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PARACONSISTENT EXTENSIONAL PROPOSITIONAL LOGICS.*

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1. Introduction

In this paper I develop some logics that are as close as possible to the classical propositional calculus (PC) but do not presuppose that the world is consistent. It will turn out that there is one obvious basic logic of this sort, which may be extended in several ways into richer systems. I shall study the main formal properties of these systems and offer some philosophical remarks about them.

The logics studied in this paper are material (and quasi-material) paraconsistent logics. I refer the reader to section 2 for a definition of «material logic» and «quasi-material logic». By a paraconsistent logic I mean, following Ayda I. Arruda (1980), a logic that «can be employed as underlying logic for inconsistent but non-trivial theories...» Incidently, the name 'paraconsistent logic' was coined recently by F. Miró Quesada, but the definition of this kind of logics was offered years ago by Newton C.A. da Costa. Quite some results on paraconsistent logics have been published recently, and it is one of the aims of the present paper to order the domain of those paraconsistent logics that are extensions of the full positive propositional calculus.

A first embrionic version of this paper was written in 1973. I got interested in the subject because I believed, and still believe, it to be relevant to several problems dealt with by Alan Ross Anderson and Nuel D. Belnap in connection with their systems E and R, to the problem of meaning relations (see my (1975a)), to the deduction problem (see my (1975b)), etc. I owe it to Leo Apostel that I later became acquainted with the wide literature on paraconsistent and dialectical logics. In this connection I refer especially to the work on

* I am indebted to Leo Apostel, Richard Routley and Etienne Vermeersch for helpful comments; I am especially indebted to Newton C.A. da Costa whose detailed comments on each section of a former draft were very valuable and stimulating and pointed out several mistakes. As usual, responsibilities remain entirely with the author.

relevant dialectical logics by Richard Routley and Robert K. Meyer, and to the numerous papers on paraconsistent logics by Newton C.A. da Costa and his collaborators. It turned out, however, that the aforementioned basic system had never been studied (as far as I could find out), that most extensional paraconsistent logics on the market are somewhere in between this basic system and PC, and that the notion of a maximal paraconsistent logic (see section 2) had not been given any attention.

Ever since Aristotle the official doctrine has been that the world is consistent. As a statement about the world-an-sich, this seems either trivial or nonsensical. However, the statement may also be taken as about the correct description of the world in a given language. As such, it is not only meaningful, but possibly false. Notice that from the fact that the correct description of the world in some language is consistent it does not follow that the same holds for the correct description of the world in another language, and *vice versa*. Whenever I use the expression 'inconsistent world', I mean a world that cannot be described consistently in a given language. I return to all this later, and shall argue that inconsistent theories may be true as well as false. I also refer the reader to Routley and Meyer (1976), to Arruda (1980), to da Costa and Wolf (1978), and especially to Routley (1979) for a discussion of the consistency principle and of the relation between inconsistency and falsehood. I refer the reader to Apostel (1979) for an extensive discussion of the relation between dialectics (Hegel, Marx) and logical systems (propositional logics of action, etc.) and of the relation between paraconsistent and dialectical logics.

The reader is prayed not to mind that the aim of this paper as well as the underlying philosophical view conflict with the still widespread but nonetheless wild dogma «Logica Una et Omnipotens est.» It is fascinating to see how many logicians explicitly or implicitly subcribe to this dogma without seeming to feel a need for justification. Even philosophically minded logicians as excellent as Anderson and Belnap seem to disregard in some of their arguments that there might very well be more than one correct analysis of entailment, depending on the context. And even Routley and Meyer (1976, 17-18) write «Whichever is the case (i.e. whether the world is consistent or inconsistent), however, the relevance position does not go wrong. This provides a major reason why the relevance position is more rational than the other positions, should it turn out that the matter of the consistency of the world cannot be definitely settled.» – an argument which seems to lead to recognizing as the most rational logic one that never «goes wrong», and this notwithstanding the fact that those relevant logics that are not led into trouble with respect to inconsistent worlds, are clearly too weak with respect to consistent worlds. Fortunately, however, the dogma is more and more disclaimed. Zinov'ev (1973, 64) wrote already: «But it is precisely the efforts to find the most adequate description of logical entailment in contemporary logic that have destroyed this prejudice. As a matter of fact, there is no single, perfect, «natural», etc., logical entailment which simply has not been adequately described up to now.»

Once we start considering inconsistent sets of beliefs or inconsistent theories, PC turns out to be completely misguided even as an extensional logic. It will be articulated in detail in sections 4 and 5 in what sense certain PC-theorems are invalid with respect to inconsistent worlds, but it is clear at once that PC turns any inconsistent theory into a trivial one in view of the theorem $p \supset (\sim p \supset q)$. As several theories are inconsistent and as our beliefs are inconsistent most of the time, PC-fanatics get a hard problem here. Of course, relying on their belief in the consistency of the world, they might go on claiming that inconsistent theories cannot be true, and hence that inconsistent sets of beliefs are bound to have false consequences. But not only does this reply rest on straightforward *petitio principii*, even if they were right that the world is consistent, their reply is missing the point. Indeed, even if the world is consistent and even if inconsistent theories have to be transformed into consistent ones by means such as, e.g., Rescher and Manor's machinary (see their (1970)), this does not help us out in the meantime. As long as we are not able to rework an inconsistent theory into a consistent one, or to replace it by a consistent one, rejecting the inconsistent theory leaves us with no theory at all in the domain.

There seem to be three main sources of inconsistencies. First of all, inconsistencies may arise from the observational criteria connected with some theory. This will be the case only if different observational criteria are available to determine whether, say, some predicate applies to some object, or also if the predicates for which independent observational criteria are available are linked with one another by means of so-called meaning postulates.(1) In such a situation we are confronted with inconsistent observational reports, and it is not obvious that the observational criteria may always be adapted in such a way as to get rid of the inconsistency. As a second case, inconsistencies may be derivable from a theory together with a set of observational reports, whereas no inconsistencies arise within the observational reports. Here we are confronted with a case of («empirical») falsification. Irrespective of the complications discussed by Quine, Grünbaum, and others, it is obvious that we shall prefer to give up the theory (or some auxiliary hypothesis), rather than replacing the logic by a weaker, paraconsistent one; and, by all means, we shall have to take care that the replacement of the logic by a weaker one does not eliminate the possibility of falsification. Finally, an inconsistency may be derivable from the theory alone, i.e. *from* the axioms of the theory *by* the underlying logic.

The paraconsistent logics studied in this paper seem to be especially suitable in situations of the first kind. They might be suitable in situations of the second kind, provided that the possibility of falsification is not eliminated. They may also be suitable in the third case, but I want to add right away some comments in this connection. As I mentioned earlier in this section I do neither believe in the existence of «the natural implication», nor in the «One God, one country, one logic» stuff and nonsense. More especially, *in as far as* a logico-mathematical theory is merely an axiomatization of a set of formulas or sentences, I cannot see why some logic should be excluded *a priori* as underlying logic. Indeed, the logic then has to provide merely a set of rules of inference that lead from axioms to theorems of the theory (see my (1975b)). A second comment is that the language of a theory, whether logico-mathematical or «empirical», may very well contain some implication that is stronger than the material implication of the logics studied in this paper. In this connection, I do not see any objection to extending these logics by introducing one or more, say, relevant implications. One of the merits of the present logics is that they are merely concerned with «what is the case» according to some inconsistent theory – a merit not shared by relevant or modal logics –

(1) See also section 10. Such situations also give rise to *vagueness*; see da Costa and Wolf (1980).

but this does by no means exclude that modal or relevant implications would belong to the vocabulary under consideration.

2. *Preliminaries*

In this section I offer a number of preliminary remarks that will facilitate the discussion in the following sections. I also list a number of definitions; the reader might skipp these, and look them up later as they are needed.

Whether a logic is paraconsistent or not will depend on the set of inferences that are correct according to this logic. However, it is quite standard to characterize a logic by an axiomatic system or a semantic system, and neither of these does determine all by itself a set of rules of inference. Notice indeed that such «*deduction rules*» as Detachment are not rules of inference. Detachment enables us to derive theorems from other theorems ('If $\vdash$ A and $\vdash (A \supset B)$, then $\vdash$ B.'), but not to derive sentences from other sentences, as does the «*rule of inference*» 'From sentences p and $(p \supset q)$, infer the sentence q.' (This distinction between 'rule of inference' and 'deduction rule' is not standard, but I shall consistently keep up with it in this paper.) Returning to the main point, in order to determine the set of inferences that are correct according to some logic, we need, apart from a semantic or axiomatic characterization, some rule that connects theorems or valid formulas to correct inferences. In this connection I shall follow the traditional conception according to which (2.1) holds if and only if (2.2) is a theorem, respectively valid, where '$\supset$' denotes the (or the main) implication of the language schema.(2)

(2.1) From sentences $A_1, \ldots, A_n$, infer sentence B.

(2.2) $A_1 \supset (A_2 \supset \ldots (A_n \supset B) \ldots)$

(2) For logical systems not containing an implication, some other conception has to be employed, e.g.: first define $\alpha \vdash A$ or $\alpha \models A$, and then link (2.1) to it. Several definitions in section 2 presuppose the choice of some such conception, but not a specific one.

In my (1975b) I have argued that the traditional view is mistaken, but the point I tried to make there is not essential to the present discussion of the paraconsistency of a logic.

I first present some definitions that concern sets of wffs and sets of sentences. I shall use $\Box$ as a variable for unary connectives. The set of sentences of a given language will be denoted by S, the set of wffs of a given language schema by F. A and $\models_L A$ or $\vdash A$ and $\models A$ in case no ambiguitiy can arise, denote as usually that A is a theorem, respectively a valid wff, of L.

Definition: $Cn_L(\alpha)$, the consequence set of α, is the set of all A that may be inferred (in the above sense) from members of α according to L.

wherein $Cn_L(\alpha)$ is a subset of S iff α is so, and is a subset of F iff α is so.

Definition: α is A-*trivial* iff all wffs, respectively sentences, of the form (3) A are members of α,

Definition: α is *negation-trivial* iff it is $\Box$p-trivial and $\Box$ is the main negation of the language schema.

In the systems studied in this paper I always consider $\sim$ as the main negation.

Definition: α is *trivial* iff it is p-trivial.

That is, iff $\alpha = S$, respectively $\alpha = F$.

Definition: α is $\Box$*-consistent* iff, for all A, either $A \notin \alpha$ or $\Box A \notin \alpha$.

Definition: α is *consistent* iff it is $\Box$-consistent and $\Box$ is the main negation of the language schema.

Definition: A *theory* is a couple $\langle\alpha, L\rangle$, wherein α is a set of sentences (the nonlogical axioms of the theory) and L is a logic (determining a set of rules of inference).

Convention: A *theory* $\langle\alpha, L\rangle$ will be said to be (...-)*trivial*, respectively (...-)*consistent*, iff $Cn_L(\alpha)$ is so.

Convention: A *valuation* v will be said to be (...-)*trivial*, respectively (...-)*consistent*, iff $\{A/v(a) = 1\}$ is so.

(3) A logical form (with respect to a language schema) is characterized by an expression containing only variables and logical constants.

Notice that I do only consider two-valued valuations, with 1 as the only designated value.

The following definitions concern logical systems. I always suppose that an axiomatic as well as a semantic characterization is available.

Convention: A *logic* L will be said to be (...-)*trivial*, respectively (...-)*consistent*, iff $\{A/\vdash_L A\}$ is so.

Notice that any logic which contains the rule of Uniform Substitution is A-trivial for any theorem A.

Definition: A logic L is A-*destructive* from α iff (i) $\nvdash_L A$, (ii) for at least some B of the form A, $B \notin \alpha$, and (iii) for any B of the form A, $B \in Cn_L(\alpha)$.

An example will clarify the matter. The three-valued logic displayed in the figure is paraconsistent in that $p \supset (\sim p \supset q)$ is not a theorem. Furthermore, this logic is $(q \vee \sim q)$-destructive from $\{p, \sim p\}$, for $p \supset (\sim p \supset (q \vee \sim q))$ is a theorem and $q \vee \sim q$ is not. As a consequence, all sentences of the form $q \vee \sim q$ are theorems of any inconsistent theory of which this logic is the second member. It follows that this logic, although it is paraconsistent, is not fit to be used as underlying logic of inconsistent theories. The distinction between the theorems of some theory and the theorems of the underlying logic makes sense, among other things, because we want a logic to describe which *logical* forms guarantee truth. The factual truth of some sentences should not mess up this distinction, and hence should not lead to the truth of all sentences of some *logical* form. (There is, of course, nothing wrong if all sentences of some non-logical form are theorems of some theory without being theorems of the underlying logic.)

⊃	1	2	3	~
*1	1	2	3	3
2	1	1	3	2
3	1	1	1	1

∨	1	2	3
1	1	1	1
2	1	2	2
3	1	2	3

Definition: A logic is *negation-destructive* iff it is $\square$p-destructive from $\{q, \square q\}$ and $\square$ is the main negation of the language schema.

Johansson's minimal logic and Curry's system D are negation-destructive.

Definition: A logic is *destructive* iff it is p-destructive from $\{q, {}^{\square}q\}$ and $^{\square}$ is the main negation of the language schema.

PC and the intuitionist propositional calculus are destructive.

Definition: A logic is $^{\square}$-*paraconsistent* iff it is not p-destructive from $\{q, {}^{\square}q\}$.

Definition: A logic is *paraconsistent* iff it is not destructive.

In other words, if L is paraconsistent, then at least some inconsistent theories of which L is the underlying logic are nontrivial. Johansson's minimal logic, Curry's system D, Anderson and Belnap's systems E and R, da Costa's calculi C_n $(1 \leq n \leq \omega)$, etc., are paraconsistent logics. It is interesting to introduce some stronger notions.

Definition: A logic L is *strictly* $^{\square}$-*paraconsistent* iff there is no wff A such that L is p-destructive from $\{A, {}^{\square}A\}$.

Definition: A logic is *strictly paraconsistent* iff it is strictly $^{\square}$-paraconsistent and $^{\square}$ is the main negation of the language schema.

Definition: A logic is *logically conservative* iff for any A and for any α, it is not A-destructive from α.

If L is logically conservative, then $Cn_L(\alpha)$ is not trivial unless α is trivial. Any logically conservative logic is strictly paraconsistent (and is strictly $^{\square}$-paraconsistent for any $^{\square}$), and any strictly ($^{\square}$-) paraconsistent logic is ($^{\square}$-) paraconsistent. It should be clear by now that the question as to whether some logic is fit to serve as underlying logic of a given theory depends in part on the question whether this logic is logically conservative, strictly $^{\square}$-paraconsistent, or $^{\square}$-paraconsistent.

Definition: A logic is *regular* iff all its theorems are PC-theorems.

Definition: L is a *maximal* (regular and) (strictly) paraconsistent logic iff L is (regular and) (strictly) paraconsistent and no extension of L is so.

In the introduction I referred to logics that are as close as possible to PC, but do not presuppose that the world is consistent. In order to make sense of this notion, I present some definitions concerning two-valued semantic systems. The meaning of some terms defined below is more restricted than their standard meaning.

Definition: A propositional logic is *extensional* iff it may be characterized semantically by a set V of valuation-functions such that (i) $\models A$ (A is valid) iff, for all $v \in V$, $v(A) = 1$, and (ii) V is defined by a set S of clauses one of which is '$v: F \rightarrow \{0,1\}$', whereas all others are of the form

If $v(A_1) = m_1$ and ... and $v(A_n) = m_n$, then $v(B) = m_{n+1}$.

in which (ii/i) any m_i $(1 \leq i \leq n+1)$ is either 0 or 1, (ii/ii) $n > 0$, and (ii/iii) no clause arrived at by dropping some conjunct from the implicans of this clause, is derivable from the members of S.

Notice that (ii/ii) and (ii/iii) are added in order to prevent that an axiom A would simply be turned into the clause '$v(A) = 1$'. It follows from the definition that an extensional propositional logic may be characterized by a semantics (i) that does not refer, e.g., to possible worlds, (ii) according to which the value assigned by some valuation function to some wff does not depend on any value assigned by some other valuation function, and (iii) according to which any clause of the form 'for all v, $v(A) = 1$' or 'for all v, $v(A) = 0$' is derivable from clauses of the form mentioned in the definition.

Definition: A variable A *occurs essentially* in a wff B iff A occurs in C whenever, for all v, $v(B) = v(C)$.

that is, iff A occurs in all wffs that are semantically equivalent to B.

The definition of a *material* propositional logic is arrived at by adding the two following conditions to the definition of an extensional propositional logic:

(ii/iv) each variable that occurs in some A_i, occurs essentially in B.
(ii/v) each A_i is a subformula of B.

This eliminates such clauses as 'If $v(\sim A) = 1$, then $v(\sim(A \& B)) = 1$'. The definition of a *quasi-material* propositional logic is arrived at by adding to the definition of an extensional propositional logic the condition (ii/iv) together with the condition that the logic is not material. This eliminates such clauses as 'If $v(\sim A) = 1$, then $v(\sim B) = 1$'. If a logic is quasi-material, then the value assigned to some wff A by a valuation function v depends entirely on the values assigned by v to wffs that contain no other variables than those occurring in A. If a logic is material, than the value assigned to A by v depends only on values assigned by v to subformulas of A. It will turn out that maximal regular and paraconsistent logics are either strictly paraconsistent or material, but not both.

Notice that any valuation-function may be seen as corresponding in an obvious way to a possible world. If a semantics excludes, for any A, that $v(A) = v(\sim A) = 1$, then I shall say that there are no inconsistent (possible) worlds according to this semantics, or, alternatively, that it presupposes that the (actual) world is consistent.

3. *The basic logic* PI

PC presupposes that the world is both consistent and complete. This is expressed semantically by the clause

$$v(\sim A) = 1 \text{ iff } v(A) = 0$$

i.e. either A or $\sim A$ is true (completeness) and one of them is false (consistency). If we drop the presupposition that the world is consistent by weakening the aforementioned clause, then we arrive at a logic which I shall call PI (it is as PC except for not excluding inconsistent worlds). This logic has the same wffs as PC and is characterized semantically by the following clauses:

C0 $v: F \rightarrow \{0,1\}$
C1 If $v(A) = 0$, then $v(\sim A) = 1$
C2 $v(A \supset B) = 1$ iff $v(A) = 0$ or $v(B) = 1$
C3 $v(A \& B) = 1$ iff $V(A) = v(B) = 1$
C4 $v(A \vee B) = 1$ iff $v(A) = 1$ or $v(B) = 1$

Validity is defined in the usual way. We shall take equivalence to be defined syntactically:

$D \equiv p \equiv q =_{df} (p \supset q) \& (q \supset p)$

One might of course also introduce this connective by means of a semantic clause, and this would result in a weaker system, but I did not find any advantage in this complication. Notice also that we would get a stronger system if disjunction were defined by $(p \vee q) =_{df} ((p \supset q) \supset q)$. Indeed, the corresponding equivalence is PI-valid, but, e.g., $\sim(p \vee q) \equiv \sim((p \supset q) \supset q)$ is not; notice indeed that $v(\sim A) = v(\sim B)$ does not follow from $v(A) = v(B)$. As I shall show later, some interesting PI-extensions are not extensions of the stronger system.

The following axiomatic system will be referred to as PI (this name is also used for the logic axiomatized by this system). It is an axiomatic system for PC^+ (the $\supset - \& - \vee -$ fragment of PC) (4) to which the law of excluded middle is added (taking care of the completeness of the considered worlds).

Axioms

PIA1	$p \supset (q \supset p)$
PIA2	$((p \supset q) \supset p) \supset p$
PIA3	$(p \supset q) \supset ((q \supset r) \supset (p \supset r))$
PIA4	$(p \& q) \supset p$
PIA5	$(p \& q) \supset q$
PIA6	$p \supset (q \supset (p \& q))$
PIA7	$p \supset (p \vee q)$
PIA8	$q \supset (p \vee q)$
PIA9	$(p \supset r) \supset ((q \supset r) \supset ((p \vee q) \supset r))$
PIA10	$p \vee \sim p$

Rules: Modus Ponens and Uniform Substitution.

I list some metatheorems on PI and PI-extensions.

Theorem 1. If $\alpha \vdash A$, then $\alpha \models A$ (soudness).
Proof: as for PC.

Theorem 2. If $\alpha \models A$, then $\alpha \vdash A$ (strong completeness).
Proof: see the appendix.

(4) PI^+ should not be confused with the *positive Logik* presented by Hilbert and Bernays (1968); the latter system is weaker, e.g., in that PIA2 is not a theorem of it.

Theorem 3. PI is decidable.

Proof: a simple truth-tabular method derives from the semantics.(5)

Lemma 1. The system resulting from adding $(p \& \sim p) \supset q$ as an axiom to PI is (an axiomatization of) PC.

Proof: all axioms and (trivially, also the rules) of Hilbert and Bernays's PC-axiomatization are derivable from this system.

Lemma 2. The rule of replacement of equivalents, henceforth called EQ, is not derivable in PI.

Proof: $\sim p \equiv \sim p$ and $p \equiv (p \& p)$ are PI-valid whereas $\sim p \equiv \sim (p \& p)$ is not.

Lemma 3. EQ, restricted to wffs outside the scope of a negation-sign is derivable in PI.

Proof: as for PC.

We now define two special kinds of conjunctive normal form. A wff A is in CNF° iff it is a continuous conjunction $B_1 \& \ldots \& B_n$, in which each B_i is either of the form $(D_1 \vee \ldots \vee D_k)$ or of the form $((C_1 \& \ldots C_m) \supset (D_1 \vee \ldots D_k))$, each C_i and each D_i being either a variable or of the form $\sim$ E. A wff A is in CNF°° iff it is in CNF° and no B_i is a PI-theorem.

Lemma 4. Any wff is PI-equivalent to a wff A which is in CNF°.

Proof: from Lemma 3 and the following equivalences:

(3.1) $((p \supset q) \supset r) \equiv ((p \vee r) \& (q \supset r))$

(3.2) $(((p \supset q) \& r) \supset s) \equiv ((r \supset (p \vee s)) \& ((q \& r) \supset s))$

(3.3) $(((p \vee q) \& r) \supset s) \equiv (((p \& r) \supset s) \& (q \& r) \supset s))$

(3.4) $((p \vee q) \supset r) \equiv ((p \supset r) \& (q \supset r))$

(3.5) $(p \supset (q \supset r)) \equiv ((p \& q) \supset r)$

(3.6) $(p \supset (q \& r)) \equiv ((p \supset q) \& (p \supset r))$

(3.7) $(p \supset ((q \supset r) \vee s)) \equiv ((p \& q) \supset (r \vee s))$

(3.8) $(p \supset ((q \& r) \vee s)) \equiv ((p \supset (q \vee s)) \& (p \supset (r \vee s)))$

(3.9) $((p \supset q) \vee r) \equiv (p \supset (q \vee r))$

(3.10) $((p \& q) \vee r) \equiv ((p \vee r) \& (q \vee r))$

Corollary 1. Any PI-non-theorem is PI-equivalent to a wff which is in CNF°°.

(5) An easy such method goes as follows: first write down all possible assignments of ones and zeros to all subwffs, and next delete those assignments that do not agree with the PI-semantics.

In the following lemmas and corollaries PI^z is either PI or an extension of PI. The proof of lemmas 5 and 6 is obvious. Consider any two-valued PI^z-semantics that agrees with all clauses of the PI-semantics.

Lemma 5. PI^z is trivial iff all valuation functions of the PI^z-semantics are trivial.

Lemma 6. PI^z is negation-trivial iff all valuation functions of the PI^z-semantics are negation-trivial.

Lemma 7. If PI^z is either material or quasi-material and if, for some A, it is A-destructive from $\{B, \sim B\}$, then $v(B \& \sim B) = 0$ for all nontrivial PI^z-valuation functions v.
Proof: suppose that PI^z is either material or quasi-material, that it is A-destructive from $\{B, \sim B\}$ (for given A), and that there is a non-trivial v such that $v(B \& \sim B) = 1$. As A is PI^z-invalid and the number of variables in B is finite, some wff of the form A, say D, is invalid and does not share any variable with B. Consequently, for some PI^z-valuation function v', $v'(D) = 0$. As B and D do not share any variable and neither v nor v' is trivial, it follows from the definition of 'material logic' and 'quasi-material logic' that there is a v'' that assigns to $B \& \sim B$ the same value as v and to D the same value as v'. Hence $v''((B \& \sim B) \supset D) = 0$. But as PI^z is A-destructive from $\{B, \sim B\}$, $V((B \& \sim B) \supset C) = 1$ for all C of the form A and for all PI^z-valuation functions v. Hence, $v''((B \& \sim B) \supset D) = 1$. As the supposition leads to a contradiction, the lemma holds true.

Corollary 2. If PI^z is material or quasi-material and, for all B, there is a nontrivial PI^z-valuation function v such that $v(B \& \sim B) = 1$, then PI^z is strictly paraconsistent.

Lemma 8. If PI^z is material or quasi-material and if it is destructive, then, for some B, $v(B \& \sim B) = 0$ for all nontrivial PI^z-valuation functions.
Proof: trivial in view of the preceding lemma.

Corollary 3. If PI^z is material or quasi-material and, for some B and some nontrivial PI^z-valuation function v, $v(B \& \sim B) = 1$, then PI^z is paraconsistent.

Lemma 9. If PI^z is material or quasi-material and, for all B, there is a nontrivial PI^z-valuation function v such that $v(B) = 1$, then PI^z is logically conservative.
Proof: the nonobvious part is wholly analogous to the proof of Lemma 7.

Corollary 4. If PI^z is material or quasi-material, then it is strictly paraconsistent iff it is logically conservative.

Theorem 4. PI is material.
Proof: trivial

Theorem 5. PI is logically conservative (hence strictly paraconsistent).
Proof: by induction on the length of formulas, from (i) lemma 9, (ii) for any formula A, there is a nontrivial valuation-function that assigns the value 1 to all variables occurring in A, and (iii) no clause of the form 'If $v(A_1) = \ldots v(A_n) = 1$, then $V(B) = 0$' is derivable from the PI-semantics.

4. *Disjunctive syllogism*

Disjunctive syllogism is the *bête noire* of relevant implication. I shall distinguish between the wff (4.1), the deduction rule (4.2), and the rule of inference (4.3), from which (4.4) follows.

(4.1) $((p \vee q) \& \sim p) \supset q$

(4.2) If $\vdash A \vee B$ and $\vdash \sim A$, then $\vdash B$

(4.3) From (premisses of the form) $p \vee q$ and $\sim p$, infer (the conclusion of the form) q.

(4.4) For all theories $T = \langle \alpha, L \rangle$, if $(p \vee q) \in Cn_L(\alpha)$ and $\sim p \in Cn_L(\alpha)$, then $q \in Cn_L(\alpha)$.

Let us first consider (4.3). With respect to a consistent world this rule of inference is correct if disjunction and negation are interpreted in the usual way. The argument goes as follows. If $\sim p$ is true and either p or q is true, then either p and $\sim p$ are true or $\sim p$ and q are true. As the world is consistent, it is impossible that p and $\sim p$ are both true; consequently $\sim p$ and q are true, and hence q is true. In an inconsistent world, however, it is quite possible that p and $\sim p$ are both true, and hence there is no way to derive q. I realize quite well that some who subscribe to the Anderson and Belnap position on entailment will balk at the argument for the validity of (4.3) in consistent worlds, but I want to point out that the proof presented by Anderson and Belnap (1975, 300-301) may be turned easily into my very argument that q is true in all consistent worlds in which both $p \vee q$ and $\sim p$ are true.

From the incorrectness of (4.3) with respect to inconsistent worlds, it follows immediately that (4.1) is not generally true in such worlds, and hence should be invalid in a logic that does not presuppose the consistency of the world. Furthermore, the system arrived at by adding (4.1) as an axiom to PI is destructive. Notice indeed that $(p \& \sim p) \supset q$ follows by the (PI-derivable) rule of transitivity from (the PI-theorem) $(p \& \sim p) \supset ((p \vee q) \& \sim p)$ and (4.1). As a consequence neither PI nor any of its paraconsistent extensions have (4.1) as a theorem.

(4.2) is trivially true about PI for the simple reason that no PI-theorem is of the form $\sim A$. However, (4.2bis) is false about PI.

(4.2bis) If $\vdash A$ and $\vdash \sim A \vee B$, then $\vdash B$.

Indeed, both $p \supset p$ and $\sim(p \supset p) \vee \sim\sim(p \supset p)$ are PI-theorems, whereas $\sim\sim(p \supset p)$ is not. On the other hand, both (4.2) and (4.2bis) are derivable in some paraconsistent PI-extensions, e.g., in PI^v (see Theorem 23). That this is so is immediately connected with the fact that PI^v is not strictly paraconsistent (anything is PI^v-derivable from certain contradictions). This, however, does not make PI^v any less paraconsistent (*simpliciter*); some inconsistent theories of which it is a member are non-trivial. Still, the corresponding rules of inference – one of them is (4.3) – are incorrect according to PI^v. [6]

[6] These rules are even incorrect in the trivial sense that they are not derivable in general for theories of which PI^v is a member, irrespective whether rules of inference are derived in the Anderson and Belnap way, i.e. as corresponding to derivable deduction rules, or with respect to theorems of the logic under consideration (see section 2).

5. *Some further* PI-*non-theorems*

The addition of any of the following wffs as an axiom to PI results in a system which is either destructive or negation-destructive. I leave the easy proofs to the reader.

(5.1) $p \supset (\sim p \supset q)$
(5.2) $(\sim p \vee q) \supset (p \supset q)$
(5.3) $((p \supset q) \& \sim q) \supset \sim p$
(5.4) $(p \supset q) \supset (\sim q \supset \sim p)$
(5.5) $(p \supset (q \& \sim q)) \supset \sim p$

There are several «variants» to these PC-theorems, e.g., the transposition-variants to (5.4) and the variant $(p \vee q) \supset (\sim p \supset q)$ to (5.2). The addition of any of these as an axiom to PI results in a destructive or negation-destructive system.

It is not difficult to see why, on the usual interpretation of the binary connectives, the rules of inference that correspond to (5.2)-(5.5) are correct with respect to consistent worlds, and hence for (true) theories about such worlds, but are incorrect with respect to inconsistent worlds, and hence for (true) theories about these. The fact that (5.1), a well-known «paradox of implication», is not a theorem of any paraconsistent PI-extension shows that the theoremhood of this wff in PC is not merely a consequence of the meaning of the implication, but also and essentially of the meaning of the negation, i.e. of the consistency presupposition. The fact that (5.2) is not a theorem of any paraconsistent PI-extension, although its converse is a PI-theorem, explains, e.g., why the correctness of modus ponens according to PI does not result in the correctness of disjunctive syllogism according to PI. Finally, it seems worth mentioning that such PI-theorems as (5.6) and (5.7) are usually «justified» with reference to respectively (5.5) and a variant of this PC-theorem.

(5.6) $(p \supset \sim p) \supset \sim p$
(5.7) $(\sim p \supset p) \supset p$

It goes without saying that these are «justifiable» by other means in PI, e.g., with reference to the law of excluded middle PIA10 and (simple) constructive dilemma, rendered here in its exported form:

(5.8) $(p \vee q) \supset ((p \supset r) \supset ((q \supset r) \supset r))$

If in (5.3)–(5.5) the horseshoe is considered as denoting a relevant implication (usually expressed by an arrow), then these wffs turn out to be theorems of numerous relevant logics, e.g., Anderson and Belnap's logics R and E and Routley and Meyer's dialectical logics DL and DK. There is nothing paradoxical about the fact that these logics are nevertheless paraconsistent, their implication being much stronger than the PI-implication.

It seems to me, however, that especially (5.3) and (5.4) are objectionable as theorems of paraconsistent logics, and this irrespective the strength of the implication. In the presence of modus ponens, (5.4) leads to the rule of inference 'From $A \supset B$ and $\sim B$ to infer $\sim A$' and (5.3) leads to the same rule in the presence of modus ponens and conjunction. I shall now present an argument against this rule. Suppose indeed that, according to some theory, a is black if it is a raven and suppose this sentence is «synthetic» in that it does not derive from the meanings of 'black' and 'raven'. Suppose furthermore that a turns out to be a raven and to be black and not to be black (this is a kind of example people interested in dialectical logics have in mind – see da Costa and Wolf (1980)). According to the aforementioned rule, viz. modus tollens, it follows that a is not a raven. But of course, this is wrong and shows that modus tollens is not correct here. It is indeed quite possible that the predicate 'raven' behaves consistently, even if a is black if it is a raven and if the predicate 'black' behaves inconsistently.

Proponents of the aforementioned paraconsistent logics might try to find a way out of this objection by claiming that the implication in 'a is black if it is a raven' is not a relevant implication, or by all means not the relevant implication of their systems, but some weaker implication. Notice, however, that this implication should guarantee the application of modus ponens, whereas none of the aforementioned logics contains an implication for which modus ponens is correct and modus tollens is incorrect.

Richard Routley (1979, 305) claims that «the weakened negation systems lack all forms of contraposition, *though surely some are correct,* and indeed *there is little basis for regarding the so-called negations of these systems as genuine negations at all* rather than, say, positive modal connectives, e.g. weird truth or necessity connectives» (my italics). The argument in the preceding paragraphs purport to show that Routley's intuitions on the matter are wrong (it is indeed easy to find counterexamples to the other forms of transposition). Also, other authors seem to have different intuitions, for da Costa and Wolf (1980) write: «we would not find any intuitive justification for contraposition». Finally, I really cannot see the point of Routley's claim. I must confess that I have (distinct) intuitions on *several* kinds of «negation», that I consider it interesting to study some of them, that I presume that some are useful in certain contexts whereas others are useful in other contexts but that I do not see any point in finding out which are «genuine» negations and which are not.

6. *Strictly paraconsistent* PI-*extensions containing* PC

Amazing as it might seem to be, PC is a fragment of several extensions of PI, and this not withstanding the fact that the PI-implication is plainly material and that *modus ponens* holds for it. This result might be interesting in view of the arguments offered by, e.g. Richard Routley (1979), in favour of the position that sound dialectical logics should contain all PC-theses. There is a second reason why this result is worth being mentioned. PC is a fragment of E as well as of several paraconsistent logics devised by Routley and Meyer (see their (1976) for DL and Routley (1979) for DK). Some readers might suppose that this is in some or other way dependent on the fact that *modus ponens* holds in such logics with respect to relevant implication only. As follows from the present results, this is a mistake.

A weak PI-extension, let us call it PI*, of which PC is a fragment is obtained by adding the two following axioms to PI:

PI*A1 $p \supset \sim\sim p$
PI*A2 $\sim p \supset (\sim q \supset \sim(p \vee q))$

The PI*-semantics is obtained by adding to the PI-semantics (7):

C5 If $v(A) = 1$, then, $v(\sim\sim A) = 1$
C7 If $v(\sim A) = v(\sim B) = 1$, then $v(\sim(A \vee B)) = 1$

(7) That this is so is trivial in view of the results mentioned in the appendix.

Theorem 6. PC is a fragment of PI*.
The proof proceeds in exactly the same way as Anderson and Belnap's proof (1975, 283 ff.) that PC is a fragment of E. (All axioms of their PC-axiomatization are theorems of PI* and their rules are derivable in it.)

In other words, the $\vee - \sim -$fragments of PI* and of PC are identical, and since this fragment is functionally complete in PC, we may define *ad hoc* connectives $\&^{o}$, $\supset^{o}$ and $\equiv^{o}$ in PI*, arriving in this way at the full PC. Hence the PI*-theorems:

(6.1) $\sim(p \&^{o} \sim p)$

(6.2) $(p \&^{o} \sim p) \supset^{o} q$

the last theorem being harmless in view of the fact that *modus ponens* is not derivable for $\supset^{o}$ in PI*. This connective is simply not an implication in PI*, but a kind of disjunction. Incidently, Anderson and Belnap have claimed *ad nauseam* (as they notice themselves) that material implication (in general) is not an implication but a kind of disjunction. By this they mean that material implication does not capture the non-corrupted intuitions on «follows from» and that it is wrong that «detachment for material «implication» is a valid mode of inference» (1975, 165). My statement that $\supset^{o}$ is not an implication has a quite different status; it simply refers to the fact that q may be false if both p and $(p \supset^{o} q)$ are true. This is immediately clear from the PI*-semantics, and there is not the faintest paradox about it (see also section 4).

The reader will wonder whether there are also paraconsistent PI-extensions the $\& - \vee - \sim -$fragment of which is identical to that of PC, and indeed there are. I shall mention at once a very strong such extension(8), which I shall call PI^{s} for reasons that will become clear later on. It is defined syntactically by adding to PI*(9):

(8) A syntactical characterization of a weaker such extension is obtained by adding D& (see some lines below this note in the text) to PI*. (Notice that PIA4-PIA6 contain defined connectives in this case.) That PI* + D& is paraconsistent follows from the fact that the stronger PI^{s} is so.

(9) An alternative (and more elegant) axiomatic system for PI^{s} is obtained by adding PI*A1 and PI^{s}A1 to PI, together with:

PI^{s}A2′ $\sim(p \vee q) \equiv (\sim p \& \sim q)$

PI^{s}A3′ $\sim(p \& q) \equiv (\sim p \vee \sim q)$

PI^sA1 $\quad \sim(p \supset q) \equiv (p \,\&\, \sim q)$
PI^sA2 $\quad \sim(p \vee q) \supset (\sim p \,\&\, \sim q)$
D& $\quad p \,\&\, q =_{df} \sim(\sim p \vee \sim q)$

and is characterized semantically by adding to the PI-semantics([10]):

$C5^s$ $\quad v(\sim\sim A) = v(A)$
$C6^s$ $\quad v(\sim(A \supset B)) = v(A \,\&\, \sim B)$
$C7^s$ $\quad v(\sim(A \,\&\, B)) = v(\sim A \vee \sim B)$
$C8^s$ $\quad v(\sim(A \vee B)) = v(\sim A \,\&\, \sim B)$

Notice that (6.3) is a theorem of PI* (US in PI*A2). By substituting in the PC^+-theorem (6.4) we obtain, by MP, (6.5). From this follows (6.6) by D&, and from (6.6) and the PC^+-theorem (6.7) we get (6.8) by transitivity.

(6.3) $\quad \sim\sim p \supset (\sim\sim p \supset \sim(\sim p \vee \sim p))$
(6.4) $\quad (p \supset (p \supset q)) \supset (p \supset q)$
(6.5) $\quad \sim\sim p \supset \sim(\sim p \vee \sim p)$
(6.6) $\quad \sim\sim p \supset (p \,\&\, p)$
(6.7) $\quad (p \,\&\, p) \supset p$
(6.8) $\quad \sim\sim p \supset p$

The converse of (6.8) is a PI*-axiom. Notice also that the converse of PI^sA2 follows from PI*A2 by importation. Finally, notice that D& cannot be replaced by the corresponding equivalence, because the rule of replacement of equivalents is not derivable in PI^s (see Corollary 8) and the addition of this rule results in a logic which is not paraconsistent (see Theorem 13).

PI^s is equivalent to the system obtained by dropping from Schütte's Φ_v (see his 1960)) the propositional constant $\wedge$ (which may be looked upon as the conjunction of all wffs). Schütte's system was published in 1960, and he does not refer to other forerunners of paraconsistent logic. PI^s is an important system, among other things because it is a regular maximally paraconsistent PI-extension (see Corollaries 6 and 7). I now turn to its metatheory.

([10]) An alternative PI^s-semantics is arrived at by dropping C3 from the PI-semantics (conjunction being considered as defined explicitly) and by adding C5 and:
$C7^o$ $\quad v(\sim(A \supset B)) = 1$ iff $v(A) = v(\sim B) = 1$
$C8^o$ $\quad v(\sim(A \vee B)) = 1$ iff $v(\sim A) = v(\sim B) = 1$

Theorem 7. PI^s is consistent and strongly complete with respect to the PI^s-semantics.
Proof: see the appendix.

Theorem 8. PI^s is decidable.
Proof: the semantics corresponds to a truth-tabular method. (11)

Theorem 9. PI^s is logically conservative (hence strictly paraconsistent).
Proof: wholly analogous to the proof that PI is logically conservative.

Theorem 10. PI^s is quasi-material.
Proof: That PI^s is not material follows from the fact that the restrictions on the value of, e.g., $\sim(p \& q)$ cannot be expressed by referring to the values of p and q only. The rest of the proof is trivial.

Lemma 10. Any wff is PI^s-equivalent to a wff in which all negation signs occur in front of variables.
Proof: trivial in view of Lemma 3 and the following PI^s-equivalences:

(6.9) $\sim\sim p \equiv p$

(6.10) $\sim(p \supset q) \equiv (p \& \sim q)$

(6.11) $\sim(p \vee q) \equiv (\sim p \& \sim q)$

(6.12) $\sim(p \& q) \equiv (\sim p \vee \sim q)$

I now define two further kinds of conjunctive normal forms. A wff is in CNF^s iff it is in CNF^o and each C_i and each D_i (see the definition of CNF^o) is an atom (either a variable or the negation of a variable). A wff is in CNF^{ss} iff it is both in CNF^{oo} and in CNF^s.

Lemma 11. Any wff is PI^s-equivalent to a wff which is in CNF^s.
Proof: from Lemma 10 and Lemma 4.

Corollary 5. Any PI^s-non-theorem is PI^s-equivalent to a wff which is in CNF^{ss}.

Theorem 11. If a PC-theorem A is not a PI^s-theorem, then the system resulting from adding A as an axiom to PI^s is equivalent to PC.
Proof. Suppose A is a PC-theorem and not a PI^s-theorem, and let

(11) First write down all assignments of ones and zeros to all subwffs and to all negations of subwffs, next delete the assignments that do not agree with the PI^s-semantics.

$PI^s + A = PI^z$. It follows from Corollary 5 that at least some wff B of either of the following forms

(6.13) $D_1 \vee \ldots \vee D_k$
(6.14) $(C_1 \& \ldots \& C_m) \supset (D_1 \vee \ldots \vee D_k)$

in which each C_i and D_i is an atom, is (i) a PC-theorem, (ii) a PI^z-theorem and (iii) not a PI^s-theorem. As the $\vee - \sim -$fragments of PC and PI^s are identical, B must be of the form (6.14). From the fact that B is not a PI^s-theorem we derive: (i) no D_i and D_j are such that $D_i = \sim D_j$, (ii) no C_i and D_j are such that $C_i = D_j$. Trivial considerations on PC-truth-tables tell us that under these conditions there is a C_i and a C_j such that $C_i = \sim C_j$. Consequently, the following substitutions can consistently be performed:

- if C_i is a variable that does not occur in the implicatum, substitute p for this variable,
- if C_i is a variable that occurs in the implicatum, substitute $\sim q$ for this variable,
- if C_i is the negation of a variable that does not occur in the implicatum, substitute p for this variable,
- if C_i is the negation of a variable that occurs in the implicatum, substitute q for this variable,
- if D_i is a variable, substitute q for it, and
- if D_i is the negation of a variable, substitute $\sim q$ for it.

It is readily seen that the result of these substitutions is PI^s-equivalent to $(p \& \sim p) \supset q$. Hence this wff is a PI^z-theorem.
Consequently, by Lemma 1, PI^z is equivalent to PC.

Corollary 6. PI^s is a maximally regular paraconsistent logic.

Theorem 12. If A is not a PC-theorem, the system resulting from adding A to PI^s is trivial.
Proof: Suppose that A is not a PC-theorem and that $PI^z = PI^s + A$. It follows from Corollary 5 that at least some wff B of the form of (6.13) or (6.14), in which each C_i and D_j is an atom, is a PI^z-theorem and not a PC-theorem. *First case:* let B have the form (6.13). As B is not a PC-theorem, there are no C_i and C_j such that $C_i = \sim C_j$. Hence the following substitutions can consistently be performed:

- if C_i is a variable, substitute p for it, and
- if C_i is the negation of a variable, substitute $\sim p$ for this variable.

The result is readily seen to be PI^s-equivalent to p. Hence p is a PI^z-theorem; hence PI^z is trivial. QED. *Second case*: let B have the form (6.14). From the fact that B is not a PC-theorem we derive (i) no C_i and C_j are such that $C_i = \sim C_j$, (ii) no D_i and D_j are such that $D_i = \sim D_j$, and (iii) no C_i and D_j are such that $C_i = D_j$. Under these conditions the following substitutions can consistently be performed:

- if C_i is a variable, substitute $\sim p$ for it,
- if C_i is the negation of a variable, substitute p for this variable,
- if D_i is a variable, substitute p for it, and
- if D_i is the negation of a variable, substitute $\sim p$ for this variable.

The result is readily seen to be PI^s-equivalent to p. Hence p is a PI^z-theorem. Hence PI^z is trivial. QED.

Corollary 7. PI^s is a maximal paraconsistent logic.

Theorem 13. The addition of the rule of replacement of equivalents to PI^s results in PC.

Proof. First notice that $\sim\sim(p \supset q) \equiv \sim(p \,\&\, \sim q)$ is PI^s-invalid and hence not a theorem of PI^s, but that it is derivable in PI^s by means of the rule of replacement of equivalents. From this the theorem follows in view of Theorem 11.

Corollary 8. The rule of replacement of equivalents is not derivable in PI^s.

In justification of a claim made in section 3 I now prove:

Theorem 14. The addition of $(p \vee q) =_{df} ((p \supset q) \supset q)$ to PI^s results in PC. ([12])

Proof. The following series of wffs is a proof in $PI^s + D\vee$ (the aforementioned definition):

(6.15)	$\sim(\sim p \vee \sim q) \supset p$	PI^s-theorem
(6.16)	$\sim((\sim p \supset \sim q) \supset \sim q) \supset p$	$D\vee$

The latter wff, however, is PI^s-invalid, and, hence, not a PI^s-theorem. Consequently, in view of Theorem 11, $PI^s + D\vee = PC$.

The system Δ of S.K. Thomason (197 +) is a sybsystem of PI^s, arrived at by a simple restriction on the wffs. The set of Δ-wffs may be defined as follows: (i) A is a pre-wff of Δ iff it is a wff of the

([12]) The disjunction sign occurring in the axioms is then to be considered as defined.

& – ∨ – ∼ – fragment of PC, (ii) A is a wff of Δ iff it is of the form B ⊃ C and both B and C are pre-wffs of Δ. In view of the semantic characterizations of Δ and PI^s it is obvious that A is a Δ-theorem iff it is a Δ-wff and a PI^s-theorem.

7. *Material* PI-*extensions containing* PC

The logics studied in the preceeding section are strictly paraconsistent, but neither of them is material. A weak material PI-extension, let us call it PI^o, is arrived at by adding to PI:

PI^oA1 $(p \& q) \supset (\sim(p \& q) \supset r)$

and to the PI-semantics:

$C6^m$ $v(\sim(A \& B)) = 1$ iff $v(A \& B) = 0$.

If we now define a strong negation:

d⌐ $\neg p =_{df} \sim(p \& p)$

then we obviously have:

Theorem 15. PI^o contains PC.

Let us pass at once at a very strong such system, which I shall call PI^v because it is nearly identical to V1, a logic forged by Ayda I. Arruda after ideas on the «imaginary logic» by Nicholas Alexandrovic Vasil'ev (see Arruda (1977)). The difference between V1 and PI^v is that the role played in V1 by the set S of «propositional letters of Vasil'ev» is played by the set of propositional variables in PI^v. PI^v may be characterized syntactically by adding to PI *either* the axiom schema

PI^vAS $A \supset (\sim A \supset B)$ if A is not a variable

or the following four axioms:

PI^vA1 $\sim p \supset (\sim\sim p \supset q)$
PI^vA2 $(p \vee q) \supset (\sim(p \vee q) \supset r)$
PI^vA3 $(p \& q) \supset (\sim(p \& q) \supset r)$
PI^vA4 $(p \supset q) \supset (\sim(p \supset q) \supset r)$

Arruda defines the PC-negation as follows (in my notation):

$$\neg p =_{df} \sim p \& \sim (p \& \sim p)$$

This definition is unnecessarily complicated in comparison with D$\neg$, but it is intuitively appealing.
PI^v is characterized semantically by adding to the PI-semantics

$C5^v$	$v(\sim\sim A) = 1$ iff $v(\sim A) = 0$
$C6^v$	$v(\sim(A \& B)) = 1$ iff $v(A \& B) = 0$
$C7^v$	$v(\sim(A \vee B)) = 1$ iff $v(A \vee B) = 0$
$C8^v$	$v(\sim(A \supset B)) = 1$ iff $v(A \supset B) = 0$

I refer to Arruda's aforementioned paper for a detailed study of PI^v and its semantics. I only mention and add here some metatheorems that seem interesting with respect to the kind of investigation I present.

Theorem 16. PI^v is consistent and strongly complete with respect to its semantics.
Proof. See the appendix (see also Arruda (1977,11)).
The proof of Theorems 17-20 is trivial.

Theorem 17. PI^v is decidable (see also Arruda (1977, 11-12)).
Theorem 18. PI^v is material.
Theorem 19. PI^v is paraconsistent but not strictly paraconsistent.
Theorem 20. PI^v contains PC.

In contradistinction to what holds for PI^s, even the $\vee - \sim -$fragments of PI^v and PC are not identical; PC is arrived at by identifying its negation with the defined strong negation of PI^v.

I now define: a wff is in CNF^v iff it is a continuous conjunction $B_1 \& \ldots \& B_n$ in which each B_i is a continuous disjunction $C_1 \vee \ldots \vee C_m$ fulfilling the following conditions: (i) each C_i is either an atom or the strong negation of an atom, and (ii)) no C_i PI^v-implies a different C_j (i.e., (ii/i) if $i \neq j$, $C_i \neq C_j$, (ii/ii) where D is a variable, if $C_i = D$, then no $C_j = \neg\sim D$, and (ii/iii) where D is a variable, if $C_i = \sim D$, then no $C_j = \neg D$). A wff is in CNF^{vv} iff it is in CNF^v and no conjunct B_i is a PI^v-theorem.

Lemma 12. Any wff is PI^v-equivalent to a wff which is in CNF^v.
Proof. Trivial in view of Lemma 3 and the following equivalences:

(7.1) $\neg A \equiv \sim A$ if A is compound (not a variable)
(7.2) $\sim\sim A \equiv A$ if A is compound.
(7.3) $\sim\sim p \equiv \neg\sim p$
(7.4) $\sim\neg p \equiv p$
(7.5) $(p \supset q) \equiv (\neg p \vee q)$
(7.6) $\sim(p \supset q) \equiv (p \,\&\, \neg q)$
(7.7) $\sim(p \,\&\, q) \equiv (\neg p \vee \neg q)$
(7.8) $\sim(p \vee q) \equiv (\neg p \,\&\, \neg q)$
(7.9) $(\sim p \vee \neg p) \equiv \sim p$
(7.10) $(p \vee \neg\sim p) \equiv p$

Corollary 9. Any PI-non-theorem is PI^v-equivalent to a wff which is in CNF^{vv}.

Theorem 21. If a PC-theorem A is not a PI^v-theorem, then the system obtained by adding A as an axiom to PI^v is equivalent to PC.
Proof. Consider a wff which is in CNF^{vv} and is equivalent to A. Some conjunct $B_i (= C_1 \vee \ldots \vee C_m)$ of this wff is a PC-theorem and not a PI^v-theorem. But then there obviously is a C_i, a C_j and a variable D such that $C_i = \neg D$, $C_j = \neg\sim D$ and D does not occur in any other C_k. As a consequence, it is possible to substitute p for D and perhaps for some other variables, and to substitute $\sim p$ for other variables in such a way that the result is equivalent to $\neg p \vee \neg\sim p$. From this $(p \,\&\, \sim p) \supset q$ is derivable. Hence the system obtained by adding A as an axiom to PI^v is equivalent to PC (by Lemma 1).

Corollary 10. PI^v is a maximally regular paraconsistent logic.

Theorem 22. If a wff A which is not a PC-theorem is added as an axiom to PI^v, then the resulting system is trivial.
Proof. Consider a wff which is in CNF^{vv} and is equivalent to A. Some conjunct $B_i (= C_1 \vee \ldots \vee C_m)$ of this wff is not a PC-theorem, and as a consequence no variable occurs in two different disjuncts C_i and C_j. Hence it is possible to perform a number of substitutions such that the result is equivalent to p. Hence the system obtained by adding A as an axiom to PI^v is trivial.

Corollary 11. PI^v is a maximal paraconsistent logic.

In support of a claim made in section 4 I now prove:

Theorem 23. The rule 'If $\vdash A$ and $\vdash \sim A \vee B$, then $\vdash B$.' holds true in PI^v.

Proof: Suppose that $\vdash A$ and $\vdash \sim A \vee B$. It follows that $v(A) = v(\sim A \vee B) = 1$ for all v. As no variable is a PI^v-theorem, $v(\sim A) = 0$ for all v. But then, for all v, $v(B) = 1$ and hence $\vdash B$.

I was somewhat amazed to find out that there is another PI-extension which is both a *material* logic and a *maximally regular paraconsistent* logic (and a maximal paraconsistent logic). This logic, let us call it PI^m, is characterized syntactically by replacing the PI-axiom PI^vA1 by

$PI^mA1 \qquad \sim\sim p \equiv p$

and semantically by making the analogous replacement. The obvious reason for the existence of two material PI-extensions which are both maximally regular, is that A as well as $\sim A$ are parts of $\sim\sim A$, and hence that the value of $v(\sim\sim A)$ may be determined in two equally material ways. PI^m is «somewhat more paraconsistent» than PI^v.

Let us now define: a wff is in CNF^m iff it is in CNF^v; a wff is in CNF^{mm} iff it is in CNF^m and no conjunct B_i is a PI^m-theorem. The analogues of the lemmas and theorems on PI^v may easily be proved for PI^m.

All of Da Costa's systems C_n $(1 \leq n < \omega)$, studied, e.g. in his (1974), turn out to be material logics *between* PI and PI^m. For these logics da Costa and Alves (1977) present a semantics, which contains the following clause:

$$\text{If } v(B^{(n)}) = v(A \supset B) = v(A \supset \sim B) = 1, \text{ then } v(A) = 0.$$

At first sight this does not look very material. However, one gets an equivalent semantics in replacing the clause by the much simpler

$$v(A^{(n)}) = 1 \text{ iff } v(A) = 0 \text{ or } v(\sim A) = 0.$$

The latter clause is not only more decent, it also shows, unlike the unnecessarily complicated former one, that the systems C_n $(1 \leq n < \omega)$ are material in view of the following definitions:

$A^o =_{df} \sim(A \,\&\, \sim A)$

$A^n =_{df} A^{oo\cdots o}$ (A followed by a times the sign ^{o})

$A^{(n)} =_{df} A^o \,\&\, \ldots \,\&\, A^n$

Incidently, da Costa's axiom scheme

$$B^{(n)} \supset ((A \supset B) \supset ((A \supset \sim B) \supset \sim A))$$

may be replaced by

$$B^{(n)} \supset ((B \,\&\, \sim B) \supset A).$$

As this reformulation makes clear, $B^{(n)}$ expresses that B behaves consistently; its behaving inconsistently is sanctionned by triviality.

The system DL of Da Costa and Wolf (197–) is not an extension of PI (excluded middle fails); the system KP of Apostel (1979) is an extension of PI but not a sublogic of PI^s, PI^v or PI^m, the specific axioms of KP containing propositional quantifiers.

8. *Some further* PI-*extensions*

I have been looking for a maximally regular material and strictly paraconsistent PI-extension, and I guess that such a system is characterized semantically by adding to the PI-semantics:

C5′ $v(\sim\sim A) = v(A)$

C6′ $v(\sim(A \supset \sim B)) = v(A \,\&\, B)$

C7′ $v(\sim(\sim A \,\&\, \sim B)) = v(A \vee B)$

C8′ $v(\sim(\sim A \vee \sim B)) = v(A \,\&\, B)$

I leave it to the reader to find a syntactical characterization of this logic (trivial in view of the results mentioned in the appendix). Obviously this logic is regular, material and strictly paraconsistent; I have no proof, however, that it is maximally so (i.e. that no extension of this logic is regular and material and strictly paraconsistent).

Let it be noticed also that there are non-regular paraconsistent and even strictly paraconsistent PI-extensions. An example of such a logic is obtained by adding «Aristotle's thesis», viz. $\sim(p \supset \sim p)$ (*cf.* Routley (1979)) to PI. Such a logic might even be material, as may be seen from the system arrived at by adding to the PI-semantics:

If $v(A) = v(B)$, then $v(\sim A \supset \sim B)) = 1$.

I mention this in order to show that Aristotle's thesis, which seems to presuppose at least a relevant implication, may very well be added to a system in which the implication is plainly *material* (and even to a material system). The semantic clause mentioned above seems even intuitively appealing.

Notice that there are also paraconsistent logics in which it is stated explicitly that at least one contradictory sentence is true, e.g., by having one of the following as an axiom (where p_0 is a propositional constant):

(8.1) $p_0 \& \sim p_0$

(8.2) $(\exists p)(p \& \sim p)$

Systems containing (8.1) are studied, e.g. by Routley and Meyer (1976), Routley (1979) and Arruda (1977). The addition of (8.1) or (8.2) to some logical system may of course have a technical use, e.g., to show that the system is indeed paraconsistent. It seems to me, however, that there are philosophical objections against having (8.1) or (8.2) as an axiom of a logical system as such. That some set of formulas is correctly considered a *logic* presupposes, among other things, that it is closed under substitution for propositional variables; this is, as Anderson and Belnap (1975, 462) say, «what makes it a logic». However, if this is correct, it is hard to see how (8.1) may be considered a theorem of logic. Furthermore, where (8.1) is a theorem of L, it cannot be related in a meaningful way to a theory $T = \langle \alpha, L \rangle$ unless p_0 is replaced by some sentence of the language in which T is formulated. In this case, however, the contradiction should obviously derive from T and not from L alone. Next, consider a logic L of which neither (8.1) nor (8.2) (nor something like) is a theorem, and let L′ be L + (8.2). If some contradiction is derivable from $T = \langle \alpha, L \rangle$, then (8.2) is superfluous anyway; and if it is derivable from T that there are true contradictions (even if none is actually derivable from T), then again (8.2) is superfluous. And there are still further arguments to the effect that (8.1) and (8.2), whether superfluous or not, are objectionable in general. Indeed, it is most natural to consider a logic as a theory of meaning of certain terms, defining some set of «formally» correct rules of inference. A logic may involve certain presuppositions about the world or about the domain described by the theory of which the logic is an element. PI does not presuppose that the world (or some domain) is consistent; it even presupposes that the world is inconsistent, in that it is too weak for consistent worlds (it fails to sanctio certain inferences that are correct with respect to consistent worlds) These presuppositions are presuppositions of any theory of meaning of the PI-connectives. But why then should it be added furthermore as a *theorem of logic* that the world is inconsistent? Nothing is added to the meaning of the connectives as defined by PI or some («normal») PI-extension if (8.1) or (8.2) are taken as axioms; neither do these have any bearing on the correctness of inferences.(13)

If an axiomatic system for a paraconsistent logic does not contain axioms such as (8.1) or (8.2), then there is no reason why disjunctive syllogism, i.e. (4.2) or (4.2bis), should not be a primitive or derivable deduction rule, provided that the corresponding rule of inference, viz. (4.3), does not hold.

Finally, it is obvious that PC can be combined with PI and with any of its extensions (and not only with those that already contain PC) in the following way. Let PIC be PIA1-PIA9 together with

PIA10′ $\neg p \supset {\sim} p$
PCA1 $p \vee \neg p$
PCA2 $(p \,\&\, \neg p) \supset q$

and the rules of modus ponens and uniform substitution. The $\sim - \supset - \& - \vee -$ fragment of PIC is PI, the $\neg - \supset - \& - \vee -$ fragment is PC. The semantics is arrived at by adding to that for PI:

C^c7 $v(\neg A) = 1$ iff $v(A) = 0$.

(see the appendix). Any PI-extension can be combined with PC in the same way.

9. System with explicit truth-predicate – the primacy of consistency

Let Vp express that p is true (V from the Latin 'verum'). The set of wffs of the system V is the smallest set S such that (i) if A is a PC-wff, then VA $\in$ S. The set of V-theorems is the set of V-wffs that are PC-theorems (where PC is trivially extended to V-wffs). Consider now the following definitions and axiom:

(13) The same objection does not apply to the use of the logical constant f in systems in which f may be interpreted, e.g., as the conjunction of all sentences.

DV ⊃ $V(p \supset q) =_{df} Vp \supset Vq$

DV & $V(p \& q) =_{df} Vp \& Vq$

DV ∨ $V(p \vee q) =_{df} Vp \vee Vq$

DV ~ $V \sim p =_{df} \sim Vp$

VIA1 $\sim Vp \supset V \sim p$

The systems VC and VI are as follows:

VC = V + DV ⊃ + DV & + DV ∨ + DV ~
VI = V + DV ⊃ + DV & + DV ∨ + VIA1

We then have: $\vdash_{PC}$ A iff $\vdash_{VC}$ VA and $\vdash_{PI}$ A iff $\vdash_{VI}$ VA.

Whereas V is obviously redundant in VC, it is not in VI. Still, VC enables us to clarify the relation between PC and PI. First of all, the same notion of truth (in the sense of V) underlies PC as well as PI. Next, the difference between DV ~ and VIA1 expresses exactly the difference between consistent and inconsistent worlds, viz, that ~p is true in a consistent world exactly in case p is not true in it, whereas both p and ~p may be true in an inconsistent world, and at least one of them is. Furthermore, the «V-formulations» of PI and its extensions enable us to translate literally the semantic clauses into a syntactical system. There is more behind this than just a technical matter. Any semantics presupposes a logic, viz. the logic that defines the meaning of the logical terms *used* in the semantical metalanguage. In this paper I always used a semantical metalanguage (for PI and its extensions) in which the meaning of the connectives is defined by PC. To this logic corresponds the system V. It is then trivial that any semantical clause expressed in the aforementioned metalanguage corresponds literally to a V-wff: $v(A) = 1$ corresponds to VA and $V(A) = 0$ to ~VA. All semantic clauses of the PC-semantics and PI-semantics are translated in this way into definitions or axioms of VC and VI. The same holds for all PI-extensions.([14]) Consider:

([14]) Semantic characterizations of V and its extensions may be obtained as follows: consider, e.g., the valuation functions of the PI^s-semantics as defining an assignment to pre-wffs, and extend it in such a way that it constitutes an suitable assignment to wffs too.

VI^vAS1. Where A is not a variable, $V \sim A \equiv \sim VA$.
and let $VI^v = VI + VI^vAS1$. then we have $\vdash_{PI^v} A$ iff $\vdash_{VI^v} VA$.
I leave it to the reader to formulate VI-extensions that correspond in the same way to other PI-extensions. Incidentally, it is trivial that any VI-extension determines (in the aforementioned way) one PI-extension, although some PI-extensions correspond to more than one VI-extension.(15)

There is a further point which I think to be philosophically important. Expressed vaguely, inconsistencies never occur at the «highest level»; however inconsistent some theory, it may always be consistently described in a metalanguage. Let us illustrate this by VI. Even if the world is inconsistent in that, for some p, both p and $\sim p$ are true, we still may (and should – see below) define some «predicate» 'true' in such a way that we never have it that p is both true and false (not true). Now, of course, one might define V in a redundant way in PI too ($Vp =_{df} p$), and in this way arrive at a logic in which $Vp \& \sim Vp$ is not a contradiction. But still, it is completely trivial that we may define a further term, say V^o, that behaves consistently in that $V^oVp \& \sim V^oVp$ is a logical falsehood, although $V^oVp \& V^o \sim Vp$ is not. Let me now show why one should introduce some notion of truth – the name of course does not matter – that behaves consistently.

Suppose that both p and $\sim p$ are true about some domain (true about «the world», or derivable from some «mathematical» or «empirical» theory). We shall adopt some paraconsistent logic in order to describe this domain, in order to formulate a theory about it. However, we only have a theory about some domain, we only have a description of some domain, if some sentences about the domain are not derivable from this theory. Notice that the 'not' should be a strong negation here. We have only a theory about some domain if some sentences are not derivable from it in this sense of 'derivable' in which 'p is and is not derivable' is logically false. The point is that one may describe an

(15) Let VI^c be the result of adding $V((p \& \sim p) \supset q)$ as an axiom to VI. We then have it that A is a PC-theorem iff VA is a VI^c-theorem, *and* that the latter is true iff VA is a VC-theorem. Nevertheless, VI^c is weaker than VC, as $V \sim p \supset \sim Vp$ is derivable in VC but not in VI^c. VC may be said to presuppose that the world is consistent, whereas VI^c may be said to presuppose that it is either consistent or trivial. This difference can be expressed semantically, but it cannot be expressed syntactically by a purely propositional logic (not containing, e.g., the predicate V).

inconsistent domain, but that something may be called a description only if its metatheory is consistent. To put it in another way, suppose that the metatheory MT of «theory» T is inconsistent (some p being both derivable and not derivable from T, being both true and false according to T). Then T is not a description of some domain. Still T might be interesting in itself, and hence it might be interesting to describe it by MT. But MT will only be a (complete or incomplete) description of T, if MT may be described consistently by some MMT, i.e. metatheory of MT. All this sounds quite old stuff. To whatever extent one might disagree with some of Popper's views, I cannot see how one could disagree with his basic insight that only those theories are informative that «forbid» something. (I return on the matter in section 10). Incidently, it is clear without further argument that the theory of truth connected with PI and its extensions in the same way as Tarski's theory of truth is connected with PC, is as Tarskian as inconsistent worlds allow it to be (the point has been made more generally by da Costa).

I finally add a remark on the strong negation. In some systems, such as PIC, $\neg$ is defined implicitly, in others, such as PI^v, it is defined explicitly. In the corresponding VI-extensions we get either as derivable or as primitive:

DV$\neg$ $\qquad V \neg p =_{df} \sim Vp.$

It seems to me that this leads to the important remark that $\neg$ is indeed the strong negation in that $\neg p$ expresses that p is false (is not the case, or what you have), i.e. if $\neg p$ is asserted, then p cannot be consistently asserted any more. Let us compare this with the negation in systems such as PI^s. PI^s has the same $\sim - \& - \vee -$ fragment as PC and in this sense contains PC. Hence $\sim(p \& \sim p)$ is a theorem of PI^s. Nevertheless, $\sim$ is not the strong negation. The easiest way to see this is by comparing the PI^s-theorem $\sim(p \& \sim p)$ with $\neg(p \& \neg p)$, which is, e.g., a theorem of Pv^v. In VI^s and VI^v these correspond respectively to :

(9.1) $\qquad V \sim(p \& \sim p)$

(9.2) $\qquad \sim(Vp \& V \neg p)$

Whereas (9.2) states that p and $\neg$p are not bot true, (9.1) does not state more than that $\sim(p \& \sim p)$ is a theorem. The only way to analyze (9.1) further is to write it as $V\sim p \vee Vp$, but this again has nothing to do with strong negation (as it even holds in VI). This shows that it is somewhat misleading to call, as Routley (1979) does, the wff $\sim(p \& \sim p)$ 'Aristotle's principle', and that one should be careful with drawing conclusions from the fact that this wff is a theorem of a paraconsistent logic (no reasonable reading of Aristotle agrees with the mere theoremhood of $\sim(p \& \sim p)$). ([16])

10. Some philosophical considerations

In PC all connectives are truth-functions and the same holds for the binary connectives in PI and its extensions. Negation, however, is not a truth-function in PI (although the PI-semantics is plainly truth-functional). Yet, some occurences of negations, viz. those occurring in front of compound wffs, behave truth-functionally in such PI-extensions as PI^s, PI^v and PI^m. In my view the philosophical point to this is the following. For any domain we think about, we may distinguish between descriptions of «facts» of the domain on the one hand, and (results of) «operations» on descriptions of facts on the other hand. By 'operations' I mean mental operations of connecting in some way or other statements about facts. I realize quite well that the distinction is put forward in a naive guise, but I prefer to do so because it seems to me that in this way my argument will be acceptable to everyone, irrespective of his or her philosophical views on the distinction under consideration. Given the distinction, it follows at once that negation functions in PC as an operation, whereas negation does not in general function as (the expression of) a mere operation in PI and its extensions; yet, in some of the extensions some occurrences of negations (see above) function as such.

([16]) Routley (1979) also writes: «In its semantical formulation the principle asserts that no statement is both true and false, that is, that it is never the case that $I(A, T) = 1$ and $I(A, T) = 0$, a point guaranteed by the bivalent features of the semantics.» This, however, simply means that his semantics is a description (in the sense explained some paragraphs ago in the next) of the logical system under consideration.

The matter can perhaps best be illustrated by means of an example. Consider a language in which we describe our observations in a given domain. If our observational criteria are such that for any object a and for any predicate P we are able to find out in a clearcut way, be it only under certain conditions, whether or not the object has the property P, then ~Pa will simply be used to express that a has not that property. Here negation functions as (the expression of) an operation, ~Pa stating that the fact expressed by Pa is not the case (just as Pa & Qa states that both the facts expressed by Pa and Qa are the case). There is no way in which a contradiction could ever arise in such a context (I disregard problems concerning time for the sake of the example). Suppose, however, that we do not dispose of unique criteria for determining whether or not an object has a certain property, e.g., because the criteria are multiple or because the meanings of the predicates of the language are related to each other in such a way as to lead, for some «observational» predicates P and Q, to a so-called meaning postulate from which it follows that ~Pa is true whenever Qa is true. In such cases it is quite possible that observations lead us to the conclusion that both Pa and ~Pa. It is clear at once then that ~Pa expresses a fact and is not the result of applying an operation on Pa. Indeed, in the cases under consideration Pa and ~Pa are accepted on the basis of different observations. Hence, the negation in '~Pa' cannot be an operation on Pa in the sense that the very criterion that might lead us to the conclusion that Pa, has lead to a negative result; for indeed, at least one criterion, be it direct or indirect, that was used to check whether or not Pa, led to a positive result (from which Pa was concluded). Incidently, it seems preferable in view of the preceding result to describe the situation by saying that both the fact that a has property P and the fact that a has property ~P are the case, rather than unnecessarily messing up our ontology by saying that the fact that a has property P both is and is not the case.

I have to add three further remarks. Even in languages in which the strong negation occurs, we may have a multiplicity of observational criteria for one and the same predicate, and the meanings of the predicates may be related in the way indicated in the preceding paragraph. However, as long as the descriptions of our observations behave consistently, it may be said that negation behaves as an operation, for if we conclude, by whatever means, to either Pa or

$\sim$Pa, then it may be said that some criterion, however complex, *would* have lead to a positive, respectively negative result. The second remark concerns the fact that p V $\sim$p is a theorem of PI. As a consequence, negation in PI functions partially as an operation in that a negative result of a criterion to check for Pa will always lead to the conclusion that $\sim$Pa (and *vice versa*). But this simply means that PI and its extensions do not admit all sets of observational criteria as suitable (or well-defined); hence, if the set of observational criteria leads to results incompatible with PI, we might be forced to move to a still weaker logic. Finally, what about the truth-functional behaviour of negations occurring in front of compound expressions in such PI-extensions as PI^v or PI^s? With respect to the present kind of example the use of such logics may be best seen as follows. Suppose that all elementary observational statements are expressed by primitive sentences of the language (as in the ideal Wittgensteinian case) or by negations of such. Even if the primitive sentences behave inconsistently, we may then, by rejecting the rule of replacement of equivalents, EQ, decide to consider negations of compound sentences as the result of operations on these sentences (as in PI^v) or as operations on related sentences (as in PI^s). The expense is clear, and the fact that all designers of paraconsistent systems came up with negations that partially behave as operations, suggests that none of them was aware of this expense.

It should be clear by now that contradictions may be true, as long as a weak negation is involved. Also, it is not obvious that inconsistent (non-trivial) theories may be replaced by consistent ones «of the same richness», or that a set of observational criteria that lead to some inconsistent observational reports, may be replaced by an equally adequate set that leads to consistent observational reports only. Hence the use of paraconsistent logics. There is, however, also an expense at employing a weak negation as I shall now try to make clear.

I have already referred to the problem of falsification in section 9. It is obvious that, where the tilde denotes the weak negation of some paraconsistent logic (whether based on a relevant or a material implication), $\sim$p cannot be used to express that p is false, for the truth of $\sim$p does not logically exclude that p too is true. As a consequence, we need either the strong negation or some term like V, in front of which negation functions as strong negation, to express that some sentence is false. Hence the use of such paraconsistent logics as PI^v, PI^m, VI (and its extensions), and PIC (and its extensions). In all such logics the strong negation is such that it turns (strongly) inconsistent sets of sentences into trivial such sets. This need not be the case for a paraconsistent logic based on a relevant implication, but it is clear that a strong negation should be added to such systems in order to express syntactically that some sentence is false. Routley and Meyer disregard this problem completely. Routley (1979) shows that «dialectical logic, DKQ, has a classical truth definition», viz. formulated in the semantical metalanguage, but still it is impossible to state within languages that have the structure of DKQ that some sentence is false.

The possibility of expressing syntactically the falsehood of a sentence seems especially important with respect to the problem of falsification. In the absence of a strong negation, no sentence will ever lead to the rejection (under whatever conditions on auxiliary hypotheses, etc.) of some theory T. Where $\& \alpha$ is the conjunction of the axioms of T, such logics as Routley and Meyer's DL might enable one to derive $\sim \& \alpha$ from some set of sentences, but $\sim \& \alpha$ does not express that $\& \alpha$ is false. (Notice that $\sim \& \alpha$ is derivable from any $T = \langle \alpha, L \rangle$, whenever $Cn_L(\alpha)$ is inconsistent.) Let me just give one example of the way in which VI *might* be interpreted in connection with the problem of falsification. The axioms of T might be considered as given in the form Vp, and the so-called meaning relations as given in such forms as $V(Ax) \supset V(\sim Bx)$. The observational criteria might be considered to lead to such conclusions as Vp or $\sim$Vp. How falsification arises in such cases is then obvious. It is easy to see that the case is analogous for PI-extensions that contain strong negation. Especially the PIC-extensions seem attractive in this connection.

I add three further remarks. First, the typical VC-theorem $V \sim p \supset \sim Vp$ cannot be expressed in a theory T under the above interpretation, as all theorems of T are of the form Vp. Next, even if the underlying logic is PI, it might be stated in a theory T that some sentence (or predicate) behaves consistently, e.g., where P is a sentence, by a theorem of the form $(P \& \sim P) \supset q$. Finally, under an interpretation as the above one, it turns out possible to falsify directly a logically true sentence. Indeed, if e.g., the underlying logic of T is VC, then the «acceptance» on observational grounds of both Vp and

$V \sim p$ (for some sentence p) leads to the very negation of a theorem of logic. Of course, several changes to T are possible ways out, but one at hand is the transition from VC to some weaker VI-extension.

It seems to me that the preceding paragraphs lead to a conclusion which goes far beyond the problem of paraconsistent logics, viz. that it is the deviser of some theory T who has to specify the underlying logic of T. The choice of an underlying logic determines indeed the set of consequences of T; it is $Cn_L(\alpha)$, and not the set α of axioms alone, that determines what the theory states. The underlying logic defines the logical structure of reality *as described* (and describable) by a given language and under the given observational criteria. In this sense the logic has factual implications. It is up to the deviser of some theory, not to some logician, to determine what he intends the theory to state. All the logician can do in this respect is to develop new logics, starting from logical or extra-logical problems, and to show the possible use of these logics.

11. Appendix

The proofs of the consistency of the axiomatic systems mentioned in this article with respect to the corresponding semantic systems (if $\alpha \vdash A$, then $\alpha \models A$) are left to the reader. They all proceed as for PC. The strong-completeness proofs (if $\alpha \models A$, then $\alpha \vdash A$) are obvious in view of my (1980). I only mention, for each system, the properties of the $\gamma \in \Gamma$ (see lemma 6 of my (1980)).

For PI:

1. $A \in \gamma$ iff $\gamma \vdash A$.
2. For some A, $A \in \gamma$.
3. $(A \supset B) \in \gamma$ iff $A \notin \gamma$ or $B \in \gamma$.
4. $(A \& B) \in \gamma$ iff $A \in \gamma$ and $B \in \gamma$.
5. $(A \vee B) \in \gamma$ iff $A \in \gamma$ or $B \in \gamma$.
6. If $A \notin \gamma$, then $\sim A \in \gamma$.

For PI*: 1 – 6 plus

7*. If $A \in \gamma$, then $\sim\sim A \in \gamma$.

8*. If $\sim A \in \gamma$ and $\sim B \in \gamma$, then $\sim(A \vee B) \in \gamma$.

For PI^s: 1 – 6 plus

7^s. $\sim\sim A \in \gamma$ iff $A \in \gamma$.
8^s. $\sim(A \supset B) \in \gamma$ iff $A \in \gamma$ and $\sim B \in \gamma$.
9^s. $\sim(A \& B) \in \gamma$ iff $\sim A \in \gamma$ or $\sim B \in \gamma$.
10^s. $\sim(A \vee B) \in \gamma$ iff $\sim A \in \gamma$ and $\sim B \in \gamma$.
For PI^v: $1-6$ plus
7^v. $\sim\sim A \in \gamma$ iff $\sim A \notin \gamma$.
8^v. $\sim(A \supset B) \in \gamma$ iff $(A \supset B) \notin \gamma$.
9^v. $\sim(A \& B) \in \gamma$ iff $(A \& B) \notin \gamma$.
10^v. $\sim(A \vee B) \in \gamma$ iff $(A \vee B) \notin \gamma$.
For PI^o: $1-6$ plus 9^v.
For PI^m: $1-6$ plus 7^s, $8^v - 10^v$.
For PIC: $1-6$ plus
7^C $\neg A \in \gamma$ iff $A \notin \gamma$.

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