2 Prospective Dynamics: Pushing the Heuristics into the Proofs

2.1 Proofs and their Explications

CL is claimed to explicate actual proofs, for example in mathematics

This presupposes:

(1) specific meaning of the logical symbols in those contexts

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CL is claimed to explicate actual proofs, for example in mathematics

This presupposes:

(1) specific meaning of the logical symbols in those contexts
not discussed here

(2) correct proofs classified as correct OK
proofs classified as correct are correct yes, but . . .

Actual proofs: result from goal-directed process

actually produced		actually published
—		—
—		—
—	⟶	—
—	skip dead ends	—
—	skip detours	—
—	skip obvious steps	—
—	...	—
—		—
—		
result of search process		presentation

Neither produced nor published proofs are explicated adequately by **CL**:

CL is too permissive, viz. not goal directed

for example

1	p	Prem
2	$p \vee q$	1; Add
3	$p \vee r$	1; Add
4	$p \vee s$	1; Add
...	...	

2.2 Instructions vs. rules

rule: preserves truth

instruction: permission/obligation to apply a rule
(depending on stage of the proof)

official proof: procedure = rules + universal permission

— not goal-directed

— does not explicate actually produced / published proofs

— is border case of procedure

some **procedures** explicate actual proofs

2.3 Prospective dynamics: idea and examples

— idea:
if one looks for A
and, e.g., $B \supset A$ was derived
then look for B

— pushing (part of) the heuristics in the proof:
if one looks for A
and, e.g., $B \supset A$ was derived
then derive $[B]\,A$
indicating that one should look for B
(given the premises, obtaining B is sufficient to obtain A)

$t \vee q,\ p \supset (q \vee {\sim}r),\ r \wedge s,\ s \supset p \ \vdash\ q$

1	$[q]\, q$	Goal	R^{14}
2	$t \vee q$	Prem	
3	$[{\sim}t]\, q$	2; $\vee$E	${\sim}t\vert$
4	$p \supset (q \vee {\sim}r)$	Prem	
5	$[p]\, q \vee {\sim}r$	4; $\supset$E	R^{11}
6	$s \supset p$	Prem	
7	$[s]\, p$	6; $\supset$E	R^{10}
8	$r \wedge s$	Prem	
9	s	8; $\wedge$E	
10	p	7, 9; Trans	
11	$q \vee {\sim}r$	5, 10; Trans	
12	$[r]\, q$	11; $\vee$E	R^{14}
13	r	8; $\wedge$E	
14	q	12, 13; Trans	

Incidentally:

algorithm: prospective proofs ⇒ Fitch-style proofs

1	$p \supset (q \vee {\sim}r)$	Prem
2	$s \supset p$	Prem
3	$r \wedge s$	Prem
4	s	3; Sim
5	p	2, 4; MP
6	$q \vee {\sim}r$	1, 5; MP
7	r	3; Sim
8	q	6, 7; DS

$\sim p \vee q \ \vdash \ p \supset q$

1	$[p \supset q]\, p \supset q$	Goal	R^{8}
2	$[q]\, p \supset q$	1; C⊃E	$q\|$ R^{8}
3	$\sim p \vee q$	Prem	
4	$[p]\, q$	3; ∨E	$p\|$
5	$[\sim p]\, p \supset q$	1; C⊃E	$\sim p\|$ R^{8}
6	$[\sim q]\, \sim p$	3; ∨E	$\sim q\|$
7	$[p]\, p \supset q$	2, 4; Trans	$p\|$ R^{8}
8	$p \supset q$	5, 7; EM	

obtain the Goal on all non-redundant conditions

2.4 Prospective dynamics: characterization

Rules (prospective proof for $\Gamma \vdash G$)

Goal To introduce $[G]\, G$.

Prem To introduce A for an $A \in \Gamma$.

Trans
$$\frac{[\Delta \cup \{B\}]\, A \qquad [\Delta']\, B}{[\Delta \cup \Delta']\, A}$$

EM
$$\frac{[\Delta \cup \{B\}]\, A \qquad [\Delta' \cup \{\sim B\}]\, A}{[\Delta \cup \Delta']\, A}$$

Note: the complement of a formula:

if A has the form $\sim B$, then $*A = B$

otherwise $*A = \sim A$

$$\begin{aligned} *p &= \sim p \\ *\sim p &= p \\ *\sim\sim p &= \sim p \end{aligned} \qquad \begin{aligned} **p &= p \\ **\sim p &= \sim p \\ **\sim\sim p &= p \end{aligned}$$

α	α_1	α_2		β	β_1	β_2
$A \wedge B$	A	B		$\sim(A \wedge B)$	$*A$	$*B$
$A \equiv B$	$A \supset B$	$B \supset A$		$\sim(A \equiv B)$	$\sim(A \supset B)$	$\sim(B \supset A)$
$\sim(A \vee B)$	$*A$	$*B$		$A \vee B$	A	B
$\sim(A \supset B)$	A	$*B$		$A \supset B$	$*A$	B
$\sim\sim A$	A	A				

Formula analysing rules:

$$\frac{[\Delta]\,\alpha}{[\Delta]\,\alpha_1 \quad [\Delta]\,\alpha_2} \qquad \frac{[\Delta]\,\beta}{[\Delta \cup \{*\beta_2\}]\,\beta_1 \quad [\Delta \cup \{*\beta_1\}]\,\beta_2}$$

Example:

$$\frac{[\Delta]\,p \wedge q}{[\Delta]\,p \quad [\Delta]\,q} \qquad \frac{[\Delta]\,p \vee q}{[\Delta \cup \{\sim q\}]\,p \quad [\Delta \cup \{\sim p\}]\,q}$$

α	α_1	α_2		β	β_1	β_2
$A \wedge B$	A	B		$\sim(A \wedge B)$	$*A$	$*B$
$A \equiv B$	$A \supset B$	$B \supset A$		$\sim(A \equiv B)$	$\sim(A \supset B)$	$\sim(B \supset A)$
$\sim(A \vee B)$	$*A$	$*B$		$A \vee B$	A	B
$\sim(A \supset B)$	A	$*B$		$A \supset B$	$*A$	B
$\sim\sim A$	A	A				

Condition analysing rules:

$$\frac{[\Delta \cup \{\alpha\}]\ A}{[\Delta \cup \{\alpha_1, \alpha_2\}]\ A} \qquad \frac{[\Delta \cup \{\beta\}]\ A}{[\Delta \cup \{\beta_1\}]\ A \quad [\Delta \cup \{\beta_2\}]\ A}$$

Example:

$$\frac{[\Delta \cup \{q \wedge r\}]\ p}{[\Delta \cup \{q, r\}]\ p} \qquad \frac{[\Delta \cup \{q \vee r\}]\ p}{[\Delta \cup \{q\}]\ p \quad [\Delta \cup \{r\}]\ p}$$

The permissions and obligations

positive part:

1. pp$(\boldsymbol{A}, \boldsymbol{A})$.
2. pp$(\boldsymbol{A}, \boldsymbol{\alpha})$ if pp$(\boldsymbol{A}, \boldsymbol{\alpha_1})$ or pp$(\boldsymbol{A}, \boldsymbol{\alpha_2})$.
3. pp$(\boldsymbol{A}, \boldsymbol{\beta})$ if pp$(\boldsymbol{A}, \boldsymbol{\beta_1})$ or pp$(\boldsymbol{A}, \boldsymbol{\beta_2})$.

A line with second element $[\boldsymbol{\Delta}]\,\boldsymbol{A}$ is marked as a dead end iff an element of $\boldsymbol{\Delta}$ is not a pp of any premise.

A line with second element $[\boldsymbol{\Delta}]\,\boldsymbol{A}$ is marked as a redundant iff
(i) $\boldsymbol{A} \in \boldsymbol{\Delta}$ (not the Goal line) or
(ii) a line with second element $[\boldsymbol{\Delta'}]\,\boldsymbol{A}$ occurs and $\boldsymbol{\Delta'} \subset \boldsymbol{\Delta}$.

more marks possible (e.g., inconsistent paths)

The target is the first formula in the condition of the last unmarked line.   (alternatives possible)

Phase 1:

– start with Goal rule

– apply FAR only to formula of line that has Prem-line in its path

– derive $[B_1, \ldots, B_n]\, A$ by FAR only if target is pp of A

– next, introduce a new premise A iff target is pp of A

– apply CAR only to target A after Prem and FAR are exhausted

– apply Trans only if Δ' is empty

Phase 2:

– only: new $[\Delta]\, G$ by EM, Trans or CAR from R-unmarked lines
next return to phase 1

2.5 Where went Ex Falso Quodlibet?

Let the logic defined by the procedure be $\mathbf{pCL}^{-}$

$p, \ {\sim}p \ \nvdash_{\mathbf{pCL}^{-}} \ q$

EFQ requires, for example:

EFQ To introduce $[{\sim}A]\, G$ for an $A \in \Gamma$.

This rule may be applied to every $A \in \Gamma$.

EFQ is an isolated, unnatural and *ad hoc* rule.

Where $\mathbf{pCL}$ is (propositional) $\mathbf{pCL}^{-}$ + EFQ:

$\boldsymbol{\Gamma} \vdash_{\mathbf{pCL}} \boldsymbol{A}$ iff $\boldsymbol{\Gamma} \vdash_{\mathbf{CL}} \boldsymbol{A}$

That is:

If $\boldsymbol{\Gamma} \vdash_{\mathbf{CL}} \boldsymbol{A}$, the procedure will lead to a proof of $\boldsymbol{A}$ from $\boldsymbol{\Gamma}$.

If $\boldsymbol{\Gamma} \nvdash_{\mathbf{CL}} \boldsymbol{A}$, the procedure will stop without $\boldsymbol{A}$ being derived.

2.6 Some properties of $\mathbf{CL}^{-}$

– natural explication of all sensible classical proofs
– EFQ is absent, whence isolated, and unnatural
– assigns same consequences as $\mathbf{CL}$ to consistent Γ
(the intended domain of application of $\mathbf{CL}$)
– derives a contradiction from all inconsistent Γ,
but not triviality (except in border cases)
$\Rightarrow$ assigns sensible consequence set to inconsistent Γ
– resulting consequence relation:
⋆ *characterized* by a semantics (and tableau method)
⋆ reflexive and monotonic
⋆ not transitive (even weak cut does not hold)
but transitive if restricted to consistent premise sets
⋆ in an interesting (specific) sense relevant
⋆ exactly the same theorems as $\mathbf{CL}$
⋆ adequate w.r.t. $\mathbf{CL}$-semantics if restricted to consistent Γ

Sensible

$$\boldsymbol{p \vee q,\ \sim p,\ \sim q\ \vdash\ p \wedge \sim p}$$

$$\boldsymbol{p \vee q,\ \sim p,\ \sim q\ \vdash\ q \wedge \sim q}$$

To derive Russell's paradox from Frege's set theory.

not sensible

$$\boldsymbol{p \vee q,\ \sim p,\ \sim q\ \vdash\ r \wedge \sim r}$$

To derive from Frege's set theory that the moon is and is not a blue cheese (or that $\wp(\emptyset) = \emptyset \wedge \wp(\emptyset) \neq \emptyset$).

In problem-solving processes, $\mathbf{CL}^{-}$ need to be applied.

A semantics (Suszko: every logic has a 2-valued semantics)

$v : \mathcal{W} \mapsto \{0,1\}$ is a partial function

1. if $v(A) \in \{0,1\}$ and sub(B, A), then $v(B), v(*B) \in \{0,1\}$
2. if $v(A \wedge B) = 1$ then $v(A) = 1$ and $v(B) = 1$.
3. if $v(A \wedge B) = 0$ then $v(A) = 0$ or $v(B) = 0$.
4. if $v(A \equiv B) = 1$ then $v(A \supset B) = 1$ and $v(B \supset A) = 1$.
5. if $v(A \equiv B) = 0$ then $v(A \supset B) = 0$ or $v(B \supset A) = 0$.
6. if $v(\sim(A \vee B)) = 1$ then $v(*A) = 1$ and $v(*B) = 1$.
7. if $v(\sim(A \vee B)) = 0$ then $v(*A) = 0$ or $v(*B) = 0$.
8. if $v(\sim(A \supset B)) = 1$ then $v(A) = 1$ and $v(*B) = 1$.
9. if $v(\sim(A \supset B)) = 0$ then $v(A) = 0$ or $v(*B) = 0$.
10. if $v(\sim\sim A) = 1$ then $v(A) = 1$.
11. if $v(\sim\sim A) = 0$ then $v(A) = 0$.
12. if $v(A \vee B) = 1$ then $v(*A) = 0$ or $v(B) = 1$.
13. if $v(A \vee B) = 1$ then $v(A) = 1$ or $v(*B) = 0$.
14. if $v(A \vee B) = 0$ then $v(A) = 0$ and $v(B) = 0$.
15. if $v(A \supset B) = 1$ then $v(A) = 0$ or $v(B) = 1$.
16. if $v(A \supset B) = 1$ then $v(*A) = 1$ or $v(*B) = 0$.
17. if $v(A \supset B) = 0$ then $v(*A) = 0$ and $v(B) = 0$.
18. if $v(\sim(A \wedge B)) = 1$ then $v(A) = 0$ or $v(*B) = 1$.
19. if $v(\sim(A \wedge B)) = 1$ then $v(*A) = 1$ or $v(B) = 0$.
20. if $v(\sim(A \wedge B)) = 0$ then $v(*A) = 0$ and $v(*B) = 0$.
21. if $v(\sim(A \equiv B)) = 1$ then $v((A \supset B)) = 0$ or $v(\sim(B \supset A)) = 1$.
22. if $v(\sim(A \equiv B)) = 1$ then $v(\sim(A \supset B)) = 1$ or $v((B \supset A)) = 0$.
23. if $v(\sim(A \equiv B)) = 0$ then $v(\sim(A \supset B)) = v(\sim(B \supset A)) = 0$.
24. if $v(A) = 0$ then $v(*A) = 1$.

Definition
$A_1, \ldots, A_n \vDash B$ (B is a semantic consequence of $A_1, \ldots, A_n$)
iff
all valuations that verify $A_1, \ldots, A_n$
and for which B is determined,
verify B.

That is:

Definition
$A_1, \ldots, A_n \vDash B$ (B is a semantic consequence of $A_1, \ldots, A_n$)
iff no valuation that verifies $A_1, \ldots, A_n$ falsifies B.

PM: three valued truth-functional semantics

Theorem
If $[A_1, \ldots, A_n]B$ is derived in a $\mathbf{pCL}^-$-proof for $\Gamma \vdash G$,
then $v(B) = 1$
whenever $v(A_1) = \ldots = v(A_n) = 1$ and $v(B) \in \{0, 1\}$.

Corollary
If G is derived in a $\mathbf{pCL}^-$-proof for $\Gamma \vdash G$,
then $\Gamma \vDash_{\mathbf{CL}^-} G$. (Soundness)

Theorem
If a prospective proof for $\Gamma \vdash G$ halts without G begin derived,
then $\Gamma \nvDash_{\mathbf{CL}^-} G$. (Completeness)

Note: Tableau method

$$\frac{\boldsymbol{T A \wedge B}}{\begin{array}{c}\boldsymbol{TA}\\ \boldsymbol{TB}\end{array}} \qquad\qquad \frac{\boldsymbol{T A \vee B}}{\begin{array}{c|c|c}\boldsymbol{F{*}A} & \boldsymbol{TA} & \boldsymbol{F{*}B}\\ & \boldsymbol{TB} & \end{array}}$$

etc. (read off from semantic clauses)

Non-logicians sometimes apply **CL** to inconsistent premises.

They consider EFQ (explosion) as a logicians trick.

Logicians know: **EFQ cannot be isolated in** $\mathbf{CL}$

avoiding EFQ requires avoiding:
– Addition or Disjunctive Syllogism
– A / $B \supset A$ or $\sim A \supset (B \wedge \sim B)$ / A
– A / $B \supset A$ or $A \supset (B \wedge \sim B)$ / $\sim A$ or $\sim\sim A$ / A
– etc.

However:
That EFQ cannot be isolated in $\mathbf{CL}$
depends on our view on logic (mere rules vs. procedures).

The upgrade to predicate logic (minus EFQ):

is straightforward

if the procedure stops (not for all Γ and A),

with A derived: then $\Gamma \vdash_{\mathbf{CL}^-} A$

with A not derived: then $\Gamma \nvdash_{\mathbf{CL}^-} A$

2.7 Afterthought

Hintikka:

distinction between rules and heuristics

comparison with game of chess

This is a mistake:

- heuristic reasoning leads to sensible proofs
- part of this reasoning can be pushed into the (object-language) proofs

Moreover: truth-in-a-model is a touchy matter: ▼

• given $\mathbf{CL}$-models one can distinguish valid consequences from sensible consequences

BUT:

• there is a semantics that is adequate for *sensible* reasoning in $\mathbf{CL}$

A is a $\mathbf{CL}^-$-consequence of Γ (no $\mathbf{CL}^-$-model of Γ falsifies A)
iff
A is a *sensible* $\mathbf{CL}$-consequence of Γ

in other words:

sensibility can be incorporated into truth