crucial property that non explosive logics share. One of the first formalization by contraction on dispensable elements is introduced in [15], and a variation on the theme of inconsistent belief bases is given in [18]; in [36] the state of the beliefs on inconsistent information is considered a distinct element from the inconsistent data itself, other treatments admit inconsistencies by the use of a paraconsistent logic as in [32] or by an adaptive logic as in [29]; another model of retraction on inconsistent bases by default reasoning is given in [5].

Constructive interpretations cannot provide such a treatment for the constraint on consistent belief sets, but are richer than classical approaches in such that they work on a system of justifications. A constructive treatment has been restricted so far to some function of revision in classical structures, which acts constructively on propositional contents, see [23]. Our aim is to illustrate now the type-theoretical interpretation of the standard revision

²³ The first remarks on the interpretability of the Ramsey Test within type theory based on the correct understanding of the notion of assumption are due to Göran Sundholm.

operators according to the conceptual frame of Constructive Type Theory introduced above.

The description of a revision procedure in the syntax of type theory is not so difficult to formulate,²⁴ but its development is based on a rather complex interpretation of the Ramsey Test and of conditionals. Notoriously, according to the Ramsey Test for conditionals,²⁵ a counterfactual conditional $A > B$ holds in a current body of knowledge K if and only if B is in K revised by A :

$$A > B \in K \leftrightarrow B \in K * A$$

with $*$ being one of the usual revision operators. From the point of view of the formal structure of type theory, a formulation of the test will require therefore an interpretation both of the conditional and of revision operations.

An interpretation of conditionals is included in the formulation of dependent judgements in terms of assertion conditions. Type theory has different ways to formalize the conditional relations for which the Ramsey Test expresses the identity with the revision operation. Among those, the case of a conditional statement (if A is true, then B is true) demands for its verification a dependent proof b of B provided that x is a proof of A or, in other words, the formulation of a construction for the type B under the alethic assumption that A is true ($b:B(x:A)$). This notion of conditional relation is the key term to interpret the Ramsey Test within the type theory: if the identity proposed by the test is to hold also in the formal language of type theory, the notions of alethic and epistemic assumptions and their role in the interpretation of revision procedures has to be clarified.

Interpreting the Revision Operators

The operation of *revision*, $K * A$ in the AGM model consists notoriously in the introduction of a new belief maintaining consistency; in general, any belief change can be defined in that model by contraction and expansion.

In the type-theoretical model, the basic interpretation of the notion of belief in terms of contextual contents, leads to a concurrent description of expansion and contraction operation. Let us start by the first one. The primitive notion of expansion is defined in terms of a pair of operations:

- *expansion*, $K + A$: it corresponds to a modification of a belief state within a belief set, by interpreting a context into another one, an operation we denote as follows:

$$(K(\Gamma \leftarrow^+ \Delta))$$

²⁴ This formal frame has been spelled out entirely in [26]. The results therein contained will be here considered from a more conceptual point of view.

²⁵ See e.g. [8] and [33].

where context Γ is expanded to context Δ in the belief set K by formulation of (at least one) new assumption $x_n:A_n$, with type A_n already contained in the starting context Γ ;

- *update*, $K \diamond A$: it corresponds to a modification of a belief state within a belief set, by interpreting a context into another one, an operation we denote as follows:

$$(K(\Gamma \leftarrow^\diamond \Delta))$$

where context Γ is update by context Δ in the belief set K by formulation of (at least one) new type declaration $B:type$. On the basis of an update a related expansion by $(x:B)$ is allowed.

Expansion and update represent in our model the two main operations performed in order to enlarge a belief state. Each operation provides conditions for the derivation of new judgements within the belief set, i.e. for a possibly monotonic extension. This distinction is strictly related to the form and nature of contexts. Constructive Type Theory deals with judgment formation, and type declarations, i.e. judgements of the form $A:type$ are considered presuppositions for any judgement of the form $a:A$. In other words, a construction for type A , showing its constructive meaning, presupposes stating that A is a meaningful type apt for predication.²⁶ On the basis of this priority, the distinction between expansion and update is formulated: the former works on existing types formulating new alethic assumptions; the latter formulates new assumptions under new type declarations; hence, an update always implies implicitly or explicitly an expansion. The distinction here intended between expansion and update obviously reflects in a new way the standard distinction settled in [19], according to which the term *revision* refers to an epistemic change due to additional information produced in a static world, whereas an *update* is a change due to a variation happening in the world. These two operations in the type-theoretical framework can also be linked to analytic and synthetic extensions of contexts:

- an expansion produces an analytic extension of the belief base, via the addition of one or more hypotheses falling within the given conceptual schema of the agent, i.e. within the types already provided, or by setting a definition for one such hypothesis;
- an update consists in a synthetic extension of the given conceptual frame, via introduction of one or more new predicable types, i.e. an extension of the meaning criteria of the body of knowledge.

The related derivation of new constructions within the belief set, is always to be intended as an analytic extension. A revision in terms of expansion or update can result nonetheless into an error; the explanation of this case requires the formulation of an appropriate type-theoretical operation:

²⁶ The distinction between constructive meaning and meaningfulness is explained in more detail in [25] and [27].

- *contraction*. $K \circ A$: is the result of removing one element in the belief set by means of a type-checking procedure performed backwards on the operations of
 1. derivation of new judgements;
 2. expansion of context;
 3. update of context.

By type-checking one identifies the failing believe by parsing any performed operations of revision; an error is categorized according to the corresponding operation that make it arise²⁷. Let us explore this issue further.

One starts the backward type-checking by analyzing performed derivation; if these are checked to be syntactically correct, the origin of a possible incoherent extension of a belief set is brought back to an operation of revision, i.e. by expansion or update. One first considers the formulation of (alethic/epistemic) hypotheses of the belief state in which an inconsistency is obtained including all of the involved type declaration: an error found at this step corresponds to an invalid formulation of an assumption within a type, hence a proof variable that cannot be verified. If any such construction is available, type-checking goes back to check updates, i.e. the staged formulation of any new type (from the more to the less dependent one).²⁸.

Ramsey Test and Minimal Change

Let us now come back to the interpretation of the Ramsey Test. Under the above displayed reading of the revision operations and provided the inter-

²⁷ I am indebted to Göran Sundholm for much of my comprehension of the problem of error in this formal context, and for the consequences I am here illustrating.

²⁸ A corresponding formal analysis has been introduced in [26, pp.182-3], as the *Restricted Monotonicity Principle I/II*:

Restricted Monotonicity Principle I: Let $(k_1 \cup \{i_1\})$ represent the *k-state* obtained by updating k_1 with the information expressed by i_1 . It holds that $(k_1 \cup \{i_1\}) \preceq k_2$, i.e. that k_2 is coherently obtained from the expanded state $(k_1 \cup \{i_1\})$, iff $i_1 \subseteq k_1$.

Restricted Monotonicity Principle II: Let $(k_1 \cup \{i_1\})$ represent the *k-state* obtained by updating k_1 with the information expressed by i_1 . It holds that $(k_1 \cup i_1) \prec k_2$, i.e. k_2 is obtained from the updated state $(k_1 \cup i_1)$ and some of the truths derived could be incoherent with what known in (k_1) , iff $i_1 \subseteq k_2$.

where the informational state i_1 expresses an extension on the belief state, and the knowledge states (belief sets) k_1 and k_2 are the starting and the resulting belief set (before and after the extension). This reflects the suggestion formulated by [4], according to which type theory provides a more direct procedure to recognize “suspect beliefs”: suspects are to be chosen among the elements of the context in which certain knowledge is derived, and the agent is allowed to choose which suspect to remove.

pretation of conditional statements as judgements of the form $b:B(x:A)$, the formula $B \in K * A$ on the right side of the Ramsey biconditional, means that a certain formal judgement $b:B$ holds within a body of knowledge, whose context has been revised by an operation concerning the propositional content A . Such revision can be of the form $\leftarrow^+ (x_n:A_n)$ – i.e. with A_n already contained in the starting context Γ and x_n anew assumption; or, rather. of the form $\leftarrow^\diamond (x_n:A_n(A_n:type))$ – i.e. an assumptions formulated under a new type A_n . In both cases, the revision by contraction of A_n is defined by comign back to a primitive expansion: $\leftarrow^+ (x_n : A_n \rightarrow \perp)$. This definition provides the type-theoretical interpretation of the Minimal Change Principle. The classical formulation of the principle is the following:

$$\neg A \notin K \rightarrow K * A = K + A \quad (\text{MCP}).$$

(MCP) says that, provided one does not hold the belief that A is not true, than the belief state is immediately revised by the belief that A is true. Some intuitive reasons challenge this principle,²⁹ but the kind of dynamics needed to solve these problems is difficult to simulate in the type-theoretical interpretation, especially in its constructive format. Our model provides in fact a far more restrictive meaning of contraction:

$$K \circlearrowleft A \leftrightarrow K(\Gamma) \leftarrow^+ (x : A \rightarrow \perp) \quad (\text{CONTR}).$$

According to this definition of contraction, the rejection of a certain belief results only from an absurdity being implied by its assumption. On the other hand, not to possess such an implication from A to the absurdity, does not imply the truth of the content at hand, at least until a proper construction for A is obtained. For this reason the standard formulation of the MCP does not hold in our model. This formulation of contraction is nevertheless equivalent to the principle of consistency for the AGM model, according to which the revision of K by A is incoherent only if it is proved in K that the negation of A holds:

$$(K * A = K \rightarrow \perp) \leftrightarrow \neg A \in K \quad (\text{AGM-REV}).$$

Under these conditions, the identity between conditional and revision proposed by the Ramsey Test is problematic because it results trivial, holding only for belief sets which are complete in respect to conditionals. Different solutions have been considered, like making the test weaker by adjoining pre-conditions, or by modifying the acceptance of conditionals and their relations to update. On the other hand, MCP is counterintuitive if interpreted for con-

²⁹ If I do not hold the believe that a God does not exist, it does not mean necessarily that I hold the opposite believe that a God exists. For example, I could be agnostic, and therefore refuse to hold any of them; or I could be willing to maintain both of them, at different stages or even at the same time, by referring to different meanings of “existence” or “godness”; I could finally refuse to accept the predication of “existence” in connection to the subject God.

ditionals, because the introduction of new information can obviously change the antecedent of a conditional. The main task of the present section is at this point to present the type-theoretical interpretation of the Ramsey Test, and to suggest how to make it non-trivial.

Type-theoretical Ramsey Test

In its type-theoretical interpretation, the right-hand expression of the Ramsey identity means essentially that the revision $K * A$ provides conditions for a certain judgement B to be apt to be predicated. Given the translation of the conditional $A > B$ with the hypothetical judgement $b:B(x:A)$, the operation of revising the context in K by the content $x:A$ is therefore enough to be able to formulate a construction for B . This *type-theoretical Ramsey Test* should be formulated as follows:

$$b:B(x:A) \in K \leftrightarrow b:B \in K(\Gamma \leftarrow^\diamond (x:A)) \quad (\text{ttRT}).$$

The requirement that the revision operation be one of update (\leftarrow^\diamond) is meant to satisfy the following additional conditions in order $B \in K * A$ to be equivalent with the related conditional $A > B$:

1. the conditional $A > B$ is valid only under formulation of the due assertion-conditions for the antecedent, in particular its type-declaration $A:\text{type}$;
2. the formulation of the due assertion conditions for A does not directly imply that correct proof-conditions b for B have been formulated, in order the truth of B to be asserted.

Under these specifications, it is not always true that a certain conditional $b:B(x:A)$ holds in K *if and only if* B holds provided the minimal change of K with $(x:A)$. Out of the formalism, it makes completely sense to modify a certain body of knowledge with some content providing conditions for other knowledge, without in fact happening to know, or being able, or even willing to formulate the consequence of those conditions. The result of the biconditional is different (and it actually corresponds to the classical interpretation) if the conditional is intended as a proper consequence, i.e. an expression of the form “ B is a consequence of A ” or “ A entails B ”, which in its complete epistemic formulation sounds “ B is known if A is known”. Such a formula says that extending a body of knowledge K by postulating an object $\text{Proof}(A)$, will imply $b:B$, i.e. an object $\text{Proof}(B)$. The way in which a corresponding revision on the body of knowledge K is performed, is completely different. One treats with extensions provided by means of an epistemic assumption, which happens to be the condition for another propositional content:

$$((A \rightarrow B)\text{holds}) \in K \leftrightarrow b : B \in K(\Gamma \leftarrow^\diamond (a:A)) \quad (\text{ttRT2}).$$

This modified version of the Ramsey Test (ttRT2) expresses the condition actually meant by the classic version: it accounts for the explicit knowledge of conditions allowing the formulation of a certain consequence. On the basis of the formulation of alethic assumptions, an agent being given with due conditions for a certain conditional, it is not constrained to the explicit knowledge of their consequences; and to know the holding of a certain conditional does not mean to possess explicit knowledge of all the needed conditions.

A Model for Kripke Semantics

The result of the restricted monotonicity, which interprets extensions of (non-)monotonic reasoning under conditions of operations on contexts, can be compared with the connected notion of forcing when belief states are intended as nodes in a Kripke semantics. Notoriously, in such a semantics if a content is forced at a certain node, say i , it will be so at every further one j . In the type-theoretical structure this means, in the simplest case, that the result of a certain (non-)dependent derivation, i.e. one obtained under a (possibly empty) context Γ , is maintained at every next step whose extending context Δ is empty:

$$i \leq j, i \models a:A(\Gamma) \rightarrow j \models a:A(\Gamma \leftarrow (\Delta = \emptyset)).$$

This simple case needs to be modified in the case of an extension of the belief set based on a non-empty operation on context. The following equation considers the alternative case when Γ and Δ differ for at least one expression; in this case, the forcing of the given content at any later node strictly depends on the monotonicity of extensions of the previous context:³⁰

$$\begin{aligned} i \leq j, i \models a:A(\Gamma) \rightarrow j \models a:A(\Gamma \leftarrow^\diamond \Delta) \leftrightarrow \\ (\Gamma \leftarrow^\diamond x : X = \Delta) \wedge (\neg(x : X \rightarrow \perp) \in \Gamma). \end{aligned}$$

Therefore, monotonicity intended as

$$i \leq j, i \models a:A \rightarrow j \models a:A$$

is restricted under revision of contexts:

$$i \leq j, (i \models a:A(\Gamma) \rightarrow j \models a:A(\Delta)) \leftrightarrow i \models a:A(\Delta).$$

This analysis can be extended to interpret dynamic operations performed on contexts as the core property of type-theory³¹: the operations here consid-

³⁰ As in the previous cases, we consider the more general update, whose result is preserved by expansion.

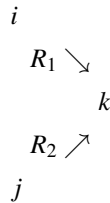
³¹ See e.g. [24] and [35].

ered allow to present the whole framework of type theory as a model of theory change, based on the constructive operation of preservation of knowability.

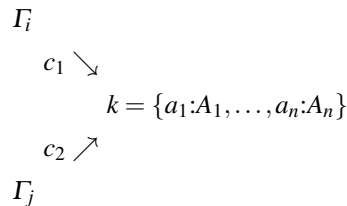
4 Belief Merging

The classical notion of merging in belief revision refers to the fusion of two or more sets of beliefs while maintaining consistency. Formal merging procedures typically refer to cases in which different sources provide the same information, inconsistent information, or in which their merging is intended to discover hidden or implicit beliefs (see e.g. [13]; for an overview see [6]).

As with belief revision, a constructive approach is more restricted and its analysis of merging processes advances with the same pro's and contra's mentioned in the previous sections. In particular, to interpret merging processes within the type-theoretical framework, one needs to provide a formalization which applies on contexts and extends the usual notion of monotonicity for expansions and derivations. The property of simple monotonicity in a (constructive) Kripke semantics applies to merging as follows: for every two stages $i \leq j$, there is an accessibility relation R_1 , such that for every literal A such that $i \models A$, then $j \models A$; and if there are two stages i and j , and they are such that $i, j \leq k$, then there are two such relations R_1 and R_2 such that for every two literals A, B such that $i \models A, j \models A$, then $k \models A, B$:



Within the type-theoretical approach, monotonicity has been interpreted on the derivations formulated on the basis of contexts. The stages i, j of the previous model of strict monotonicity can be now interpreted as distinguished context Γ_i, Γ_j ; A, B corresponds to (eventually distinguished) constructions c_1, c_2 for corresponding propositional contents; and k is one and the same knowledge state obtained by those operations:



This formalizes a basic (without any protocol being defined) merging; moreover, this schema is always satisfied *if and only if* no operations of revision are performed on the contexts, and provided those contexts present both sufficient conditions for deriving the judgements in k . This is once again due to the conditions expressed by the restricted monotonicity, according to which only extensions executed without revisions on the informational contents of contexts lead with certainty to monotonic extensions of knowledge states. But a proper operation of merging between different belief states must be given under condition of revisions performed on the contexts of beliefs, i.e. precisely in the case when the Principle of Restricted Monotonicity may fail. Such a structure can be represented as a schema commuting under those explicit conditions of revision.³² The two starting belief states Γ_i and Γ_j are now defined by operations of revisions r_1, r_2 (i.e. expansion or update) on a given state Γ :³³

$$\begin{array}{ccc} & & \Gamma_i \\ & r_1 \nearrow & \\ \Gamma & & \\ & r_2 \searrow & \\ & & \Gamma_j \end{array}$$

Provided these revisions, there is a merging if and only if a certain state k of the form $\{a_1:A_1, \dots, a_n:A_n\}$ is the result of (eventually distinguished) derivations c_1, c_2 from Γ_i, Γ_j , i.e. that $\Gamma_i, \Gamma_j \vdash k$:

$$\begin{array}{ccccc} & & \Gamma_i & & \\ & r_1 \nearrow & & c_1 \searrow & \\ \Gamma & & & & k \\ & r_2 \searrow & & c_2 \nearrow & \\ & & \Gamma_j & & \end{array}$$

³² It is easy to recognize the property illustrated in the following as the well-known “diamond property” for λ -calculi, which is usually defined as follows: a binary relation \succ on the lambda terms satisfies the diamond property if for all terms M, M_1, M_2 for which $M \succ M_1$ and $M \succ M_2$ we have a term M_3 such that $M_1 \succ M_3$ and $M_2 \succ M_3$; the property is used especially for the reductions among lambda terms. See e.g. [3]. Martin-Löf in a talk given at the workshop “Mathematics, Algorithms and Proofs 2007”, Lorentz Center, Leiden, made use of this basic property to show the type-theoretical interpretation of the inductive limit and logical operators in a comparison with Topos Theory.

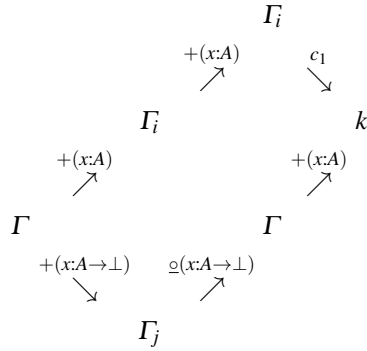
³³ In the following schemas, we keep the direction of arrows to the right, but they actually correspond to revision operations among contexts, with left-oriented arrows, from the previous sections.

This means in turn that the operation of belief merging for the type-theoretical structure is interpreted in a rather strict way: for two belief sets to be mergeable, they need to provide sufficient assertion conditions to derive a unique knowledge state. This condition is too restrictive if it means that the merging is impossible provided the two contexts are equipped with contradictory elements. In fact, one interesting operation which often is presented in connection to merging is that of preference information. The basic case is the following: given two belief sets equipped with contradictory information among other, a preference operation should be performed, such that it makes possible to reject one of the two contradictory information (or both, if necessary), thus allowing the merging of the remaining informations in a new belief set.³⁴ A similar operation should be representable also in our type-theoretical frame and it has to be harmonized with the given formal representation of merging. To obtain this, one starts by considering a unique Γ : contradictory states are represented by expansions on such context leading respectively to $\Gamma_i = \Gamma \leftarrow^+ (x:A)$ and to $\Gamma_j = \Gamma \leftarrow^+ (x:A \rightarrow \perp)$. The merging of Γ_i, Γ_j is at this stage impossible. A revision is needed in order to reject one or both of the judgements: such operation can be performed following the already mentioned type-checking procedure (any step requires that the checking has been performed on the previous ones, provided no error has occurred in the output):

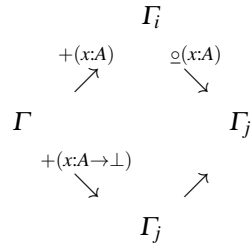
1. analyze the set of inferences valid for k , performed on the basis of Γ_i :
 - [1a.] if any judgement derivable from $k(\Gamma_i)$ is contradictory w.r.t. Γ , reject the extension provided by Γ_i ;
 - [1b.] if no contradiction is found, the checking procedure goes to the next step;
2. analyze the result of the substitution $(x/a : A)$ in Γ_i :
 - [2a.] if an error is found in such a construction, reject the truth of type A , accept the extension of Γ_j ;
 - [2b.] if no error is found by assuming a construction in the type A , the extension provided by Γ_i is accepted and Γ_j will be submitted to contraction;
3. a counter-proof for the last step is represented by checking the presupposition $A: type$ in Γ_i : an error found at this stage means that the concept represented by that type is not admissible for predication, and indirectly implies the acceptance of Γ_j .

Once this type-checking is completed, one can accept one of the two extensions. The acceptance of the expansion given by Γ_i is expressed by the following schema:

³⁴ Among other methods, a preference on contradictory belief bases that satisfies majority is defined in [30].



Because Γ_i provides at least one new assumption, the schema is here completed by a new construction c_1 which is meant to lead to a new knowledge state k . The merging on the basis of Γ_j is instead given by the following schema, in which the expansion provided by the new assumption $(x : A)$ is rejected, and it is accepted the extension by means of $x : A \rightarrow \perp$:



The operation of merging allows therefore only a procedure for checking the admissibility of one side of two contradictory informations, and it does not allow a real internal dynamics of preference for informative contents at different stages.

5 Some Remarks

In this last section, the interpretation of type theory in the context of theories of change and the dynamics of knowledge system shall be considered in the light of some of its properties³⁵. The conceptual order of conditions described at the beginning results here essential to interpret some of the most interesting features of standard models of dynamic reasoning.

³⁵ These properties are partially extracted from the problems suggested in [17].

5.1 Admitting Beliefs

A first remark concerns the basic cognitive abilities of the rational agent which implements our system of belief change. Submitting beliefs to revision has been often represented in terms of finite belief bases (i.e. on calculi enjoying completeness and decidability) and finite ability of extensions. The type-theoretical model establishes stricter conditions on such abilities: this is obtained by the switch from a simple finite belief state, to a model in which beliefs are intended as conditions for proving knowledge. The criterion of predicativity for types, interpreted in terms of the definition of types from below by means of their constructions, represents the justification of the entire model of knowledge. Beliefs, rather than being admitted with an upper limit bound, are restricted by a double criterion:

1. beliefs are admitted if meaning criteria are satisfied, by admission of predication aptness of the concepts involved;
2. beliefs are admitted if their formulation is needed by further judgements construction.

In this way, beliefs express a specific epistemic status for a rational agent who is carrying on a knowledge process, and extensions are accepted only if coherent with the starting knowledge base, i.e. only if the newly introduced expressions satisfy similar conditions. Moreover, the formulation of belief contents does not imply the logical closure of the related belief set, because the consequences require the formulation of due constructions. This analysis leads directly to a second topic, namely that of setting criteria for the degrees of beliefs.

5.2 Degrees of Belief

The degree of a certain belief, which is a natural property in every-day reasoning (“how much do you believe in this?”) is formally expressed by the complementing notions of degree of confidence and resistance to change. The notion of degree of confidence is usually intended as a static criterion, held by the agent with respect to a certain belief: this notion is central in those approaches based on probabilistic accounts of beliefs. On the other hand, the notion of resistance to change is a dynamic criterion, directly depending on the degree given to a certain belief: this latter relation is mainly explained in the standard models by referring to the already introduced relation of epistemic entrenchment. The first of the two relations holds in the models maintaining a distinction between sentences representing beliefs and those that do not (so called dichotomous models), whereas the distinction fails when resistance to change is called upon.

The epistemic structure described for the type theory can simply simulate the two relations here at hand, obviously in terms of the relation of provability for propositional contents and assertion-conditions for judgements. The main consequence is to set different epistemic states for categorical and dependent judgements. The static and the dynamic notions of degree of belief and resistance to change are therefore in this model completely connected, applying to the possible judgemental contents in the following way:

- a propositional content whose truth is asserted on the basis of its proof-conditions (a known content provided with its justification and related identity on proof terms, categorical judgement) provides the highest degree of confidence (certainty); its degree of resistance to change depends on the correctness of the justification, therefore can vary only after revision of this construction;
- a propositional content whose truth is asserted depending on a context of assumptions (known content under believed justifications, dependent judgement) has a degree of resistance to change depending on the degree of confidence held by the agent with respect to the content of those assumptions;
- the contents of a certain context (belief contents, assumptions) provide a degree of confidence depending on the coherence and correctness of the judgements derived on the basis of this context; their degree of resistance to change is the lowest, because these contents can always be submitted to revision, and it is also dependent on a certain degree of confidence in the meaningfulness the agent ascribes to the involved concepts (type-introductions).

The conceptual order establishes a fixed degree of confidence (comparable to value 1 in probabilistic models) for propositional contents whose proof-conditions are satisfied. This value is variable when referred to assertion-conditions of some proved judgement, i.e. when it is referred to the belief state of a certain knowledge process: in this case, the degree of confidence determines the resistance to change for dependent judgements and, viceversa, the degree of confidence in the derived judgements depends on the resulting coherence with the entire knowledge state and determines the resistance to change for their conditions. This latter value is the lowest, because assumptions are the most simple contents to revise, and they only depend on the meaningfulness of the types involved.

This model indirectly establishes also the degree of revision of contents, each content being epistemically determined by its proper justifications and conditions:

- a justified content is maintained as long as the justification is proved to hold, i.e. up to an error in the construction is found;
- a justified content given under conditions is maintained as long as those conditions are not proven false;

- conditions for knowledge are maintained as long as the derived content is coherent with the entire state, and the presuppositions of meaning for those conditions are satisfied.

The entire structure of type theory is essentially based on the conditional relation of justification, and the conceptual order provides in this way the conditions to revise contents. It seems clear that the constructive version of type theory, along with a number of other frameworks focusing on justifications³⁶, provides a structure in which the notion of belief and a certain degree of dynamic for reasoning can be successfully simulated. The major property (and possibly its weakness) consists in a strong epistemic basis, which defines belief states in the context of a proper definition for knowledge, but which also imposes strict conditions on the possibility of revision, information preference and knowledge change.

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³⁶ See e.g. [2].

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