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Note added in proof, but not inserted by the editors or publisher:

A much nicer semantics is to be found in D. Batens, "Dialectical dynamics within formal logics", *Logique et Analyse* 114, 1986, pp. 161-173.

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VI
Dynamic Dialectical Logics*

D. Batens

1. Introduction

The logics I present in this paper have some quite unusual properties. When I enumerated them in a lecture, some logicians started laughing, apparently because they thought I was fooling the non-logicians among the audience. To whatever extent the properties of these logics clash with the traditional logician's prejudices, I claim that they are meaningful, useful, and even indispensable to reconstruct important deductive thought processes of a specific kind. I realize of course that I shall have to offer some arguments in order to motivate the reader not to drop out on me, and I shall indeed offer two distinct ones. My first argument will at best convince those who take dialectics to be meaningful and important: I shall show that dynamic dialectical logics capture some essential features of the by now traditional notion of dialectics. My second argument, however, is meant to convince all. I shall indeed consider a paradigm case for the application of these logics. Apart from its motivational use, this paradigm case is interesting because I arrived at the present logics by analysing it.

Before I try to convince you to buy dynamic dialectical logics, let me informally sketch their main features. They are *dynamic* in the following sense: in constructing proofs by means of these logics from certain sets of sentences we run into cases where (i) the rules of inference are modified in view of the sentences derived up to that point of time, i.e. at that stage of the proof, and (ii) certain sentences that are derivable and even derived at some point in time, are not derivable any more at some later point in time, and *vice versa*. Furthermore, I call these logics *dialectical* because their dynamics depends essentially on the occurrence of inconsistencies in the set of sentences that are derived at some point in time.

In recent years a large number of logics have been developed that are called paraconsistent or static dialectical logics. Paraconsistent logicians disagree about the usefulness of and motivation for some of these logics, but I shall not enter into this debate here. An essential property of such logics is that they do not sanction as correct such rules of inference as 'From A and ~A to infer B' or 'From A & ~A to infer B'.[1] As a consequence, paraconsistent logics capture some aspects of dialectics, viz. that inconsistency does not lead to triviality, that contradictions (and inconsistent theories) are not necessarily false, and that inconsistent sets of sentences may be handled in a sensible way by deductive logics. Dynamic dialectical logics share the aforementioned properties with paraconsistent logics, but in contradistinction to the latter they are dynamic in the sense explained above. As a consequence, the derivability of some sentence from some set of premisses cannot be defined without referring to a stage in the proof, which for simplicity's sake I shall often call a *time*. It will turn out, however, that it is possible to define a notion of *final derivability* with respect to dynamic logics; A is finally derivable from α if and only if it is possible to construct a proof of A from members of α in such a way that A will remain derivable at any stage in any continuation of this proof from α. It is of course strongly desirable that such a notion of final derivability may be defined and that at least a number of sets of sentences lead to a set of final consequences, i.e. to a set of sentences that are finally derivable from the set under consideration. If this were not so, the status of dynamic dialectical logics as inferential devices would be utterly questionable.

In view of the fact that my logics are dynamic but nevertheless lead to a set of final consequences of sets of premisses, they may also be described as *adaptive* logics: if we construct a proof from the set α in terms of a dynamic dialectical logic, the logic adapts itself to α until we reach a set of final consequences of α. Immediately connected to the adaptive character of dynamic dialectical logics is their property, quite peculiar for deductive logics, that A may be a final consequence of α without being a final consequence of $\alpha \cup \beta$. This conflicts with the widespread conviction which Massey, 1981, p. 490 formulates as follows: "... as everyone knows, valid [deductive] arguments remain valid no matter what other premisses are added." Once one realizes that a deductive logic may adapt itself to α and adapt itself to $\alpha \cup \beta$ differently because $\alpha \cup \beta$ differs from α, then it is not difficult to see why, as with so many other things "everyone knows", it is prejudice that is really at stake. Needless to say that static paraconsistent logics are not adaptive.

Another main distinction between static paraconsistent logics and dynamic dialectical logics is related to their semantic presuppositions. Paraconsistent logics give up the presupposition of the consistency of "the world" in general, or at least give up this presupposition except for certain categories of sentences, e.g., the conjunctive ones.[2] The semantic idea behind dynamic dialectical logics is radically different: the world is supposed to be consistent, except for those parts which need be inconsistent in order for the premisses to be true. More exactly, but still in non-technical terms: dynamic dialectical logics presuppose the consistent behaviour of all sentences, except for those that have to behave inconsistently, i.e. have to be true together with their negation, in order for the premisses to be true. The matter may be easily illustrated by means of the rule of disjunctive syllogism, the famous *bête noire* of relevant logicians: From $A \vee B$ and $\sim A$ to infer B. According to the classical propositional calculus (PC) this rule is correct, according to paraconsistent logics it is incorrect. Indeed, if both A and $\sim A$ are true, then both $A \vee B$ and $\sim A$ are true, even if B is false. Hence, B cannot be derived from $A \vee B$ and $\sim A$ according to static paraconsistent logics. According to dynamic dialectical logics this derivation is correct except if the premisses require that both A and $\sim A$ are true. In other words, if the premisses may be true without both A and $\sim A$ being true, then it will be supposed that one of them is false; under these conditions B cannot be false if both $A \vee B$ and $\sim A$ are true, and hence B is a consequence of $A \vee B$ and $\sim A$. The relation of this semantic idea to the adaptive character of dynamic dialectical logics is obvious.

The stand dynamic dialectical logics take with respect to, e.g., disjunctive syllogism, may be summarized as follows: if $A \vee B$ and $\sim A$ are derivable from some set α, then B is derivable from this set unless A is derivable from it. It is obvious that a certain circularity is involved here. In a subsequent section I return to this problem and show that this circularity may be fully eliminated by characterizing dynamic dialectical logics in terms of instructions for constructing proofs. I shall even show that the notion of a final consequence may be defined in a systematic and time independent way, and why this is desirable although seemingly paradoxical. I now get to the motivational bits.

It seems to me that I do justice to those who take the idea of a deductive dialectical thought process to be meaningful, by saying that such a process is mainly characterized by the following properties:

(i) Deductive dialectical thought processes contain inconsistencies: for some A, both A and $\sim A$ are derived from (or stated in) the premisses. It is essential that the premisses have not been changed in a relevant way between the time at which one member of the inconsistent pair is derived and the time at which the other is derived; if they had been changed, the inconsistency would be apparent only.

(ii) The occurrence of an inconsistency leads neither to the mere rejection of the premisses, nor to the derivability of all sentences from the premisses, nor to the mere ending of the deductive process.

(iii) The occurrence of an inconsistency constitutes a problem the solution of which leads to certain structural changes, to a structural dynamics.

(iv) This dynamics plays at the level of the rules of inference that may be applied at some time, at the level of the set of sentences that are to be taken as (still) derived at some time, and at the level of the set of sentences that are derivable at some time.

(v) This dynamics also leads to a change in the premisses: the inconsistency under consideration is not any more derivable, and the new set of premisses is essentially richer than the previous one.

If applied to inconsistent sets of premisses, static paraconsistent logics lead to proofs that display properties (i) and (ii), whereas dynamic dialectical logics lead to proofs that display properties (i)-(iv). I shall comment on (v) in a subsequent section. If applied to consistent sets of premisses, dynamic dialectical logics lead to standard PC-proofs, whereas static paraconsistent logics lead to proofs that are considerably poorer. This feature of dynamic dialectical logics is an advantage on which I return in the sequel.

Even readers who mistrust the notion of a deductive dialectical thought process will have to agree that the logical reconstruction of the following paradigm case is a sensible enterprise. Consider a theory $T = \langle \alpha, \text{PC} \rangle$, i.e. a couple the first element of which is a set of non-logical axioms and the second element of which is a logic, viz. the classical propositional calculus. The theorems of T are the sentences derivable from α by PC, i.e. the members of $Cn_{PC}(\alpha)$, the PC-consequence set of α. For the sake of simplicity we may suppose that PC is given as a set of rules of inference, but this is of minor importance with respect to my present point. From the fact that the second element of T is PC we know that the theory is or was meant and believed to be consistent. Suppose, however, we are able to derive some inconsistency from α, and hence we find out that any sentence is a theorem of T as it stands. The trouble is what we should do next. A first alternative is that we simply reject T because it is false and trivial, i.e. contains all sentences as theorems. If, however, T has an important function with respect to other theories or with respect to our knowledge of or action in some domain, then merely giving up T has disastrous consequences. Another alternative is that we replace PC, the second element of the theory, by some paraconsistent logic, and hence move to a theory which has the same non-logical axioms as T but a considerably poorer logical basis. This theory, however, will be awfully poor with respect to T—by which I do not only mean that it is not trivial, but also that it is much poorer than "what T was intended to be", much poorer than "T *except for the pernicious consequences of its inconsistency*". I realize that such expressions are vague, but shall show in the sequel that we can make them precise in terms of dynamic dialectical logics.

Given the failures of the two aforementioned alternatives, let us look for something better.[3] In doing so we should keep in mind that we have two problems. The first problem is that for the time being, i.e. as long as we have no decent alternative for T, we should find a way to keep on employing T in its full richness but avoiding the pitfalls of its triviality. The second problem is that we should look for a theory T′ that can replace T. T itself is heuristically important in this connection: we want a large number of theorems of T, viz. all "good" ones, to be theorems of T′. In order to make sense of this requirement we need to know which sentences are theorems of T. Alas, if we stick to PC, then *all* sentences are theorems of T, and if we replace PC by some paraconsistent logic, we end up with a set of theorems which is *too poor* for the reasons explained earlier. Summarizing the situation: with respect to both problems we need "T except for the pernicious consequences of its inconsistency", and neither PC nor any static paraconsistent logic is able to provide us with this.

The next section contains some information on static paraconsistent logics, which I shall need to articulate dynamic dialectical logics and to explain how I arrived at them. On several occasions I shall refer to my (1980b), but I did my utmost to make the present paper as self-contained as possible.

2. Regular paraconsistent extensional propositional logics (RPEPLs)

Dynamic dialectical logics of the kind presented in this paper are in a very specific sense intermediate between PC and some RPEPL, and are defined with reference to some RPEPL. Rather than repeating here the definitions from my (1980b), I shall clarify by means of an example and some hints when a logic is a RPEPL, and afterwards mention some technical results which I need for the articulation of dynamic dialectical logics. The semantics of the basic or minimal RPEPL, PI, is arrived at by simply dropping the consistency requirement from the PC-semantics. Validity and the semantic consequence relation are defined as usual with respect to the set of valuations that have the following properties:[4]

C0. $v: \mathbf{F} \rightarrow \{0, 1\}$ ($\mathbf{F}$ is the set of all wffs)
C1. $v(A \supset B) = 1$ iff $v(A) = 0$ or $v(B) = 1$
C2. $v(A \& B) = 1$ iff $v(A) = v(B) = 1$
C3. $v(A \vee B) = 1$ iff $v(A) = 1$ or $v(B) = 1$
C4. If $v(A) = 0$, then $v(\sim A) = 1$ (not conversely)

PI is well axiomatized in the sense of Anderson and Belnap (1975) by the following axiomatic system, which is sound and strongly complete with respect to the above semantics.

Axioms:

A 1. $p \supset (q \supset p)$
A 2. $(p \supset (q \supset r)) \supset ((p \supset q) \supset (p \supset r))$
A 3. $((p \supset q) \supset p) \supset p$
A 4. $(p \& q) \supset p$
A 5. $(p \& q) \supset q$
A 6. $p \supset (q \supset (p \& q))$
A 7. $p \supset (p \vee q)$
A 8. $q \supset (p \vee q)$
A 9. $(p \supset r) \supset ((q \supset r) \supset ((p \vee q) \supset r))$
A10. $(p \supset \sim p) \supset \sim p$

Rules: *modus ponens* and uniform substitution.

Notice that A10 is contextually equivalent to $p \vee \sim p$ as well as to $(p \supset q) \supset ((\sim p \supset q) \supset q)$. PI is the full positive fragment of PC to which A10 is added. By a RPEPL I mean a paraconsistent logic between PI (included) and PC (not included). It is provable that a logic between PI and PC is paraconsistent if and only if $p \supset (\sim p \supset q)$ is not a theorem. As the addition of the latter formula as an axiom to PI results in PC, all logics between PI and PC (not included) are RPEPLs. None of the following formulas is a theorem of any RPEPL: $(\sim p \vee q) \supset (p \supset q)$; $((p \vee q) \& \sim p) \supset q$; $((\sim p \supset q) \& \sim q) \supset p$; $(\sim p \supset q) \supset (\sim q \supset p)$; $(\sim p \supset q) \supset ((\sim p \supset \sim q) \supset p)$.

On the other hand each of the following PI-non-theorems is a theorem of some RPEPL: $\sim(p \& \sim p)$; $\sim\sim(p \vee \sim p)$; $\sim p \supset (\sim\sim p \supset q)$; $(p \& q) \supset (\sim(p \& q) \supset r)$; $\sim\sim p \supset p$; $\sim(p \supset q) \supset (p \& \sim q)$; $\sim(p \& q) \supset (\sim p \vee \sim q)$; $\sim(p \vee q) \supset (\sim p \& \sim q)$; and the converses of the last four formulas. Needless to say that none of these two lists is exhaustive. In my (1980a) I showed that the transition from an axiomatic formulation of a RPEPL to its semantic formulation, and *vice versa*, is straightforward.

Some RPEPLs, e.g., the one arrived at by adding $\sim p \supset (\sim\sim p \supset q)$ to PI, are not *strictly paraconsistent*. Although these are paraconsistent (*simpliciter*), some inconsistencies lead to triviality, i.e. anything may be derived from inconsistencies of certain forms. Also, some RPEPLs are *maximally paraconsistent* in that they have no proper extension which is paraconsistent; in other words, there are no logics between them and PC. Some, but not all, maximally paraconsistent logics are strictly paraconsistent. For all RPEPL I stipulate that the syntactic consequence relation is determined by:

(1) $B_1, \ldots, B_n \vdash A$ iff $(B_1 \& \ldots \& B_n) \supset A$ is a theorem.

Traditionalists held the position that dialectics is nonsense because anything is derivable from an inconsistency. Naive paraconsistent logicians might hold the position that dialectics is nonsense because inconsistencies do not constitute a problem from a logical point of view. It may be demonstrated, however, that anytime we are able to replace an inconsistent but nontrivial theory T by a consistent theory T′ in such a way that any theorem of T corresponds by some translation relation to a theorem of T′, then we made a net gain in that T′ will be richer, e.g., with respect to its conceptual system, than T.

I now prove two theorems that are important for the articulation of dynamic dialectical logics.

Theorem 1. For any RPEPL PIz, if $\vdash_{PIz}(C_1 \& \sim C_1) \vee \ldots \vee (C_m \& \sim C_m) \vee A$, then $\vdash_{PC}A$.

Proof. Consider an extension of the PI-semantics that is adequate for PIz. (It is obvious that there is one in view of my [1980a].) All PC-valuations are PIz-valuations, and hence $(C_1 \& \sim C_1) \vee \ldots \vee (C_m \& \sim C_m) \vee A$ is PC-valid. But then A is PC-valid and hence a PC-theorem. □

Theorem 2. For any RPEPL PIz, if $\vdash_{PC} A$, then there is a finite number of formulas $C_1, \ldots, C_n$ such that $\vdash_{PIz}(C_1 \& \sim C_1) \vee \ldots \vee (C_m \& \sim C_m) \vee A$.

Proof. As for PC, the value assigned to A by some PI-valuation depends only on the values assigned by this valuation to subformulas of A. Let $\{C_1, \ldots, C_m\}$ be the (obviously finite) set of all C_i for which $\sim C_i$ is a subformula of A. If some PI-valuation v assigns the value 0 to A, then there are C_i such that $v(C_i) = v(\sim C_i) = 1$, as may easily be seen from the PI-semantics. Hence $v((C_1 \& \sim C_1) \vee \ldots \vee (C_k \& \sim C_k) \vee A) = 1$. It follows that this wff is PI-valid and hence a PI-theorem. As all RPEPL are PI-extensions, the wff is also PIz-valid. □

Corollary 1. For all RPEPL PIz, $\vdash_{PC} A$ iff, for some $C_1, \ldots, C_m$, $\vdash_{PIz}(C_1 \& \sim C_1) \vee \ldots \vee (C_m \& \sim C_m) \vee A$.

This corollary deserves careful attention because it suggests the central idea behind the articulation of dynamic dialectical logics. We may paraphrase it as follows: if A is PC-valid, then, according to any RPEPL, either one of a finite number of formulas behaves inconsistently, i.e. is true together with its negation, or else A is true. E.g., although (2) is not PI-valid, (3) is PI-valid.

(2) $((p \vee q) \& \sim p) \supset q$
(3) $(p \& \sim p) \vee (((p \vee q) \& \sim p) \supset q)$

which reads: either p behaves inconsistently, or else $(p \vee q) \& \sim p$ implies q. Notice that (3) is not equivalent to (4) in PI.

(4) $\sim(p \& \sim p) \supset (((p \vee q) \& \sim p) \supset q)$

Indeed, (4) is not PI-valid, which may be easily seen by assigning the value 1 to p, $\sim p$ and $\sim(p \& \sim p)$ and the value 0 to q. In other words, that p behaves *inconsistently* is correctly expressed by saying that $p \& \sim p$ is *true*, and that p behaves *consistently* is correctly expressed by saying that $p \& \sim p$ is *false*; but the latter is by no means guaranteed by the fact that $\sim(p \& \sim p)$ is true: $p \& \sim p$ and $\sim(p \& \sim p)$ may be true together. Notice also that not every formula the negation of which is a subformula of A, should be turned into a contradictory disjunct. Not only (5) but also (6) and (7) are PI-valid.

(5) $(p \& \sim p) \vee (q \& \sim q) \vee (r \& \sim r) \vee ((r \supset (\sim p \& \sim q)) \supset ((p \& q) \supset \sim r))$
(6) $(p \& \sim p) \vee ((r \supset (\sim p \& \sim q)) \supset ((p \& q) \supset \sim r))$
(7) $(q \& \sim q) \vee ((r \supset (\sim p \& \sim q)) \supset ((p \& q) \supset \sim r))$

The fact that both (6) and (7) are PI-valid shows that the second disjunct of (6) is true in case p behaves consistently, but also in case q behaves consistently.

Let us return to the validity of (3): either p behaves inconsistently, or else q is true in case both $p \vee q$ and $\sim p$ are true. In other words, whenever p behaves consistently, q is derivable from $p \vee q$ and $\sim p$. To consider the general case, suppose that (8) is PI-valid.

(8) $(C_1 \& \sim C_1) \vee \ldots \vee (C_m \& \sim C_m) \vee ((B_1 \& \ldots \& B_n) \supset A)$

If each C_i behaves consistently, then the first m disjuncts are false and hence the last disjunct is true. In other words, whenever all C_i behave consistently, A is derivable from $B_1, \ldots, B_n$. In view of these results it seems reasonable to say that any RPEPL determines two kinds of rules of inference, viz. unconditional ones such as (9) and conditional ones such as (10).

(9) From $A \supset B$ and A to infer B.
(10) Given that A behaves consistently, from $A \vee B$ and $\sim A$ to infer B.

In some RPEPLs the fact that A behaves consistently cannot be expressed syntactically, i.e. within the formal system. As a consequence, conditional rules such as (10) cannot be applied in proofs based upon such RPEPLs. In other RPEPLs the fact that A behaves consistently can be expressed syntactically, but it is then provable that whatever may be derived by means of a conditional rule may also be derived by means of some unconditional rule. In other words, conditional rules of inference are either useless or superfluous with respect to proofs based on some RPEPL. However, as I shall show in the next section, the fact that an RPEPL determines conditional rules of inference enables us to use it as a basis for the articulation of a dynamic dialectical logic.

3. Enters dynamics

As I explained in section 1, dynamic dialectical logics should enable us to find, for any inconsistent theory $T = \langle \alpha, \text{PC} \rangle$, "T except for the pernicious consequences of its inconsistency". The fact that any RPEPL determines conditional rules of inference suggests that we might tackle the problem by the following convention:

(11) A sentence behaves consistently according to some theory (or some set α of premisses) if and only if either the sentence itself or its negation is not derivable from α.

Whenever (8) is valid, we might formulate the corresponding conditional rule of inference as follows:

(12) Given that, for each C_i, either C_i or $\sim C_i$ is not derivable from the set of premisses under consideration, from $B_1, \ldots, B_n$ to infer A.

Although (12) is intuitively appealing, it does not tell us how to define a dynamic dialectical logic from, say, PI. All it says is that the dynamic dialectical logic based on PI should enable us to infer certain sentences from a set of premisses only in case certain other sentences are not derivable from the premisses. In order to define a dynamic dialectical logic we need a way to circumvent this circularity.

A well-tried procedure to avoid circularity is to come down from the heaven of systematic and abstract definition to the earthly level of concrete actions. As far as logic is concerned this means that we should concentrate on actual proofs instead of on the notion of derivability. The construction of proofs is determined by instructions, viz. commands and permissions: if you have written down this and that, then (you may) do so and so. It is obvious, however, that (11) is altogether inappropriate to formulate instructions for proof construction, if these instructions themselves are to be based upon concrete matters such as the lines of which the proof consists at some time. Looking for an appropriate substitute for (11) we gain some hints again by keeping in mind the object we pursue: T, "in its full richness", "except for the pernicious consequences of its inconsistency". We want all rules of inference that are validated by PC to apply, *except* in those cases where they lead to triviality. In other words, we *suppose* that the theory (or set of premisses) is consistent, *except for* those sentences the inconsistency of which is unavoidable. This suggests that we replace (11) by:

(13) A sentence behaves consistently unless and until proved otherwise.

As a consequence, whenever (8) is valid, we get the following corresponding conditional rule of inference:

(14) Unless both some C_i and its negation have actually been derived, from $B_1, \ldots, B_n$ to infer A.

This dramatic move from the syntactic and semantic level to the pragmatic level eliminates all circularity.

The reference to time in the term 'until' is essential. Suppose p has not been proved at some time (stage in the proof) and hence behaves consistently at that time; suppose furthermore that, given the consistent behaviour of p, we derive q from $p \vee q$ and $\sim p$. Yet it is possible that we later derive p, e.g., from r and $r \supset p$. From this time on, q is not any more derivable from $p \vee q$ and $\sim p$. A very simple example is displayed in the following proof:

(*1*) $p \vee q$ premiss
(*2*) $\sim p$ premiss
(*3*) $r \supset p$ premiss
(*4*) r premiss
(*5*) q from (*1*) and (*2*) and the consistent behaviour of p
(*6*) p from (*3*) and (*4*)

At time (*4*), viz. after line (*4*) has been written and before line (*5*) has been written, q is derivable from the premisses; at time (*6*) q is not any more derivable, because p behaves inconsistently. Obviously, it cannot merely depend on the accidental way in which we construct a proof, whether or not some sentence belongs to "T except for the pernicious consequences of its inconsistency". Hence, from time (*6*) on, we should consider q as not any more derivable, and we should keep in mind not to use it for further inferential steps. To assist our memory, we might *delete* line (*4*) after line (*6*) has been written, or even add 'deleted at time (*6*)'. In the sequel I shall say that I delete some line, but, to avoid a mess for the printer, I shall put double square brackets around the elements of the line.

Some readers probably still wonder whether something sensible is going to come out. I beg their patience. I shall prove some nice properties of the logics I am at the point of articulating, but first I have two further clarifications. The first concerns speech. The "times" I need are merely members of an ordered series of intervals, each of which starts "at the moment" some line in the proof has been written down, and is named after the line number of this line. In the above proof line (*4*) should be deleted at time (*6*). The second clarification concerns notation. The lines of a standard (explicit) proof consist of four elements:

(i) a line number
(ii) the sentence (or formula) derived
(iii) the line numbers of the sentences from which (ii) is derived, and
(iv) the rule of inference that justifies the derivation

In dynamic dialectical logic proofs it is preferable to add a fifth element to each line:

(v) the sentences that have to behave consistently in order for (ii) to be derivable *by* (iv) *from* the second elements of the lines enumerated in (iii).

By adding (v) it will be easy to detect at any time which lines have to be deleted. For simplicity's sake I shall talk about the third and fifth elements of a line as about *sets*; this will enable me to say, e.g., that the intersection of two fifth elements is empty. So much for preparation. Let's move ahead to the first dynamic dialectical logic.

4. The dynamic dialectical logic DPI*

This logic is based on PI, whence its name—the reason for the superscripted star—is explained later. Its instructions read as follows:

I1* If, at some time, both A and $\sim$A occur as the second element of a line, then all lines the fifth element of which contains A should be deleted.

I2 At any time you may write down a line consisting of (i) an appropriate line number, (ii) a premiss, (iii) 'premiss', (iv) a dash, (v) '$\emptyset$'.

I3 If $(B_1 \,\&\, \ldots \,\&\, B_n) \supset A$ is PI-valid and if each B_i occurs as the second element of some line, then one may write down a line consisting of (i) an appropriate line number, (ii) A, (iii) for each B_i, the line number of a line at which it occurs as the second element, (vi) '$B_1, \ldots, B_n/A$', and (v) the union of the fifth elements of all lines listed in (iii).[5]

I4* If $(C_1 \,\&\, \sim C_1) \vee \ldots \vee (C_m \,\&\, \sim C_m) \vee ((B_1 \,\&\, \ldots \,\&\, B_n) \supset A)$ is PI-valid, and if each B_i occurs as the second element of a line whereas, for each C_i, either C_i or $\sim C_i$ does not occur as the second element of a line, then one may write down a line consisting of (i) an appropriate line number, (ii) A, (iii) for each B_i, the line number of a line at which it occurs as the second element, (iv) '$B_1, \ldots, B_n/A$', and (v) the union of $\{C_1, \ldots, C_m\}$ and the fifth elements of all lines listed in (ii).[6]

The specification of the fifth element in I3 and I4 may cause some amazement on the part of the reader, which will be removed by the following example.

(1)	$\sim p \,\&\, r$	premiss	—	$\emptyset$
(2)	$q \supset p$	premiss	—	$\emptyset$
(3)	$q \vee \sim r$	premiss	—	$\emptyset$
(4)	$r \supset p$	premiss	—	$\emptyset$
(5)	$\sim p$	*(1)*	$A \,\&\, B/A$	$\emptyset$
(6)	r	*(1)*	$A \,\&\, B/B$	$\emptyset$
⟦*(7)*	$\sim q$	*(2)*, *(5)*	$A \supset B, \sim B/\sim A$	p⟧ deleted at time *(9)*
⟦*(8)*	$\sim r$	*(3)*, *(7)*	$A \vee B, \sim A/B$	p, q⟧ deleted at time *(9)*
(9)	p	*(4)*, *(6)*	$A \supset B, A/B$	$\emptyset$
(10)	q	*(3)*, *(6)*	$A \vee \sim B, B/A$	r

Both *(5)* and *(6)* are derivable unconditionally from *(1)*, whence the fifth element of these lines is empty. *(7)* is derivable from *(2)* and *(5)* because p behaves consistently at time *(6)*. *(8)* is derivable from *(3)* and *(7)* because q behaves consistently at time *(7)*. *However*, as $\sim$q was only derivable because p behaved consistently—see line *(7)*—we should add p in the fifth element of line *(8)*; and indeed I4* forces us to do exactly so. If p had not behaved consistently, then we would not have been able to derive $\sim$r in the way we did. At time *(9)* p does *not* behave consistently any more, and hence I1* forces us to delete both line *(7)* and line *(8)*.

This proof illustrates quite nicely the dynamic character of DPI*. From time *(2)* to *(8)* $\sim$q is derivable, and at times *(7)* and *(8)* it is derived. From time (*9*) on it is not derived any more, and not derivable either. On the other hand, q is not derivable from (*3*) and (*6*) at time (*8*), because r behaves inconsistently at that time. But at time (*9*) r behaves consistently again, and consequently q is derivable. Hence, the dynamics with respect to the sentences that are *derived* at some time and with respect to the sentences that are *derivable* at some time actually occurs. The dynamics with respect to the rules of inference may be illustrated as follows: up to time (*8*) the rule of *modus tollens* may be applied to $q \supset p$ and $\sim p$ in order to infer $\sim q$. From time (*9*) on, however, this rule becomes restricted and cannot any more be applied to those formulas. In general, the dynamics with respect to the rules concerns their range of application. This range may change several times as the proof proceeds, and depends on the inconsistencies that are (still) derived at some time. It is obvious that DPI* is a dynamic dialectical logic in the sense explained in section 1. It is easily demonstrated that the set of DPI*-consequences of some set α is generally richer than the PI-consequence set of α. E.g., q is not a PI-consequence of the premisses (*1*)-(*4*). If α is consistent, all PC-consequences of α are DPI*-consequences of α and *vice versa*. If α is inconsistent, its set of DPI*-consequences does not contain all sentences and hence is poorer than its set of PC-consequences.

The reader might wonder what might happen if we were to continue the proof after line (*10*). The answer is that nothing worth mentioning will happen. Indeed, no line that is not deleted in the present proof will be deleted in any of its extensions, and no further atoms (propositional variables and their negations) will be derived. The only moves that are still possible are applications of such rules as conjunction, addition, commutativity, and likewise uninteresting stuff. In view of the fact that PI validates the rule of irrelevance, viz. $A/B \supset A$, as well as addition, the DPI*-consequence set of premisses (*1*)-(*4*) is identical to the PI-consequence set of $p \,\&\, \sim p \,\&\, q \,\&\, r$. I do not prove all this, because I shall show that DPI* has a serious drawback, which is a very decent reason to replace it by the improvement DPI. Before doing so, however, I stress again the dynamic character of DPI*: *no static logic leads to the same set of consequences from the premisses* (*1*)-(*4*). Indeed, in the preceding proof q is derived from $q \vee \sim r$ and r, and will not be deleted in any extension of this proof; s to the contrary is not derivable in any extension of this proof, although both $s \vee \sim p$ and p are derivable in it.

As I announced, DPI* has a serious drawback. Consider the set of premisses $\{\sim p, \sim q, p \vee q\}$ and the following proof:

(*1*)	$\sim p$	premiss	—	$\emptyset$
(*2*)	$\sim q$	premiss	—	$\emptyset$
(*3*)	$p \vee q$	premiss	—	$\emptyset$
(*4*)	p	(*2*), (*3*)	$A \vee B, \sim B/A$	q

As p behaves inconsistently at time (*4*), it is obvious that it will not be possible to derive q in any extension of this proof, and hence it is obvious also that line (*4*) will not be deleted in any extension of this proof. However, compare the preceding proof with the one that contains (*1*)-(*3*) as its first three lines, but as its fourth line:

(*4*) q (*1*), (*3*) $A \vee B, \sim A/B$ p

Here q behaves inconsistently at time (*4*) and ever thereafter in any extension of *this* proof, whereas p is not derivable in any such extension. By comparing the two proofs, we find out that it depends merely on the accidental way in which we start off the proof whether p is derivable and q is not, or the other way around. This situation is most unpalatable. If you agree, skip the next paragraph, if you take the situation to be palatable or, even worse, desirable, let me try to convince you.

Some people who consider dialectical deductive thought processes interesting might argue—some did orally to me—that this situation, this indeterminacy, displays the creative aspects of dialectics. It seems to me, however, that sheer accident should not be confused with creativity. No doubt, if you have some good reason to prefer the derivation of p to the derivation of q, and if the logic tells you whenever you have the choice, then it is quite all right to derive p. Alas, none of these conditions applies. DPI* as it stands does not tell you when you have the choice. Furthermore, the good reasons you might have to prefer to derive p (and hence its inconsistency) instead of q, are by no means *logical* reasons connected with DPI*; as far as DPI* is concerned the choice for either of the two proofs is equally unjustifiable. Consequently, the argument from dialectic creativity does not hold water. From the more general point of view, which is that DPI* should provide us, for any $T = \langle \alpha, \text{PC} \rangle$, with "T except for the pernicious consequences of its inconsistency", the situation is even worse. From the premisses $\sim p$, $\sim q$ and $p \vee q$ we may arrive at two different results, from other sets of premisses we may arrive at two hundred different results. Indeterminacy might be acceptable if at least we were guaranteed an overview of the alternatives; but DPI* does not even provide us with such an overview. Whence we should look for something better.

5. The (decent) dynamic dialectical logic DPI

It turns out that the indeterminacy to which DPI* leads is very simple in nature. Suppose that a line of the following form occurs as the last line in a DPI*-proof:

(*i*) A (line numbers) '$D_1, \ldots, D_n, B/A$' $C_1, \ldots, C_m, B$

Suppose furthermore that line (i) has been written by application of I4* and in view of the PI-validity of (15).

$$(15)\quad (C_1 \mathbin{\&} \sim C_1) \vee \ldots \vee (C_m \mathbin{\&} \sim C_m) \vee (B \mathbin{\&} \sim B) \vee ((D_1 \mathbin{\&} \ldots \mathbin{\&} D_n \mathbin{\&} B) \supset A)$$

As (15) is valid, so is (16). This may easily be seen by applying the oblique method: proceed as for PC but remember that A and $\sim$A may be both true, although not both false.

$$(16)\quad (C_1 \mathbin{\&} \sim C_1) \vee \ldots \vee (C_m \mathbin{\&} \sim C_m) \vee (A \mathbin{\&} \sim A) \vee ((D_1 \mathbin{\&} \ldots \mathbin{\&} D_m \mathbin{\&} \sim A) \supset \sim B)$$

Suppose finally that $\sim$A occurred already as the second element of a line in the proof. Hence, the following line might have been written down instead of the preceding line (i):

(i) $\sim$B (line numbers) '$D_1, \ldots, D_n, \sim A/\sim B$' $C_1, \ldots, C_m$, A

In the original proof we established at time (i) the *inconsistent* behaviour of A by relying on the *consistent* behaviour of B. If we replace the original line (i) by the above second line (i), then at time (i) we establish the *inconsistent* behaviour of B by relying on the *consistent* behaviour of A. Notice indeed that B did already occur as the second element of a line, as it was used to derive A at the original line (i). By considering all cases where the inconsistent behaviour of some sentence may be established in a DPI*-proof by relying on the consistent behaviour of some other sentences, one readily demonstrates the following:

Theorem 3. If at some time in a DPI*-proof the inconsistency of A is proved by relying on the consistent behaviour of B (and possibly of some other sentences), then the inconsistency of B might have been proved by relying on the consistent behaviour of A (and possibly of some other sentences).

The situation may be described as follows: in view of the premisses certain sentences are connected with respect to their consistency; for each such sentence there is a DPI*-proof in which it behaves inconsistently and in which the other sentences (that are so connected) behave consistently. The diagnosis of the trouble is that *the premisses do not provide sufficient information to decide which of the sentences behaves inconsistently and which consistently.* Reconsider the previous example with premisses $\sim$p, $\sim$q and p $\vee$ q. As either p or q is true, one of them behaves inconsistently, but the premisses do not provide enough information to decide which of the two behaves inconsistently (nor, obviously, to decide that both behave inconsistently). So, a first remedy is straightforward: *prevent the derivation of* p *as well as the derivation of* q. Alas, this is not the end of our worries. As a consequence of this move, both p and q behave consistently, and this is not quite all right either. Indeed, $(r \vee p) \vee q$ is (unconditionally) derivable from $p \vee q$, and in view of the consistent behaviour of p and q, r is derivable from $(r \vee p) \vee q$, $\sim p$ and $\sim q$. Hence, anything is derivable from $p \vee q$, $\sim p$ and $\sim q$, which is simply a restricted but still unacceptable form of *ex falso quodlibet.* As the premisses do state that either p or q behaves inconsistently, to take both as behaving consistently is mistaken and abortive. The problem we are facing is then to define a set of instructions which, in case some sentences are connected with respect to their consistency, prevent us from deriving the inconsistency of one of them, but at the same time prevent us from relying on the consistent behaviour of one of them in deriving some other sentence. At first sight the derivability of (17) might be taken as the expression of the fact that $A_1, \ldots, A_n$ are connected with respect to their consistency.

(17) $(A_1 \& \sim A_1) \vee \ldots \vee (A_n \& \sim A_n)$

However, it is obvious that (18) is derivable whenever (17) is, even if the consistency of B is not in any sense related to the inconsistency of $A_1, \ldots, A_n$.

(18) $(A_1 \& \sim A_1) \vee \ldots \vee (A_n \& \sim A_n) \vee (B \& \sim B)$

On the other hand, whenever (18) is derivable by relying on the consistent behaviour of certain sentences, then (17) is derivable by relying on the consistent behaviour of B and of those other sentences. This suggests that we take (18) to be the expression of the fact that $B, A_1, \ldots, A_n$ are connected with respect to their consistency, *unless* (17) is derivable *without* relying on the consistent behaviour of B.

In order to avoid the indeterminacy of DPI*-proofs, we have to modify both I1* and I4*. Instead of modifying both instructions in view of the above (somewhat complicated) criterion, I shall replace I1* by an instruction which does not specifically refer to the difference between DPI* and DPI:

I1 If, at some time, a line that occurs in the proof cannot be repeated, with only its first element adjusted, as the next line in the proof, then delete that line.

In order to simplify the formulation of I4 I stipulate:

Definition. $\text{DK}(\alpha)$ is the set of all formulas $(A_1 \& \sim A_1) \vee \ldots \vee (A_k \& \sim A_k)$ such that $\{A_1, \ldots, A_k\} \subseteq \alpha$ (hence $\text{DK}(\emptyset) = \emptyset$).

I4 If $(C_1 \& \sim C_1) \vee \ldots \vee (C_m \& \sim C_m) \vee ((B_1 \& \ldots \& B_n) \supset A)$ is PI-valid, if each B_i occurs as the second element of some line, and if, for each C_i and, for all $D_1, \ldots, D_k$ $(0 \leq k)$ either $(D_1 \& \sim D_1) \vee \ldots \vee (D_k \& \sim D_k) \vee (C_i \& \sim C_i)$ does not occur as the second element of a line the fifth element of which is empty, or else some member of $\text{DK}(D_1, \ldots, D_k)$ occurs as the second element of a line the fifth element of which is empty, then one may write down a line consisting of (i) an appropriate line number, (ii) A, (iii) for each B_i, the line number of some line at which B_i occurs as the second element, (iv) '$B_1, \ldots, B_n/A$', and (v) the union of $\{C_1, \ldots, C_m\}$ and the fifth elements of the lines listed in (iii).

DPI-proofs are constructed by application of I1-I4. An inessential difference between DPI* and DPI is that the inconsistent behaviour of some sentence A was taken to be expressed by the occurrence of both A and $\sim A$ as the second element of a line in the former system, whereas it is taken to be expressed by the occurrence of $A \& \sim A$ as the second element of a line in the latter system. This difference is inessential both from a systematic point of view and from the practical point of view of proof construction: one might fail to see an "obvious" inconsistency in both systems. Notice, however, that I4 might be rephrased, albeit in a more complex way, without referring to formulas that contain conjunctions; to do so will be necessary when dealing with conjunctionless fragments of DPI. I4 cannot be rephrased without referring to formulas that contain disjunctions. This, however, is not objectionable because in disjunctionless fragments of DPI no sentences will ever be connected with respect to their consistency (implications cannot lead to such connection because *modus ponens* is unconditionally correct); disjunctionless fragments of DPI are equivalent to the corresponding fragments of DPI*. Notice, finally, that I4 does never allow one to write down a line if, for some C_i, $C_i \& \sim C_i$ occurs as the second element of a line the fifth element of which is empty. In order to write such a line under these conditions, some other line should contain nothing as its second element, which is impossible.

Let us return for a moment to the two proofs discussed at the end of section 4. As soon as we realize that $(p \& \sim p) \vee (q \& \sim q)$ is unconditionally derivable from the premisses, we shall write down this formula as the second element of a line the fifth element of which is empty. As soon as such a line is added to either proof, line (*4*) is deleted by application of I1, which is exactly what we want. It is also instructive to consider some more complicated sets of premisses. In order to save space I do not list any more proofs, but I counsel the reader to write some out. Consider, e.g., the set of premisses $\{p, \sim p \vee q, \sim q \vee r, \sim r\}$. As soon as we realize that p, q and r are connected with respect to their consistency, by deriving $(p \& \sim p) \vee (q \& \sim q) \vee (r \& \sim r)$ unconditionally we shall have to delete all lines the fifth element of which contains r, for $(p \& \sim p) \vee (q \& \sim q)$ cannot be derived unless by relying on the consistent behaviour of r; analogously for p and q. Next, consider the union of the above set of premisses with, e.g., $\{s, s \supset \sim p\}$. In this case p is given as a premiss and $\sim p$ is derivable without relying on the consistent behaviour of either r or q, viz. from the two added premisses. Hence, we may write a line the second element of which is $p \& \sim p$ and the fifth element of which is empty. From this we see that p, q and r are *not* connected with respect to their consistency on these premisses. After p & ~p has been written down, q will not any more be derivable and *a fortiori* r will not any more be derivable. Hence r behaves consistently from this time on, and by relying on this we may derive ~q from ~q ∨ r and ~r. This is exactly as it should be: the latter set of premisses cannot be true unless p & ~p is true, whereas no further contradiction need be true in order for the premisses to be true.

In I4 there are two references to lines the fifth element of which is empty. That the latter requirement is "harmless" may be understood from the aforementioned relation between (17) and (18). If, however, we were to reformulate I4 so as not to refer to the emptiness of the fifth element of certain lines, we would arrive at awful complications. It is indeed possible that p, q and r are connected with respect to their consistency, whereas also p, s and u are connected with respect to their consistency. This is the case, e.g., on the following set of premisses: {p, ~q, ~u, ~p ∨ q, ~q ∨ r, ~p ∨ s, ~s ∨ u}. (p & ~p) ∨ (q & ~q) ∨ (r & ~r) is derivable unconditionally from these premisses, but at the same time p & ~p is derivable from them by relying on the consistent behaviour of s and u, and hence by not relying on the consistent behaviour of either q or r.

A possible objection to DPI might be that we might neglect to derive some sentence A. It is then possible that certain sentences that would have been derivable if A were derived, are not derivable; and that certain sentences that would not have been derivable if A had been derived, are derivable. All this is correct but not an objection to DPI. The instructions do indeed not prevent us from writing out proofs which are uninteresting, redundant, messy, or stupid. But here DPI is on a par with static logics. It is also true that the instructions as such do not constitute an algorithm for 'A is derivable from α' or 'A is finally derivable from α'. However, they were never intended to constitute such algorithms. Furthermore, such algorithms do exist, as is shown in the next section.

6. Some metatheory

In preparation for the proof of some interesting properties of DPI I list some definitions.

Definition. A DPI-*proof from* α is a finite ordered series of lines arrived at by consecutive applications of I1-I4, with I2 restricted to members of α, and with I1 applied after the last line has been written.

It need not be remembered that deleted lines do not belong to a proof.

Definition. $A_1, \ldots, A_k$ are *simultaneously derived in a* DPI-*proof from* α iff each A_i occurs as the second element of a line in that proof.

Analogously for 'A is *derived in a* DPI-*proof from* α'.

Definition. $A_1, \ldots, A_k$ are *simultaneously* DPI-*derivable from* α iff there is a DPI-proof from α in which they are simultaneously derived.

Analogously for 'A is DPI-*derivable from* α'.

Definition. A is *finally* derived at some line in a* DPI-*proof from* α iff it occurs as the second element of this line and this line will not be deleted in any extension of the proof.

Definition. $\alpha \vdash^{*}_{\mathrm{DPI}} A$, A is *finally** DPI-*derivable from* α, iff there is a DPI-proof from α in which A is finally derived at some line.

The terms 'finally derived' and 'finally derivable' (without asterisk) will be defined later.

Theorem 4. PI is decidable.

Proof. See my (1980b). A simple truth-tabular method as well as an oblique method derives from the PI-semantics; mainly remember that both A and $\sim A$ may receive the value 1, and that at least one of them should; all other connectives behave as for PC.

Lemma 1. If in a DPI-proof from α A occurs as the second element and $C_1, \ldots, C_m$ $(0 \leq m)$ occurs as the fifth element of line (i), then $\alpha \vdash_{\mathrm{PI}} A \vee (C_1 \,\&\, \sim C_1) \vee \ldots \vee (C_m \,\&\, \sim C_m)$.

Proof is by induction. The lemma obviously holds if (i) is the first line in the proof, for then either $A \in \alpha$ or $\vdash_{\mathrm{PI}} A \vee (C_1 \,\&\, \sim C_1) \vee \ldots \vee (C_m \,\&\, \sim C_m)$. Suppose the lemma holds for all lines that precede (i).

Case 1: the third element of line (i) is 'premiss'; then $m = 0$, $A \in \alpha$, and hence $\alpha \vdash_{\mathrm{PI}} A$.

Case 2: line (i) has been written by application of I3 or I4 and its third element is empty in view of the fact that A is a PC-theorem (if A is also a PI-theorem, then possibly $m = 0$); consequently, $a \vdash_{\mathrm{PI}} A \vee (C_1 \,\&\, \sim C_1) \vee \ldots \vee (C_m \,\&\, \sim C_m)$.

Case 3: line (i) has been written by application of I3 or I4, its third element is, say, $(j_1), \ldots, (j_n)$, where $n > 0$ and the second elements of lines $(j_1), \ldots, (j_n)$ are respectively $B_1, \ldots, B_n$. But then $\vdash_{\mathrm{PI}} (C_1 \,\&\, \sim C_1) \vee \ldots \vee (C_m \,\&\, \sim C_m) \vee ((B_1 \,\&\, \ldots \,\&\, B_n) \supset A)$. Furthermore, as the union of the fifth elements of lines $(j_1), \ldots, (j_n)$ is a subset of $\{C_1, \ldots, C_m\}$, it follows from the supposition that $\alpha \vdash_{\mathrm{PI}} (B_1 \,\&\, \ldots \,\&\, B_n) \vee (C_1 \,\&\, \sim C_1) \vee \ldots \vee (C_m \,\&\, \sim C_m)$. Consequently, $\alpha \vdash_{\mathrm{PI}} A \vee (C_1 \,\&\, \sim C_1) \vee \ldots \vee (C_m \,\&\, \sim C_m)$. □

Lemma 2. If $\alpha \vdash^{*}_{\mathrm{DPI}} A$, then there are $C_1, \ldots, C_m$ $(0 \leq m)$ such that (i) $\alpha \vdash_{\mathrm{PI}} A \vee (C_1 \,\&\, \sim C_1) \vee \ldots \vee (C_m \,\&\, \sim C_m)$ and (ii) for all $C_i \in \{C_1, \ldots, C_m\}$ and for all $D_1, \ldots, D_k$, either $\alpha \nvdash_{\mathrm{PI}} (D_1 \,\&\, \sim D_1) \vee \ldots \vee (D_k \,\&\, \sim D_k) \vee (C_i \,\&\, \sim C_i)$ or $\alpha \vdash_{\mathrm{PI}} (D_1 \,\&\, \sim D_1) \vee \ldots \vee (D_k \,\&\, \sim D_k)$.

Proof. Suppose $\alpha \vdash^{*}_{\mathrm{DPI}} A$, viz. that A is finally* derivable at some line (j) in a DPI-proof from α. Suppose furthermore the fifth element of line (j) is $C_1, \ldots, C_m$. It follows from lemma 1 that $\alpha \vdash_{\mathrm{PI}} A \vee (C_1 \,\&\, \sim C_1) \vee \ldots \vee (C_m \,\&\, \sim C_m)$. Suppose finally that, for some $C_i \in \{C_1, \ldots, C_m\}$ and for some $D_1, \ldots, D_k$, $\alpha \vdash_{\mathrm{PI}} (D_1 \,\&\, \sim D_1) \vee \ldots \vee (D_k \,\&\, \sim D_k) \vee (C_i \,\&\, \sim C_i)$, whereas $\alpha \nvdash_{\text{PI}} (D_1 \& \sim D_1) \vee \ldots \vee (D_k \& \sim D_k)$. Hence it is possible to add to the proof a line the second element of which is $(D_1 \& \sim D_1) \vee \ldots \vee (D_k \& \sim D_k) \vee (C_i \& \sim C_i)$ and the fifth element of which is empty, whereas no member of $\text{DK}(D_1, \ldots, D_k)$ occurs as the second element of a line the fifth element of which is empty. Consequently, line (j) is deleted by application of I1. This, however, is impossible in view of the supposition we made at the outset. □

Lemma 3. If $B_1, \ldots, B_n \vdash_{\text{PI}} (D_1 \& \sim D_1) \vee \ldots \vee (D_k \& \sim D_k)$ and, for any D_i, $B_1, \ldots, B_n \nvdash_{\text{PI}} (D_1 \& \sim D_1) \vee \ldots \vee (D_{i-1} \& \sim D_{i-1}) \vee (D_{i+1} \& \sim D_{i+1}) \vee \ldots \vee (D_k \& \sim D_k)$, then each $\sim D_i$ is a subformula of some B_j.

Proof. Suppose the *implicans* of the lemma holds true, and hence so does its semantic counterpart. Hence there is a PI-valuation such that $v(B_1) = \ldots = v(B_n) = 1$ and $v(D_j \& \sim D_j) = 0$ whenever $j \neq i$, and, for all PI-valuations v, if $v(B_1) = \ldots = v(B_n) = 1$ and if $v(D_j \& \sim D_j) = 0$ whenever $j \neq i$, then $v(D_i) = v(\sim D_i) = 1$. It is obvious in view of the PI-semantics that this is only possible if $\sim D_i$ is a subformula of some B_j or of some $\sim D_j$. Let $1(A)$ denote the number of symbols occurring in A.

Case 1: there is no D_j such that $1(D_j) > 1(D_i)$. Hence $\sim D_i$ cannot be a proper subformula of some $\sim D_j$. If $\sim D_i$ were identical to $\sim D_j$, then we would have $v(D_i \& \sim D_i) = v(D_j \& \sim D_j)$, which is false. Consequently, $\sim D_i$ is a subformula of some B_j.

Case 2: there are D_j such that $1(D_j) > 1(D_i)$. If $\sim D_i$ is a subformula of some $\sim D_j$, then it is readily shown by induction on $1(D_j)$ that $\sim D_i$ is a subformula of some $\sim D_j$ to which case 1 applies, and hence that $\sim D_i$ is a subformula of some B_j. □

Theorem 5. $B_1, \ldots, B_n \vdash^*_{\text{DPI}} A$ iff there are $C_1, \ldots, C_m$ $(0 \leq m)$ such that (i) $B_1, \ldots, B_n \vdash_{\text{PI}} A \vee (C_1 \& \sim C_1) \vee \ldots \vee (C_m \& \sim C_m)$ and (ii) for all $C_i \in \{C_1, \ldots, C_m\}$ and for all $D_1, \ldots, D_k$, either $B_1, \ldots, B_n \nvdash_{\text{PI}} (D_1 \& \sim D_1) \vee \ldots \vee (D_k \& \sim D_k) \vee (C_i \& \sim C_i)$ or $B_1, \ldots, B_n \vdash_{\text{PI}} (D_1 \& \sim D_1) \vee \ldots \vee (D_k \& \sim D_k)$.

Proof. One direction follows from lemma 2. For the other suppose that there are $C_1, \ldots, C_m$ $(0 \leq m)$ such that (i) and (ii) are fulfilled. I shall show that a DPI-proof from $B_1, \ldots, B_n$ may be constructed in which A is finally* derived at some line. First write down all premisses by application of I2 (we need n lines to do so). Let σ be the (finite) set of all formulas the negation of which is a subformula of B_1 or ... or B_n. For any $\{E_1, \ldots, E_l\} \subseteq \sigma$, consider:

(19) $(E_1 \& \sim E_1) \vee \ldots \vee (E_l \& \sim E_l)$

and check whether $B_1, \ldots, B_n \vdash_{\text{PI}} (19)$; whenever the outcome is positive, add to the proof a line the second element of which is (19). As σ is finite and PI is decidable (theorem 4), this will result in the addition of a finite number of lines to the first n ones. It follows immediately from the supposition that a further line may be added which has A as its second element, some or all members of the first n line numbers as its third, some rule as its fourth, and $C_1, \ldots, C_m$ as its fifth element; let (j) be its first element. I now show that line (j) will not be deleted in any extension of this proof. Suppose that $C_i \in \{C_1, \ldots, C_m\}$ and that (20) occurs as the second element of a line the fifth element of which is empty.

$$(20)\quad (D_1 \,\&\, {\sim}D_1) \vee \ldots \vee (D_k \,\&\, {\sim}D_k) \vee (C_i \,\&\, {\sim}C_i)$$

From the supposition made at the outset, it follows that (21) is PI-derivable from $B_1, \ldots, B_n$.

$$(21)\quad (D_1 \,\&\, {\sim}D_1) \vee \ldots \vee (D_k \,\&\, {\sim}D_k)$$

From lemma 3 we know that the result obtained from (21) by deleting all $(D_j \,\&\, {\sim}D_j)$ for which ${\sim}D_j$ is not a subformula of either B_1 or $\ldots$ B_n, is PI-derivable from $B_1, \ldots, B_n$. However, this result is of the form of (19), and hence occurs already as the second element of some line the fifth element of which is empty. Consequently, some DK$(D_1, \ldots, D_k)$ occurs as the second element of a line the fifth element of which is empty, and hence line (j) cannot be deleted on account of the presence of (20) as the second element of a line. □

Theorem 5 as it stands cannot be proved for infinite sets, because for them σ will not in general be finite. It is indeed possible that there is an infinite set β of formulas which have the properties of (19), each of which is PI-derivable from the infinite set α, and none of which is derivable from the other members of β. It follows that, after line (j) has been written, it is always possible to add a line the second element of which has the properties of (20), the fifth element of which is empty, and for which no member of DK$(D_1, \ldots, D_k)$ occurs as the second element of a line the fifth element of which is empty. Nevertheless, it seems obvious that, under the conditions stated in theorem 5, A is finally DPI-derivable from infinite sets in some sense: whenever line (j) has been deleted because (20) has been derived from some finite subset of α, we may derive some DK$(D_1, \ldots, D_k)$ from the same subset, and hence reintroduce line (j) with its line number adjusted, and this line will not any more be deleted for the same reasons; "in the end" A will not any more be deleted. These considerations refer to infinite proofs, but we may arrive at the desired result without reference to such animals too.

Definition. An *intelligent extension* of a DPI-proof from α is an extension such that, if the result of dropping some disjunct from $(D_1 \,\&\, {\sim}D_1) \vee \ldots \vee (D_k \,\&\, {\sim}D_k)$ is PI-derivable from α, then this longer formula does not occur occur as the second element of a line in the extension unless the shorter formula occurs as the second element of a previous line the fifth element of which is empty.

Definition. A is *finally derived at some line in a* DPI-*proof from* α iff it occurs as the second element of this line and this line will not be deleted in any intelligent extension of the proof.

Definition. $A_1, \ldots, A_k$ are *simultaneously finally derived at some lines in a* DPI-*proof from* α iff each of them is finally derived at some line in that proof.

Definition. $\alpha \vdash_{DPI} A$, A is *finally* DPI-*derivable from* α, iff there is a DPI-proof from α at a line of which A is finally derived.

Definition. $Cn_{DPI}(\alpha)$, the DPI-*consequence set of* α, is $\{A|\alpha \vdash_{DPI} A\}$.

Theorem 6. $\alpha \vdash_{DPI} A$ iff there are $C_1, \ldots, C_m$ $(0 \leq m)$ such that (i) $\alpha \vdash_{PI} A \vee (C_1 \& \sim C_1) \vee \ldots \vee (C_m \& \sim C_m)$ and (ii) for all $C_i \in \{C_1, \ldots, C_m\}$ and for all $D_1, \ldots, D_k$, either $\alpha \nvdash_{PI} (D_1 \& \sim D_1) \vee \ldots \vee (D_k \& \sim D_k) \vee (C_i \& \sim C_i)$ or $\alpha \vdash_{PI} (D_1 \& \sim D_1) \vee \ldots \vee (D_k \& \sim D_k)$.

Proof. One direction: as for lemma 2; the other: obvious in view of the definition of an intelligent extension.

Corollary 2. For all finite α, $\alpha \vdash^*_{DPI} A$ iff $\alpha \vdash_{DPI} A$.

The proof of theorems 7-9 is obvious or wholly analogous to previous proofs.

Theorem 7. If α is consistent, then $Cn_{DPI}(\alpha) = Cn_{PC}(\alpha)$.

Theorem 8. If $Cn_{PI}(\alpha)$ is inconsistent but non-trivial, then $Cn_{PI}(\alpha) \subseteq Cn_{DPI}(\alpha) \subset Cn_{PC}(\alpha)$. Notice that $Cn_{PI}(\alpha) = \{A|\alpha \vdash_{PI} A\}$ is decidable for all finite and for certain infinite α.

Theorem 9. If $Cn_{PI}(\alpha)$ is decidable and $\alpha \vdash_{DPI} A$, then any DPI-proof from α may be extended in such a way that A is finally derived at some line in the extended proof.

Corollary 3. If $Cn_{PI}(\alpha)$ is decidable then all members of any finite $\beta \in Cn_{DPI}(\alpha)$ are simultaneously finally derivable from α.

In other words, DPI avoids the unwanted properties of DPI*.

Lemma 4. If $B_1, \ldots, B_n \vdash_{PI} A \vee (C_1 \& \sim C_1) \vee \ldots \vee (C_m \& \sim C_m)$ and $B_1, \ldots, B_n \nvdash_{PI} A \vee (C_1 \& \sim C_1) \vee \ldots \vee (C_{i-1} \& \sim C_{i-1}) \vee (C_{i+1} \& \sim C_{i+1}) \vee \ldots \vee (C_m \& \sim C_m)$, then $\sim C_i$ is a subformula of either A or B_1 or ... or B_n.

Proof as for lemma 3.

Theorem 10. For all finite α, $Cn_{DPI}(\alpha)$ is decidable.

Proof. Let β be the set of all formulas $(D_1 \& \sim D_1) \vee \ldots \vee (D_k \& \sim D_k)$ that are PI-derivable from α, whereas no result of dropping one or more of the disjuncts is PI-derivable from α. It follows from lemma 3 and theorem 4 that β is finite and decidable. Let γ_A be the set of all formulas $(C_1 \& \sim C_1) \vee \ldots \vee (C_m \& \sim C_m)$ such that $\alpha \vdash_{PI} A \vee (C_1 \& \sim C_1) \vee \ldots \vee (C_m \& \sim C_m)$, whereas, for all $C_i \in \{C_1, \ldots, C_m\}$, $\alpha \nvdash_{PI} A \vee (C_1 \& \sim C_1) \vee \ldots \vee (C_{i-1} \& \sim C_{i-1}) \vee (C_{i+1} \& \sim C_{i+1}) \vee \ldots \vee (C_m \& \sim C_m)$. It follows from lemma 4 and theorem 4 that γ_A is finite and decidable, and that $\alpha \vdash_{PI} A \vee (E_1 \& \sim E_1) \vee \ldots \vee (E_\ell \& \sim E_\ell)$ iff some member of $DK(E_1, \ldots, E_\ell)$ is a member of γ_A. It is obvious that $A \in Cn_{DPI}(\alpha)$ iff, for some $F \in \gamma_A$, no disjunct of F is a disjunct of any member of β. As β and γ_A are finite, it is decidable whether or not $A \in Cn_{DPI}(\alpha)$. □

All preceding lemmas and theorems concern either PI or *final* DPI-derivability. DPI-derivability as such is indeed not very interesting. In order not to bore the reader with unimportant results, I merely list two easily provable theorems concerning DPI-derivability which are of some interest.

Theorem 11. If α is consistent, then $A_1, \ldots, A_n$ are simultaneously DPI-derivable from α iff they are all PC-derivable from α.

Theorem 12. If α is inconsistent, $\beta \subset \alpha$, $\gamma \subset Cn_{PC}(\beta)$, γ is finite, and neither β nor γ contain some formula of the form $(A_1 \& \sim A_1) \vee \ldots \vee (A_n \& \sim A_n)$ $(1 \leq n)$, then all members of γ are simultaneously DPI-derivable from α.

7. Theorems, axiomatizations, elegance

It seems to me that the common view on the role of axiomatic systems with respect to logic is largely mistaken.[7] I cannot elaborate on this here, but the reader will easily see some general consequences of my treatment of DPI.

We dispose of two characterizations of DPI, one in terms of proof construction, the other—a partial characterization only—in terms of final DPI-consequences (*cf.* theorem 6). With respect to such characterizations of logics, there are at least two different ways to define the set of DPI-theorems, and each way leads to a different set of DPI-theorems. On the one hand we might define DPI-theorems as the formulas that are finally DPI-*derivable from the empty set* of premisses. If we take the DPI-instructions as they stand, viz. not requiring that $n > 0$ in I3 and I4, then the set of DPI-theorems is *identical to the set of* PC-*theorems* on the above definition. The empty set of premisses will indeed never lead to the inconsistent behaviour of some sentences, and hence any PC-theorem may be written as the second element of a line in a DPI-proof, and such line will not be deleted in any extension of the proof. However, we may also define DPI-theorems as the formulas that are finally DPI-*derivable from any set of premisses.* On this definition the set of DPI-theorems is *identical to the set of* PI-*theorems.*

The second definition has an advantage over the first. If we define 'theorem' in the second way, we may easily demonstrate that DPI is equivalent to the system obtained by adding the instruction that any theorem may be written at any time in any proof as the second element of a line with an appropriate line number, 'theorem' as its third member, a dash as its fourth and '$\emptyset$' as its fifth element. Hence, on this view theorems may be unconditionally asserted at any time, which sounds attractive. On the other hand, the first definition has advantages over the second. If A is a theorem on the first definition, A may be proved from no premisses by means of the instructions I1-I4; furthermore, any inferential step then corresponds to an implicative theorem, even if this step presupposes the consistent behaviour of certain sentences. Finally, all these theorems may be affirmed at any time, unless some relevant sentence behaves inconsistently. Notice that the sense in which we use the term 'DPI-theorem' is merely a matter of linguistic convention. More important is that DPI defines two sets of logical truths, and that each of the two kinds of logical truths has its special features.

The fact that at least in one sense of the term all PC-theorems are DPI-theorems is interesting for the following reason. A large number of non-classical propositional logics *contain* PC: Anderson and Belnap's E and R (see their 1975), Routley and Meyer's static dialectical logic DL (see their 1976), numerous extensional paraconsistent logics, which may contain PC in several senses (see my 1980b). Except for some RPEPLs in which the strong (classical) negation is either primitive or defined (*sic*), and which for that reason are not strictly paraconsistent, all those logics contain PC only in that all theorems of some functionally complete PC-fragment, e.g., all theorems of its disjunction-negation fragment, are theorems of those logics. However, most of these PC-theorems do not have their inferential force in those logics; e.g., although $((p \vee q) \& \sim p) \supset q$ is a theorem of E, q cannot be inferred from $p \vee q$ and $\sim p$ according to E, because *modus ponens* fails for material implication. According to DPI all PC-theorems have their inferential force, except in case some inconsistency prevents an application to specific sentences. Notice also that DPI contains PC in the connected sense that all PC-proofs from consistent sets of premisses are DPI-proofs from those sets of premisses.

It is obvious that DPI is neither characterized by some standard PC-axiomatization nor by some standard PI-axiomatization. DPI simply cannot be characterized by some axiomatization which does not contain rules that take care of its dynamic character. It is even more obvious that DPI cannot be characterized with mere reference to some set of theorems—but even static logics cannot be characterized in this way.

I realize that some readers will complain about my formulation of DPI. DPI is defined with respect to the set of theorems of PI (and the set of PI-valid formulas). It would be more elegant to define DPI with respect to some set of axioms and rules of inference. My main aim, however, was to show that there are dynamic dialectical logics that display some nice properties, and it seems to me that I succeeded in showing this. Nevertheless, it is quite trivial that there are formulations of DPI which are more elegant according to traditional standards. To get one such formulation (i) turn the PI-axioms listed in section 2 into axiom schemes, thus obtaining an infinite supply of axioms, (ii) add a fifth instruction to I1-I4, which allows one to write down a line with an axiom as its second element and '$\emptyset$' as its fifth element, (iii) modify I3 in such a way that it only leads to applications of *modus ponens*, and (iv) modify I4 in such a way that it only leads to applications of "conditional *modus ponens*", viz. 'from $(C_1 \& \sim C_1) \vee \ldots \vee (C_m \& \sim C_m) \vee (B \supset A)$ and B to infer A'—or instead, (iv') modify I4 in such a way that it only leads to applications of 'from $(C \& \sim C) \vee A$ to infer A'.

Apart from its "elegance" this formulation throws some light on the nature of a DPI-proof.

8. Semantics

It is well-known that valuations $v : F \to \{0, 1\}$ are in one-one relation with model sets, definable from them, e.g., by $\{A \mid v(A) = 1\}$. From an intuitive point of view, it is sometimes desirable to talk about "worlds" rather than about valuations or model sets.[8] PC-worlds correspond to maximal consistent sets; PI-worlds to non-trivial, deductively closed, implication saturated sets (a set is implication saturated iff, for all B, it contains $A \supset B$ whenever it does not contain A; *cf.* my 1980a). PI-worlds may be inconsistent, but they are maximally non-trivial in that the addition of any non-member renders them trivial. The usual criterion for the semantic consequence relation is that $\alpha \vDash A$ iff A is true in all worlds in which all members of α are true. Needless to say, this criterion will not enable us to arrive at an adequate definition of $\alpha \vDash_{DPI} A$, among other things because DPI is an adaptive logic, viz. because the set of worlds we will consider will depend on α. We might, however, replace the above criterion by the following one: $\alpha \vDash_{DPI} A$ iff A is true in all worlds in which all members of α are true, and that are as consistent as possible with respect to the members of α. $\alpha \vDash_{DPI} A$ is meant here to be the semantic counterpart of $\alpha \vdash_{DPI} A$. It is obvious that the worlds referred to in this criterion are PI-worlds. Expressed in terms of valuations this leads to:

Definition. $\alpha \vDash_{DPI} A$ iff $v(A) = 1$ for all valuations v that fulfill the following conditions (i) $v(B) = 1$ for all $B \in \alpha$, and (ii) for all C, if $v(C \,\&\, {\sim}C) = 1$, then, for some $D_1, \ldots, D_k$, $\alpha \vDash_{PI} (D_1 \,\&\, {\sim}D_1) \vee \ldots \vee (D_k \,\&\, {\sim}D_k) \vee (C \,\&\, {\sim}C)$ and $v(D_1 \,\&\, {\sim}D_1) = \ldots = v(D_k \,\&\, {\sim}D_k) = 0$.

Condition (ii) makes sure that $v(C \,\&\, {\sim}C) = 0$ unless, for some $D_1, \ldots, D_k$, $(D_1 \,\&\, {\sim}D_1) \vee \ldots \vee (D_k \,\&\, {\sim}D_k) \vee (C \,\&\, {\sim}C)$ is a PI-consequence of α *and* $(D_1 \,\&\, {\sim}D_1) \vee \ldots \vee (D_k \,\&\, {\sim}D_k)$ is not a PI-consequence of α. The DPI-semantics consists of the set of PI-valuations together with the above definition.

Theorem 13. If $\alpha \vdash_{DPI} A$, then $\alpha \vDash_{DPI} A$ (soundness).

Theorem 14. If $\alpha \vDash_{DPI} A$, then $\alpha \vdash_{DPI} A$ (strong completeness).

The *proofs* are obvious in view of the preceding definition and the remark by which it is followed, lemma 1, theorem 6, and the soundness and strong completeness of PI. It seems important to stress that the dynamic dialectical logic has a very simple two-valued semantics: we need no reference to accessibility relations or the Kripkean worlds.

I add a final remark concerning the difference between DPI* and DPI from a semantic point of view. Suppose that $\alpha \vDash_{PI} (p \,\&\, {\sim}p) \vee (q \,\&\, {\sim}q)$, $\alpha \nvDash_{PI} p \,\&\, {\sim}p$ and $\alpha \nvDash_{PI} q \,\&\, {\sim}q$, and suppose that no other contradictions are semantic PI-consequences of α. In this case A will be finally DPI-derivable from α if and only if A is true in all worlds in which all members of α and *either* $p \& \sim p$ *or* $q \& \sim q$ are true, but not both.[9] On the other hand, A is finally derivable in some DPI*-proofs if it is true in all worlds in which all members of α and $p \& \sim p$ are true and in which all other contradictions are false, and will be finally derivable in some other DPI*-proofs if it is true in all worlds in which the members of α and $q \& \sim q$ are true and in which all other contradictions are false. In other words, the "choice" for the inconsistent behaviour of p and the consistent behaviour of q corresponds directly to a choice of PI-worlds.

9. Some other dynamic dialectical logics of the type of DPI

There is an infinite number of RPEPLS between PI and PC (see my 1980b). If in the instructions I1-I4 we replace the reference to PI by a reference to some RPEPL PIz, then we obtain a dynamic dialectical logic DPIz. I shall point to some differences between DPI and those DPIz.

Some RPEPLs are strictly paraconsistent, others are not; e.g., none of da Costa's logics C_n is strictly paraconsistent. (See, e.g., his [1974].) If PIz is not strictly paraconsistent, then the dynamic and adaptive character of DPIz is severely narrowed down. Let us consider at once a very strong RPEPL, viz. PIv, which is a maximal paraconsistent logic. Its semantic characterization is obtained by adding to the PI-semantics:

If A is complex (not a variable), then $v(\sim A) = 1$ iff $v(A) = 0$. Notice that, e.g., $(\sim p \& \sim\sim p) \supset q$ is PIv-valid, and hence that $\sim A, \sim\sim A/B$ holds unconditionally according to PIv; analogously for other complex formulas. In other words, *any* sentence is finally DPIv-derivable from complex contradictions (or complex inconsistencies). In general, whenever PIz is not strictly paraconsistent, then there are certain forms of contradictions such that, if a contradiction of one of these forms is derivable from α, then $Cn_{PIz}(\alpha) = \mathbf{F}$ (remember that **F** is the set of all sentences, respectively formulas). This suggests that dynamic dialectical logics based on RPEPLs that are not strictly paraconsistent, are not very interesting. There is another reason why they are not very interesting; reconsider PIv as an example. The only way in which PIv is richer than PI is that certain PI-theorems of the form of (22) are PIv-theorems even if we drop those disjuncts $C_i \& \sim C_i$ in which C_i is complex.

(22) $(C_1 \& \sim C_1) \vee \ldots \vee (C_m \& \sim C_m) \vee (B \supset A)$

As a consequence, DPI and DPIv define exactly the same set of rules of inference, except in that DPIv will not require the consistent behaviour of any complex formula because PIv already presupposes that those formulas behave consistently (and sanctions their inconsistent behaviour with triviality). In other words, DPI-proofs are identical to DPIv-proofs, except in that the fifth elements of the lines in the DPIv-proofs need not contain any complex formulas that might be contained in the fifth elements of the lines of the DPI-proof. Suppose, however, that the inconsistent behaviour of no such formula is derivable; then no line in the DPI-proof will be deleted, unless it will also be deleted in the DPIv-proof (because some variable behaves inconsistently). Suppose on the other hand that some complex formulas behave inconsistently; then any sentence is derivable in the DPIv-proof, whereas some formulas will not be derivable in the DPI-proof. Hence, there is no reason why one might prefer DPIv above DPI: by using DPIv we lose in safety against triviality, and we gain nothing. In general, whenever PIz is not strictly paraconsistent, then there is a strictly paraconsistent PIy such that $Cn_{PIz}(\alpha) = Cn_{PIy}(\alpha)$, unless $Cn_{PIz}(\alpha) = \mathbf{F}$.

I now turn to strictly paraconsistent RPEPLs. These are closer to PC without in any sense presupposing the consistency of certain formulas. Nevertheless, the way in which they are richer than PI could be characterized by saying that they link the (in)consistent behaviour of certain sentences to the (in)consistent behaviour of other sentences. If, e.g., we add $p \supset \sim\sim p$ as an axiom to PI, then we get a system according to which $\sim p$ behaves inconsistently whenever p behaves inconsistently. Again, I consider at once a maximal paraconsistent logic, viz. PIs, which is interesting in several respects. We obtain a semantic characterization of PIs by adding the following four conditions to the PI-semantics:

C5 $v(\sim\sim A) = v(A)$
C6 $v(\sim(A \supset B)) = v(A \,\&\, \sim B)$
C7 $v(\sim(A \,\&\, B)) = v(\sim A v \sim B)$
C8 $v(\sim(AvB)) = v(\sim A \,\&\, \sim B)$

As PIs is strictly paraconsistent, it is easily provable that, for any finite α, $Cn_{PIs}(\alpha)$ is non-trivial, which certainly is an important standard to be met by dynamic dialectical logics.

In view of the fact that PIs is stronger than PI, one might be tempted to believe that DPIs is stronger than DPI and that $Cn_{DPI}(\alpha)$ will in general be a proper subset of $Cn_{DPIs}(\alpha)$. This, however, is false. It is true, e.g. that p is only finally DPI-derivable from $\{\sim\sim p\} \cup \alpha$ in case $\sim p$ behaves consistently, whereas p is finally DPIs-derivable from that set irrespective of the consistent or inconsistent behaviour of $\sim p$. However, exactly for this reason it is possible that $\sim p$ behaves inconsistently in some DPI-proof, whereas p behaves consistently in it, and this may be relied upon to infer, say $\sim q$ from $q \supset p$ and $\sim p$. In a DPIs-proof, to the contrary, p will behave inconsistently whenever $\sim p$ behaves inconsistently, and hence $\sim q$ is not finally

DPIs-derivable from $q \supset p$, $\sim p$, and $\sim\sim p$. All this suggests, it seems to me, that DPI is preferable to DPIs with respect to the paradigm case described in section 1. DPI localizes inconsistencies as much as possible. If $\sim p$ behaves consistently, then p is finally DPI-derivable from $\sim\sim p$ anyway; and if $\sim p$ behaves inconsistently, then, it seems to me, it will in general be more interesting to derive other sentences from $\sim p$ (and other sentences) by relying on the consistent behaviour of p, rather than to derive p from $\sim\sim p$. I do realize, however, that the relations between DPI-proofs and DPIs-proofs should be studied in more detail in order to reach a well-argued conclusion on their respective merits.[10]

10. Final comments and some open problems

Nicholas Rescher and Ruth Manor (see their 1970) have proposed a theory on "inference from inconsistent premisses" which among other things was intended to solve the kind of problem I described in the paradigm case in section 1. My dynamic dialectical logics are clearly distinct from their logical machinery, both with respect to proof procedure and with respect to the sets of consequences defined. I cannot argue in full why I take the dynamic dialectical logics to be preferable to the logical machinery, but offer a few comments (that will obviously be more significant to the reader who is familiar with Rescher and Manor's work). The Rescher-Manor approach has the disadvantage of splitting up inconsistent sets of premisses into maximal consistent subsets. As a consequence, certain premisses may get totally disconnected: no consequences for the derivation of which all premisses are needed will be attained. Furthermore, the outcome of the Rescher-Manor approach is highly dependent on the way in which the premisses are given; some premisses together with p & q will lead to different consequences than the same premisses together with p and q separately, and a premiss $(p \& \sim p) \& q$ will have no consequences at all, not even q. As the maximal consistent subsets of some set of premisses are consistent, the occurrence of some inconsistency will never block some inference in the way it does according to DPI. This is why, to my view, the set of inevitable consequences is in some sense too weak, and in another too strong; and analogously for the set of weak consequences and for the set of preferred consequences. Rescher and Manor adapt the premisses to the logical requirement of consistency, whereas dynamic dialectical logics adapt themselves to the set of premisses.

I realize quite well that relevant logicians will not be satisfied with my dynamic dialectical logics, which are based on material implication. I agree with them that there are important implications that differ drastically from material implication, but I disagree with their claim that the latter is not really an implication and hence that *modus ponens* fails to hold for it. Not being able to argue for my position here at some length, I only make two comments relevant to the merits of logics based on material implication. Relevant logicians have claimed *ad nauseam* that material implication is just a kind of disjunction, and that disjunctive syllogism is an inferential mistake. With respect to extensional paraconsistent and dynamic dialectical logics, however, material implication cannot be defined in terms of the (classical) disjunction: $p \supset q$ is not derivable from $\sim p \vee q$; the first means that p is *false* or q is true, whereas the second means the $\sim p$ is *true* or q is true—remember that the falsehood of p is not derivable from the truth of $\sim p$ according to these logics. With respect to disjunctive syllogism, I fully agree that this is an inferential mistake if we do not presuppose the world to be consistent. This is why disjunctive syllogism is incorrect according to all RPEPLs, although *modus ponens* for material implication is correct according to all of them. If, however, we presuppose that the world is consistent—we might after all have some good reasons to presuppose so—then disjunctive syllogism is correct. Apart from these comments "in defense", I want to point at least to one advantage of extensional logics. Given the standard interpretation of the binary connectives, the presuppositions we make about consistency (partial consistency, connections between inconsistencies) will have immediate consequences on the correctness of certain rules of inference; different presuppositions lead to different sets of correct rules. It is precisely because of the existence of these relations that I was able to articulate the present dynamic dialectical logics; and it is by no means obvious that such logics may be defined on the basis of static relevant logics, for all relevant logics give up the consistency presupposition from the very start: they "play safe". Relevant logicians realize this, and even argue it is an advantage (see Routley and Meyer, 1976, section IV), but of course a logic that "plays safe" cannot adapt itself to the set of premisses in the same way as PI does within DPI.

Returning to extensional dynamic dialectical logics, I want to point quickly at some possible *variations.* DPI informs us whenever certain sentences are connected with respect to their consistency. Hence, it may easily be adapted in such a way as to take account of non-logical preferences which might enable one to choose for the inconsistent behaviour of one of the connected sentences and for the consistent behaviour of the others. A more important point concerns the elimination of inconsistencies. Here again non-logical preferences might be taken into account to eliminate one "half" of the inconsistency. This might be done in several sensible ways, e.g., by eliminating the non-preferred half from the set of final consequences, but nevertheless using it in the course of the proof to combine it with other sentences in order to derive certain consequences. In some contexts, viz. in case the contextual preferences may be expressed in terms of formal properties of inconsistencies, one might eliminate inconsistencies on "logical grounds" alone. Some interesting suggestions to this effect were made by Leo Apostel (in his 1979), viz. to eliminate first the strongest inconsistencies derived (those from which other inconsistencies may be derived, given "the rest" of the context). Another possibility, appropriate in some contexts, would be to eliminate both "halves" of the inconsistency. For example, DPI might be adapted along the following lines: if, at some time, you have derived A & ~A, then (i) believe neither A nor ~A, (ii) believe no proposition from which A or ~A is derivable, (iii) believe no proposition which is derivable from A alone or from ~A alone, and which is not derivable from some other proposition, (iv) do not believe that A behaves consistently, and (v) continue to believe other propositions that are derivable from either A or ~A together with other propositions. A star might discriminate a non-believed proposition which nevertheless is still used for inferential purposes.

Although I do believe that some logics may take care of the elimination of inconsistencies, I do not believe that the "enrichment" of the premisses can be taken care of by any deductive logic. This problem, it seems to me, is typically a heuristic problem. Although possible enrichments may very well be arrived at by algorithmic means, no algorithm could possibly lead to "the best enrichment". (It is well-known that the interesting enrichments are conceptual in nature; whence it follows that the best enrichment will probably always be the best among the set of enrichments discovered.)

I saved a kind of paradox for my final comment. A DPI-proof from some finite set of premisses α proceeds dynamically. If constructed in an intelligent way, it will lead to a set of sentences which are finally derivable from α. If constructed in a very intelligent way, it will even lead to a set of sentences β from which all final consequences of α are PI-derivable; i.e. $Cn_{PI}(\beta) = Cn_{DPI}(\alpha)$. Yet, the set of all final DPI-consequences of α, $Cn_{DPI}(\alpha)$, may be defined in a systematic way, without reference to any dynamics, and is determined before any (dynamic) proof has been constructed. It seems to me that this does not indicate a disadvantage of my logics, but rather that it reveals some inescapable property of dialectical thought processes. I write 'inescapable' because (i) to give way to mere accident is not a sensible escape, and (ii) if we referred to non-logical preferences, still the final outcome would be determined by the logic, given those preferences. Notice, however, that this determinism will sound paradoxical mainly to non-dialecticians. Most dialecticians were determinists indeed.

Finally, I list some open problems:

1. The properties of the dynamic dialectical logics based on RPEPLs deserve more study; an important problem is whether DPI is indeed preferable in all contexts, or only in some.
2. The "variations" mentioned in this section should be studied more thoroughly in order to prove that they are adequate in specific contexts.

3. Is it possible to define dynamic dialectical logics in terms of sets of "natural" rules of inference only?
4. Are there any special problems concerning the corresponding predicative logics, especially problems concerning decidability?
5. Is it possible to define logics that are even more adaptive in that their static paraconsistent basis (PI in case of DPI) is not given from the start, but determined by the inconsistencies that are derivable from the set of premisses?
6. Relevant logics might be used as a basis for dynamic dialectical logics, e.g., by defining (non-relevant) conditional rules as follows: if $(B_1 \& \ldots \& B_n) \rightarrow (A \vee (C_1 \& \sim C_1) \vee \ldots \vee (C_m \& \sim C_m))$ is a theorem, then, given the consistent behaviour of $C_1, \ldots, C_m$, from $B_1, \ldots, B_m$ to infer A. What are the properties of such logics? For which relevant logics RL does $\emptyset \vdash_{DRL} A$ iff $\vdash_{PC} A$ hold? For which $Cn_{DRL}(\alpha) = Cn_{PC}(\alpha)$, if α is consistent?
7. Same problems as in 6, but for dynamic dialectical logics based on some RPEPL to which a relevant implication has been added.

Notes

* The main results of this paper were first presented in my "Dialectische processen en dialectische logica's" (1979), written for the "Werkgroep Dialectiek" of the Vrije Universiteit Brussel, directed by Jean-Pierre De Waele. I am especially indebted to Leo Apostel, once my teacher and a friend ever since, were it only because he cured me very patiently from a severe infection of classicitis. Finally, I want to express my gratitude to Graham Priest who made detailed comments on a former draft, which enabled me to correct several definitions and proofs; in his 1979 he defended an approach that comes very close to the one developed here.

[1] These rules are not equivalent in all systems; see, e.g., Diego Marconi's introduction to his 1979.

[2] I refer to sections 7 and 10 of my 1980b for technical details and for a discussion of a sensible interpretation of such partial rejection of the consistency of the world.

[3] The alternative approach proposed by Nicholas Rescher and Ruth Manor is discussed in section 10.

[4] There is no clause for $\equiv$, material equivalence, because it is quite uninteresting and furthermore might be introduced by means of its usual definition.

[5] The formulation of element (iv) is quite awkward. I shall actually write metaformulas corresponding to the formulas $B_1, \ldots, B_n$ and A, with '/' between the next to last and last one.

[6] I4 leads to the same results as I3 in case $m = 0$. If $n = 0$, then I3 enables one to write the PI-theorem A as the second element of a line the third and fifth elements of which are empty, and I4 enables one to write the PC-theorem A as the second element of a line which has an empty third element and an appropriate fifth element $C_1, \ldots, C_m$, if the condition on the C_i is satisfied.

[7] Notice in this connection that an axiomatic system as such does not determine a set of rules of inference, that dynamic logics cannot be characterized in the standard way by an axiomatic system, and that some logics have no theorems at all (I shall publish some results on such logics in the future).

[8] A referee once objected to a paper of mine that I talked about worlds without presenting a "possible worlds" semantics. Apparently he was corrupted by Kripke's presentation of his semantics for modal logics. One may easily rephrase this semantics by considering valuations $v: F \to \{0, 1\}$ only, defining the accessibility relation as a relation between valuations, and keep on associating worlds with valuations.

[9] See the definition of $\alpha \vDash_{\text{DPI}} A$. The following example is helpful to realize why the 'not both' has to be added: $p \vee q, \sim p, \sim q \vdash_{\text{DPI}} (p \,\&\, q) \supset r$ but $v((p \,\&\, q) \supset r) = 0$ for all PI-valuations v for which $v(p \vee q) = v(p \,\&\, \sim p) = v(q \,\&\, \sim q) = 1$ and $v(r) = 0$.

[10] I discovered a nice graph that relates all DPIs-inconsistencies in two variables. I do not reproduce it because I do not yet understand which interesting consequences derive from this graph, if any.

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