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Theory and Applications
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A Pedagogically Useful Relevant Implication and Some General Lessons

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Outline

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Aim

relevant implication in elementary logic course

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- relevant logics too complex
- relevant logics too far away from **CL**

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How to combine a (simple) relevant implication with **CL**?

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How to combine a (simple) relevant implication with **CL**?

· combination with more sophisticated implication is not more difficult

· **PCR** (1992, 1978?)

· Ghent lecture by David Makinson

Possible motivation

paradoxes of classical logic
break down into:

(i) consequence relation: derivable *given* vs. derivable *from*
$p \vdash_{\mathbf{PC}} q \supset q$
cf. semantics

(ii) contradictory theories no models
logically indistinguishable
no sensible reasoning from them

(iii) meaning of the implication in **CL** vs. natural languages
$p \vdash_{\mathbf{PC}} q \supset p, \quad p \vdash_{\mathbf{PC}} \neg p \supset q, \quad \neg(p \supset q) \vdash_{\mathbf{PC}} p \wedge \neg q$

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official relevance tradition (A & B) removes all paradoxes in single move

however: derivable given, . . . sensible

This lecture

PCR: **PC** + specific simple relevant implication

pedagogically useful

theoretical problems similar to **PC** + other relevant implications

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propositional level: where paradoxes surface

relevant implication: no obvious approach for formalizing predicative statements

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Aim

Fitch-Style Rules: **PC**

Relevant Consequence Relation

Eliminating Nested Arrows

Stars and Carrying them Over

Fitch-Style Rules: **PCR**

Worlds Semantics and Tableaux

Algebraic Semantics

Embarrassing Strength?

A General Recipe and a Lesson

Peter's Complaint

Fitch-Style Rules: **PC**

'structural' rules: PREM, HYP, REIT

deduction rules:

MP	$A, A \supset B / B$
CP	From a subproof starting with the hypothesis A and ending with B, to infer $A \supset B$.
ADJ	$A, B / A \wedge B$
SIM	$A \wedge B / A$ and $A \wedge B / B$
ADD	$A / A \vee B$ and $B / A \vee B$
DIL	$A \vee B, A \supset C, B \supset C / C$
EI	$A \supset B, B \supset A / A \equiv B$
EE	$A \equiv B / A \supset B$ and $A \equiv B / B \supset A$
DN	$\neg\neg A / A$
RAA	$A \supset B, A \supset \neg B / \neg A$

subproof is *closed* iff a formula was derived from it by CP

1	$p \supset \neg q$	PREM
2	$(r \supset r) \supset q$	PREM
3	$\mid p$	HYP
4	$\mid \mid r$	HYP
5	$\mid r \supset r$	4, 4; CP
6	$\mid (r \supset r) \supset q$	2; REIT
7	$\mid q$	5, 6; MP
8	$p \supset q$	3, 7; CP
9	$\neg p$	1, 8; RAA

paradox in line 8; $p \supset \neg q, (r \supset r) \supset q \vdash_{\mathbf{PCR}} \neg p$ not paradoxical

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Definition A **PC**-*proof* of *A from* Γ . . .

Definition $\Gamma \vdash_{\mathbf{PC}} A$ iff there is a **PC**-proof of A from Γ.

Definition $\vdash_{\mathbf{PC}} A$ iff $\emptyset \vdash_{\mathbf{PC}} A$.

In pedagogical context: derivable rules of inference
(equilibrium between heuristic facility and set of rules)

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0-worlds consistent and $\neg$-complete (**PC**-valid $\Rightarrow$ **L**-valid)

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role and status of theorems (and valid formulas) unusual

· if (2), then $n < 0$; $\emptyset$ **L**-entails nothing (not even theorems)

· **L**-theorems bring one from premises to conclusion (by MP and ADJ)

· (2) is Tarski (refl., mon., trans.)

Outline

Eliminating Nested Arrows

$\mathcal{W}$	formulas of language of **PC** ($p, q, \ldots, \neg, \vee, \wedge, \supset, \equiv$)
$\mathcal{W}^{\rightarrow}$	formulas of language of usual relevant logics
$\mathcal{W}^{1}$	(no nested arrows) formulas of language of **PCR**

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$\mathcal{W}^{1}$	(no nested arrows) formulas of language of **PCR**

to formalize statements from natural languages into $\mathcal{W}^1$ hardly hindrance

most sentences of form

$A \rightarrow (B \rightarrow C)$

equivalent to sentence of form

$(A \wedge B) \rightarrow C$

or to metalinguistic

$A \vdash B \rightarrow C$

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Stars and Carrying them Over

RHYP introduce any member of $\mathcal{W}$ with a star attached to it

RHYP starts a (new) *starred subproof*

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the intention:

$\vdots$	$\vdots$	
i	$\mid A^*$	RHYP
$\vdots$	$\mid \vdots$	$\vdots$
j	$\mid B^*$	$\dots$
$j{+}1$	$A \rightarrow B$	i, j; RCP
$\vdots$	$\vdots$	

restriction:
no subproof can be started within a starred subproof

Carrying over stars

we better have an interpretation of the arrow

(M→) $A \to B$ means that reasons to accept A constitute reasons to accept B *and*
that reasons to reject B constitute reasons to reject A.

Carrying over stars

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(M$\rightarrow$) $A \rightarrow B$ means that reasons to accept A constitute reasons to accept B *and* that reasons to reject B constitute reasons to reject A.

(M$\neg$) One has reasons to accept $\neg A$ iff one has reasons to reject A.
One has reasons to reject $\neg A$ iff one has reasons to accept A.

(M$\wedge$) One has reasons to accept $A \wedge B$ iff one has reasons to accept A as well as reasons to accept B.
If one has reasons to reject A or reasons to reject B, then one has reasons to reject $A \wedge B$.

(M$\vee$) If one has reasons to accept A or reasons to accept B, then one has reasons to accept $A \vee B$.
One has reasons to reject $A \vee B$ iff one has reasons to reject A as well as reasons to reject B.

this gives us:

$$\text{we have reasons to accept } \neg(A \wedge B)$$
$$\Updownarrow$$
$$\text{we have reasons to reject } A \wedge B$$
$$\Uparrow$$
$$\text{we have reasons to reject } A \text{ or to reject } B$$
$$\Updownarrow$$
$$\text{we have reasons to accept } \neg A \text{ or to accept } \neg B$$
$$\Downarrow$$
$$\text{we have reasons to accept } \neg A \vee \neg B$$

modifying the ‘implications’ to ‘equivalences’ creates problem:

we have reasons to accept $\neg(A \wedge B)$

$\Updownarrow$

we have reasons to reject $A \wedge B$

$\Updownarrow$

we have reasons to reject A or to reject B
or to merely reject $A \wedge B$

$\Updownarrow$**?**

we have reasons to accept $\neg A$ or to accept $\neg B$
or to merely accept $\neg A \vee \neg B$

$\Updownarrow$

we have reasons to accept $\neg A \vee \neg B$

answer depends on the *merely* parts

positive answer is justifiable by

if our knowledge became total *and would still constitute a reason to reject* $A \wedge B$, then it would constitute a reason to reject A or to reject B or to reject both A and B.

if our knowledge became total *and would still constitute a reason to accept* $\neg A \vee \neg B$, then it would constitute a reason to accept $\neg A$ or to accept $\neg B$ or to accept both $\neg A$ and $\neg B$

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these justify a further meaning postulate:

(M$\wedge\vee$) One has reasons to merely reject $\neg A \wedge \neg B$ iff one has reasons to merely accept $A \vee B$.
One has reasons to merely reject $A \wedge B$ iff one has reasons to merely accept $\neg A \vee \neg B$.

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don't suppose total knowledge: $A \rightarrow (B \vee C) \nvdash_{\mathbf{PCR}} (A \rightarrow B) \vee (A \rightarrow C)$

the meaning postulates justify four conventions for carrying over stars:

(C1) same meaning in view of the meaning postulates:
$A \supset B$ and $\neg A \vee B$
$A \equiv B$ and $(A \supset B) \wedge (B \supset A)$
$A \wedge (B \vee C)$ and $(A \wedge B) \vee (A \wedge C)$
$\neg\neg A$ and A
$\neg(A \vee B)$ and $\neg A \wedge \neg B$ etc.

mutual tautological entailments

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(C3) rules with a major and a minor (local) premise
ex. YES: RMP: $A^*, A \to B / B^*$
ex. NO: DS: $\neg A^*, A \vee B / \neg A$

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(C4) ADJ-like steps: only starred if *both* local premises starred

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Fitch-Style Rules: **PCR**

RHYP

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RREIT (only) formulas of the form $A \rightarrow B$ may be reiterated into a starred subproof

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ADJ $A^*, B^* / A \wedge B^*$

SIM $A \wedge B^* / A^*$ and $A \wedge B^* / B^*$

ADD $A^* / A \vee B^*$ and $B^* / A \vee B^*$

MI $A \supset B^* \,\|\, \neg A \vee B^*$

ME $A \equiv B^* \,\|\, (A \supset B) \wedge (B \supset A)^*$

DN $\neg\neg A^* \,\|\, A^*$

ND $\neg(A \vee B)^* \,\|\, \neg A \wedge \neg B^*$

NC $\neg(A \wedge B)^* \,\|\, \neg A \vee \neg B^*$

DIST $A \wedge (B \vee C)^* / (A \wedge B) \vee (A \wedge C)^*$

Fitch-Style Rules: **PCR**

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ADJ $A^*, B^* / A \wedge B^*$

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ME $A \equiv B^* \parallel (A \supset B) \wedge (B \supset A)^*$

DN $\neg\neg A^* \parallel A^*$

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DIST $A \wedge (B \vee C)^* / (A \wedge B) \vee (A \wedge C)^*$

RMP $A^*, A \to B / B^*$

RDIL $A \vee B^*, A \to C, B \to C / C^*$

RMT $\neg B^*, A \to B / \neg A^*$

RCP From a subproof $\langle A^*, \ldots, B^* \rangle$, to infer $A \to B$.

also OK without stars

some properties:

(1) **PCR** is a conservative extension of **PC**: if $A \in \mathcal{W}$, then $\vdash_{\mathbf{PCR}} A$ iff $\vdash_{\mathbf{PC}} A$.
(2) For all $A \in \mathcal{W}$, $\vdash_{\mathbf{PCR}} A$ iff $\vdash_{\mathbf{PCR}} \neg A \to A$.
(3) $A \to B$ is a **PCR**-theorem iff it is a tautological entailment.
(4) If $\vdash_{\mathbf{PCR}} A \leftrightarrow B$ and D is obtained by replacing the subformula A in C by B, then $\vdash_{\mathbf{PCR}} C \leftrightarrow D$. (Replacement of Relevant Equivalents)
(5) Replacement of (Material) Equivalents does not hold in **PCR**. Example $\nvdash_{\mathbf{PCR}} (p \to q) \equiv ((p \vee (r \wedge \neg r)) \to q)$.
(6) Derivable rules: $A \to (B \wedge C) \parallel (A \to B) \wedge (A \to C)$
$(A \vee B) \to C \parallel (A \to C) \wedge (B \to C)$
(7) Negative results:
$B \nvdash_{\mathbf{PCR}} A \to B$ $\qquad \neg A \nvdash_{\mathbf{PCR}} A \to B$
$\neg(A \to B) \nvdash_{\mathbf{PCR}} A$ $\qquad \neg(A \to B) \nvdash_{\mathbf{PCR}} \neg B$
In general no *implication* paradox for arrow.

that REIT need to be restricted:

1	$\neg p$	PREM	
2	$\neg p \vee (p \rightarrow q)$	1; ADD	
3	$\mid p^*$	RHYP	
4	$\mid \neg p \vee (p \rightarrow q)$	2; REIT	!
5	$\mid p \rightarrow q$	3, 4; DS	
6	$\mid q^*$	3, 5; RMP	
7	$p \rightarrow q$	3, 6; RCP	

Outline

Worlds Semantics and Tableaux

$M = \langle W, w_0, v \rangle$

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$w_0 \in W$

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$M = \langle W, w_0, v \rangle$

W a set

$w_0 \in W$

$v : \mathcal{W}^1 \times W \to \{0, 1\}$ fulfils:

SPCR1 $v(\neg A, w_0) = 1$ iff $v(A, w_0) = 0$

SPCR2 $v(A \to B, w_0) = 1$ iff, for all $w_i \in W$, $v(A, w_i) \leq v(B, w_i)$ and $v(\neg B, w_i) \leq v(\neg A, w_i)$

SPCR3 $v(A \vee B, w_i) = \max(v(A, w_i), v(B, w_i))$

SPCR4 $v(\neg\neg A, w_i) = v(A, w_i)$

SPCR5 $v(\neg(A \vee B), w_i) = \min(v(\neg A, w_i)), v(\neg B, w_i))$

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$\{A \mid v(A, w_0) = 1\}$ is consistent and negation-complete

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$M \Vdash A$ iff $v(A, w_0) = 1$
M model of Γ iff M verifies all members of Γ
$\Gamma \vDash A$ iff all models of Γ verify A
$\vDash A$ iff all models verify A.

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M model of Γ iff M verifies all members of Γ
$\Gamma \vDash A$ iff all models of Γ verify A
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tableau-method: PM

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Algebraic Semantics

PM

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Embarrassing Strength?

examples:

$A \rightarrow B \vdash_{\mathbf{PCR}} A \rightarrow (A \wedge B)$

$A \rightarrow B \vdash_{\mathbf{PCR}} (A \wedge C) \rightarrow (B \wedge C)$

Embarrassing Strength?

examples:
$A \to B \vdash_{\textbf{PCR}} A \to (A \wedge B)$
$A \to B \vdash_{\textbf{PCR}} (A \wedge C) \to (B \wedge C)$

cause: that inference relation is not relevant: for all Γ,
$\Gamma \vdash_{\textbf{PCR}} A \to A$
$\Gamma \vdash_{\textbf{PCR}} C \to C$

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independent arguments for relevance of the arrow in **PCR**

(i) $\emptyset \vdash_{\mathbf{PCR}} A \to B$ iff $A \to B$ is tautological entailment

(ii) the valid statements
$\Gamma \vdash_{\mathbf{PCR}} A \to B$ iff $\Gamma \vdash_{\mathbf{PCR}} (A_1 \to B_1) \wedge \ldots \wedge (A_1 \to B_1)$
are identical to those for first degree entailments

so blame on inference relation, not on implication

Outline

A General Recipe and a Lesson

R-M semantics

remember:

- $A_1, \ldots, A_n$ **L**-entail B iff, for all M, B is verified by every world that verifies $A_1, \ldots, A_n$
- M verifies A iff $v(A, 0) = 1$
- every model verifies every **L**-theorem

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· every model verifies every **L**-theorem

extending **PC** with **L**: $\Gamma \vDash_{\mathbf{PCL}} A$ iff, every model of Γ verifies A

Fitch-style rules

remember:
originally for **L**-theorems only

A General Recipe and a Lesson

R-M semantics

remember:

- $A_1, \ldots, A_n$ **L**-entail B iff, for all M, B is verified by every world that verifies $A_1, \ldots, A_n$
- M verifies A iff $v(A, 0) = 1$
- every model verifies every **L**-theorem

extending **PC** with **L**: $\Gamma \vDash_{\mathbf{PCL}} A$ iff, every model of Γ verifies A

Fitch-style rules

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weakest Tarski inference relation $\vdash_{\mathbf{PCL}}$ that extends $\Gamma \vdash_{\mathbf{PC}} A$ and ($A_1, \ldots, A_n$ **L**-entail B) and (if A is a **L**-theorem, then $\emptyset \vdash_{\mathbf{PCL}} A$)

Outline

Peter's Complaint

Peter wanted: if $A_1, \ldots, A_n$ and B belong to the language of **L**, then $A_1, \ldots, A_n \vdash_{\mathbf{PC}^{\mathbf{L}}} B$ iff $A_1, \ldots, A_n B_n$ **L**-entail B.

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trouble (for **R** and many other relevant logics):

$\Gamma \vdash_{\mathbf{PC}^{\mathbf{R}}} A \vee \neg A$ for all Γ

and

$(A \vee \neg A) \rightarrow (((A \vee \neg A) \rightarrow B) \rightarrow B)$

so

$\Gamma \vdash_{\mathbf{PC}^{\mathbf{R}}} ((A \vee \neg A) \rightarrow B) \rightarrow B$

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intriging, although not exactly the original intention

Thanks.

Questions?