An Adaptive Characterization of Signed Systems for Paraconsistent Reasoning*

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Abstract

In this paper we characterize the six (basic) signed systems from [18] in terms of adaptive logics. We prove the characterization correct and show that it has a number of advantages.

1 Aim of This Paper

In [18], signed propositional systems for paraconsistent reasoning were introduced and six central consequence relations were defined and studied. Three of the consequence relations are called signed, the three others are called unsigned. Some further consequence relations were defined in [18] and extended and studied in later works, for example [19].

In this paper we characterize the six central consequence relations in terms of adaptive logics. Doing so has a number of advantages. First, it avoids several complications (the preparation of the premises in negation normal form, their translation to a signed language, reasoning in terms of extensions) and hence makes the consequence relations more transparent. Next, it provides them with dynamic proofs, with a characteristic semantics, and with decision methods at the propositional level. Third, it makes several extensions obvious—we shall present the extension with a detachable implication and the extension to the predicative level. The extensions are absolutely straightforward whereas they are tiresome (if at all possible) on the signed approach. Even at the predicative level, there are partial decision methods and criteria (see [14] and [15] for tableau methods and [11] for procedural proofs). Fourth, it gives the consequence relations a place in a unified framework, which facilitates the comparison with other inconsistency-handling logics. Fifth, it provides them with easy proofs of many metatheoretic properties.

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After introducing some notational conventions in Section 2, we present the consequence relations in Section 3. For reasons that become clear later, we shall deviate from the original definitions for the prudent consequence relations. We introduce the required paraconsistent logic in Section 4 and explain the working of adaptive logics in standard format and introduce the required adaptive logics in Section 5. In Section 6, the signed systems are characterized by adaptive logics and the characterization is proved adequate. In Section 7 the original prudent consequence relations are discussed. The unsigned one coincides with the modified one presented in Section 3; the signed one will be shown to be defective (but will nevertheless be characterized by an adaptive logic). In Section 8, the consequence relations are extended, first to a language including the detachable implication of **CL**, next to the full predicative language. Finally, in Section 9, we briefly survey what was realized and comment on the consequence relations and the signed approach.

We shall deviate in several respects from the symbolism used in [18] and [19]. Mere notational differences will not be mentioned, but only clarified where this is useful.

2 Notational Conventions

$\mathcal{S}$ will be the set of sentential letters[1] and $\mathcal{S}^{\pm} = \{\sigma^+, \sigma^- \mid \sigma \in \mathcal{S}\}$. From now on we shall use σ (possibly with a subscript) as a metavariable for members of $\mathcal{S}$. A *literal* is a member of $\mathcal{S} \cup \{{\sim}\sigma \mid \sigma \in \mathcal{S}\}$. We shall say that a literal ${\sim}\sigma$ *occurs in* a formula A iff it is a subformula of A and that a literal σ *occurs in* A iff there is an occurrence of σ in A outside the literal ${\sim}\sigma$. So the literals ${\sim}p$ and q occur in ${\sim}p \vee q$, but the literal p does not occur in ${\sim}p \vee q$.

We shall need three propositional languages, which we characterize in the following table

language	letters	connectives	set of formulas
$\mathcal{L}$	$\mathcal{S}$	${\sim}$, $\neg$, $\wedge$, $\vee$, $\supset$, $\equiv$, $\rightarrow$, $\leftrightarrow$	$\mathcal{W}$
$\mathcal{L}^*$	$\mathcal{S}$	${\sim}$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$	$\mathcal{W}^*$
$\mathcal{L}^{\pm}$	$\mathcal{S} \cup \mathcal{S}^{\pm}$	${\sim}$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$	$\mathcal{W}^{\pm}$

Occasionally we shall need $\mathcal{W}^{\not\neg}$ and $\mathcal{W}^{\not\sim}$ which comprise all members of $\mathcal{W}$ except for those in which occurs $\neg$, respectively ${\sim}$.

The duplicated connectives of $\mathcal{L}$ deserve a comment. The meaning of (the standard negation) ${\sim}$ varies according to the context, whereas $\neg$ always denotes classical negation, viz. the negation of **CL** (Classical Logic). The implication $\supset$ is a detachable implication in all contexts, whereas $A \rightarrow B =_{df} {\sim}A \vee B$. So $\rightarrow$ is not detachable where ${\sim}$ is paraconsistent.[2] Similarly, the equivalence $\equiv$ is detachable in both directions, whereas $A \leftrightarrow B =_{df} (A \rightarrow B) \wedge (B \rightarrow A)$.

As we consider different sets of formulas, let $\mathrm{Cn}^x_{\mathbf{L}}(\Gamma) = \{A \in \mathcal{W}^x \mid \Gamma \vdash_{\mathbf{L}} A\}$, in which $\mathbf{L}$ is a logic and x is either nothing or $*$ or $\pm$. For example, $\mathrm{Cn}^*_{\mathbf{CL}}(\Gamma)$ denotes the set of the **CL**-consequences of Γ that belong to $\mathcal{W}^*$.

[1] In [18] and [19], $\mathcal{S}$ is taken to be a finite set. We shall at once consider the general case where $\mathcal{S}$ is infinite.

[2] A negation † is paraconsistent (in a context) iff (in that context) $A, \dagger A \nvdash B$ for some A and B.

In Section 8, we shall consider the predicative case. So there we need the predicative extensions of the languages and sets of formulas. Before that section, we only consider the propositional case, whence, for example, **CL** refers to propositional classical logic.

3 The Signed Systems

The signed systems concern premise sets $\Gamma \subseteq \mathcal{W}^*$,[3] and require that Γ is transformed into $\Gamma^\pm$. Consider two relations on the set $\mathcal{W}^\pm$: A is a positive part of A; if A is a positive (negative) part of B, then A is a positive (negative) part of $B \wedge C$, of $C \wedge B$, of $B \vee C$, of $C \vee B$, and of $C \rightarrow B$; if A is a positive (negative) part of B, then A is a negative (positive) part of $\sim B$ and of $B \rightarrow C$. $A^\pm$ is obtained by replacing in A every σ that is a positive part of A by σ^+ and every σ that is a negative part of A by $\sim\sigma^-$.[4]

As is justly noted in [18], a more convenient approach proceeds in two steps: A is first transformed to its negation normal form (NNF), and $A^\pm$ is defined from this. The NNF of a formula $A \in \mathcal{W}^*$ is obtained by applying all of the following transformations to subformulas of A until no further application is possible: replace $\sim\sim B$ by B, $\sim(B \vee C)$ by $(\sim B \wedge \sim C)$, $\sim(B \wedge C)$ by $(\sim B \vee \sim C)$, $B \leftrightarrow C$ by $(B \rightarrow C) \wedge (C \rightarrow B)$, and $B \rightarrow C$ by $\sim B \vee C$. Obviously, the order of the transformations is immaterial. The resulting formula in NNF contains only literals, $\wedge$, $\vee$ and parentheses. Where B is the NNF of A, $A^\pm$ is obtained by replacing in B every literal σ by σ^+ and every literal $\sim\sigma$ by σ^-. Thus $(r \vee \sim(p \rightarrow q))^\pm$ is $r^+ \vee (p^+ \wedge q^-)$.[5] We shall follow the more convenient approach to $A^\pm$. Where $\Gamma \subseteq \mathcal{W}^*$, let $\Gamma^\pm = \{A^\pm \mid A \in \Gamma\}$. Obviously $\Gamma^\pm$ is a consistent set of formulas (because σ^+ and σ^- are different sentential letters).

The signed systems are defined in terms of $\mathrm{Ext}(\Gamma^\pm)$, the set of extensions of $\Gamma^\pm$. In [18] extensions are obtained by means of defaults. A simpler approach is this:

Definition 1 $\Delta \in \mathrm{Ext}(\Gamma^\pm)$ *iff there is a* $\Sigma \subseteq \mathcal{S}$ *such that (i)* $\Delta = \mathrm{Cn}^{\pm}_{\mathbf{CL}}(\Gamma^\pm \cup \{(\sigma^+ \leftrightarrow \sigma) \wedge (\sigma^- \leftrightarrow \sim\sigma) \mid \sigma \in \mathcal{S} - \Sigma\})$ *is consistent and (ii) for all* $\sigma \in \Sigma$, $\sim(\sigma^+ \leftrightarrow \sim\sigma^-) \in \Delta$.[6]

For the two prudent consequence relations, we shall slightly deviate from [18] in the present section, and discuss the original versions in Section 7. Where $\Delta \in \mathrm{Ext}(\Gamma^\pm)$, $\mathrm{nor}(\Delta) = \{(\sigma^+ \leftrightarrow \sigma) \wedge (\sigma^- \leftrightarrow \sim\sigma) \in \Delta \mid \sigma \in \mathcal{S}\}$ (the *normal part of* the extension Δ) and $\mathrm{nor}(\Gamma^\pm) = \bigcap\{\mathrm{nor}(\Delta) \mid \Delta \in \mathrm{Ext}(\Gamma^\pm)\}$ (the normal part of $\Gamma^\pm$). Let $\mathcal{T} = \{\sigma^+ \vee \sigma^- \mid \sigma \in \mathcal{S}\}$ (the $\mathcal{T}$ refers to *Tertium non datur*).

Definition 2 *Where* $\Gamma \subseteq \mathcal{W}^*$ *and* $A \in \mathcal{W}^*$, *the six consequence relations are defined as follows:*

[3] In [18] and subsequent papers, premise sets are required to be finite. We present at once the general case.

[4] Unlike what is suggested in [18], the definition of positive part and of negative part is not completely general in that $\leftrightarrow$ is not covered. If it were, both σ_1 and σ_2 would be a positive part as well as a negative part of $\sigma_1 \leftrightarrow \sigma_2$. So the definition of $A^\pm$ requires that $\leftrightarrow$ is eliminated from A.

[5] On the original definition from the previous paragraph in the text, $(r \vee \sim(p \rightarrow q))^\pm$ is $r^+ \vee \sim(p^+ \rightarrow \sim q^-)$, but this is indeed **CL**-equivalent to $r^+ \vee (p^+ \wedge q^-)$.

[6] There are Γ for which $\sim(\sigma^+ \leftrightarrow \sim\sigma^-) \in \mathrm{Cn}^{\pm}_{\mathbf{CL}}(\Gamma^\pm)$ for all $\sigma \in \mathcal{S}$. In this border case $\mathrm{Ext}(\Gamma^\pm) = \{\mathrm{Cn}^{\pm}_{\mathbf{CL}}(\Gamma^\pm)\}$ and $\Sigma = \mathcal{S}$ for the unique extension of $\Gamma^\pm$.

prudent unsigned consequence: $\Gamma \vdash_p A$ *iff* $A \in \mathrm{Cn}^{\pm}_{\mathbf{CL}}(\Gamma^{\pm} \cup \mathrm{nor}(\Gamma^{\pm}) \cup \mathcal{T})$
skeptical unsigned consequence: $\Gamma \vdash_s A$ *iff* $A \in \bigcap(\mathrm{Ext}(\Gamma^{\pm}))$
credulous unsigned consequence: $\Gamma \vdash_c A$ *iff* $A \in \bigcup(\mathrm{Ext}(\Gamma^{\pm}))$
prudent signed consequence: $\Gamma \vdash^{\pm}_p A$ *iff* $A^{\pm} \in \mathrm{Cn}^{\pm}_{\mathbf{CL}}(\Gamma^{\pm} \cup \mathrm{nor}(\Gamma^{\pm}) \cup \mathcal{T})$
skeptical signed consequence: $\Gamma \vdash^{\pm}_s A$ *iff* $A^{\pm} \in \bigcap(\mathrm{Ext}(\Gamma^{\pm}))$
credulous signed consequence: $\Gamma \vdash^{\pm}_c A$ *iff* $A^{\pm} \in \bigcup(\mathrm{Ext}(\Gamma^{\pm}))$

The prudent unsigned consequences are the unsigned formulas (members of $\mathcal{W}^*$) that are **CL**-derivable from the union of $\Gamma^{\pm}$ and the formulas of the form $(\sigma^+ \leftrightarrow \sigma) \wedge (\sigma^- \leftrightarrow {\sim}\sigma)$ that belong to every extension of $\Gamma^{\pm}$. The skeptical unsigned consequences are the unsigned formulas that are a member of every extension of $\Gamma^{\pm}$. The credulous unsigned consequences are the unsigned formulas that are a member of some extension of $\Gamma^{\pm}$.

An unsigned formula A is a prudent (respectively skeptical, respectively credulous) consequence of Γ iff $A^{\pm}$ is **CL**-derivable from the union of $\Gamma^{\pm}$ and the formulas of the form $(\sigma^+ \leftrightarrow \sigma) \wedge (\sigma^- \leftrightarrow {\sim}\sigma)$ that belong to every extension of $\Gamma^{\pm}$ (respectively $A^{\pm}$ is a member of every extension of $\Gamma^{\pm}$, respectively $A^{\pm}$ is a member of some extension of $\Gamma^{\pm}$).

While unsigned consequence sets are consistent (for all Γ), the signed ones are inconsistent iff Γ is inconsistent.

4 Paraconsistent Preliminaries

In subsequent sections we shall need adaptive logics (see Section 5) that have the propositional fragment of the paraconsistent logic **CLuNs** as their lower limit logic.[7] For reasons that become clear later on, we formulate **CLuNs** for the language $\mathcal{L}$. Let us start with an axiomatic system. It comprises a rule, axioms and definitions—see [13] for another axiomatic system and for semantic systems not discussed in the present paper.

MP	From A and $A \supset B$ to derive B
A⊃1	$A \supset (B \supset A)$
A⊃2	$((B \supset A) \supset A) \supset A$
A⊃3	$(A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))$
A∨1	$A \supset (A \vee B)$
A∨2	$B \supset (A \vee B)$
A∨3	$(A \supset C) \supset ((B \supset C) \supset ((A \vee B) \supset C))$
A∼	$(A \supset {\sim}A) \supset {\sim}A$ (alternative: $A \vee {\sim}A$)
A¬1	$(A \supset \neg A) \supset \neg A$
A¬2	$A \supset (\neg A \supset B)$
A∼∼	${\sim}{\sim}A \equiv A$
A∼⊃	${\sim}(A \supset B) \equiv (A \wedge {\sim}B)$
D∧	$A \wedge B =_{df} {\sim}({\sim}A \vee {\sim}B)$
D≡	$A \equiv B =_{df} (A \supset B) \wedge (B \supset A)$

[7] **CLuNs** is an extension of **CLuN**, which is like **CL**, except that it allows for gl*u*ts with respect to the standard *n*egation. The “s” in **CLuNs** refers to the fact that it (its propositional version) was first presented by Schütte in [22]; see also [1] and see [13] for the full predicative logic.

D$\rightarrow$ $\quad A \rightarrow B =_{df} {\sim}A \vee B$
D$\leftrightarrow$ $\quad A \leftrightarrow B =_{df} (A \rightarrow B) \wedge (B \rightarrow A)$

$\Gamma \vdash_{\mathbf{CLuNs}} A$ and $\vdash_{\mathbf{CLuNs}} A$ are defined as usual. **CLuNs** comprises two kinds of connectives: (i) the connectives of $\mathcal{L}^*$, which are all defined in terms of $\sim$ and $\vee$ and (ii) classical negation $\neg$, the detachable implication $\supset$, and the (in both directions) detachable equivalence $\equiv$. Incidentally, $A \supset B$ can be defined in **CLuNs** by $\neg A \vee B$.

In **CLuNs**, all complex inconsistencies entail truth-functions of elementary contradictions, for example $(p \wedge q) \wedge {\sim}(p \wedge q) \vdash_{\mathbf{CLuNs}} (p \wedge {\sim}p) \vee (q \wedge {\sim}q)$.

Replacement of Equivalents is invalid in **CLuNs**,[8] it is even invalid if we restrict **CLuNs** to the language $\mathcal{L}^*$.[9] However, **CLuNs** validates Replacement of Equivalents outside the scope of the standard negation $\sim$.

We present the semantics for the propositional systems in terms of valuation functions. A **CLuNs**-valuation $v : \mathcal{W} \mapsto \{0, 1\}$ fulfils the following conditions:

C1 $\quad v(A \supset B) = 1$ iff $v(A) = 0$ or $v(B) = 1$
C2 $\quad v(A \vee B) = 1$ iff $v(A) = 1$ or $v(B) = 1$
C3 $\quad v(\neg A) = 1$ iff $v(A) = 0$
C4 $\quad$ if $v(A) = 0$, then $v({\sim}A) = 1$
C5 $\quad v({\sim}{\sim}A) = v(A)$
C6 $\quad v({\sim}(A \vee B)) = 1$ iff $v({\sim}A) = v({\sim}B) = 1$
C7 $\quad v({\sim}(A \supset B)) = 1$ iff $v(A) = v({\sim}B) = 1$

Clause 4 is derivable for complex A and hence may be restricted to "if $v(\sigma) = 0$, then $v({\sim}\sigma) = 1$." $\Gamma \vDash_{\mathbf{CLuNs}} A$ and $\vDash_{\mathbf{CLuNs}} A$ are defined as usual.

It is instructive to see what happens if one distinguishes the assignment and the valuation function determined by a model. A **CLuNs**-assignment should assign a truth-value to all literals. Clauses C1–3 and C5–7 may then be reformulated about valuations v_M determined by a model $M = \langle v \rangle$, and C4 may then be rephrased as

C4 $\quad v_M({\sim}\sigma) = 1$ iff $v_M(\sigma) = 0$ or $v({\sim}\sigma) = 1$

It is easily seen that the above axiomatic system is equivalent to that for the propositional fragment of **CLuNs** from [13] and similarly for the above semantics. So the following theorem follows from theorems proved in [13]:

Theorem 1 $\Gamma \vdash_{\mathbf{CLuNs}} A$ *iff* $\Gamma \vDash_{\mathbf{CLuNs}} A$. *(Soundness and Completeness)*

We now prove some properties of **CLuNs** that will be useful below.

Fact 1 *Where* $\circ \in \{\vee, \wedge\}$, *if* $\vdash_{\mathbf{CLuNs}} A \equiv B$ *then* $\vdash_{\mathbf{CLuNs}} (A \circ C) \equiv (B \circ C)$ *and* $\vdash_{\mathbf{CLuNs}} (C \circ A) \equiv (C \circ B)$.

Fact 2 *All transformations used to obtain the NNF of a formula* $A \in \mathcal{W}^*$ *correspond to valid* **CLuNs**-*equivalences.*

Theorem 2 *If* B *is the NNF of* $A \in \mathcal{W}^*$, *then* $\vdash_{\mathbf{CLuNs}} A \equiv B$.

[8] For example, $\vdash_{\mathbf{CLuNs}} {\sim}(A \supset B) \equiv (A \wedge {\sim}B)$ but $\nvdash_{\mathbf{CLuNs}} {\sim}{\sim}(A \supset B) \equiv {\sim}(A \wedge {\sim}B)$. Indeed, $\vdash_{\mathbf{CLuNs}} {\sim}{\sim}(A \supset B) \equiv (A \supset B)$ and $\vdash_{\mathbf{CLuNs}} {\sim}(A \wedge {\sim}B) \equiv ({\sim}A \vee B)$, but $\nvdash_{\mathbf{CLuNs}} (A \supset B) \equiv ({\sim}A \vee B)$.

[9] For example, $\vdash_{\mathbf{CLuNs}} (A \vee {\sim}A) \equiv (B \vee {\sim}B)$, but $\nvdash_{\mathbf{CLuNs}} {\sim}(A \vee {\sim}A) \equiv {\sim}(B \vee {\sim}B)$.

Proof. If one drives negations inwards from the outside, the proof is obvious in view of Facts 1 and 2. ■

Corollary 1 *If B is the NNF of $A \in \mathcal{W}^*$, then a* **CLuNs***-model verifies A iff it verifies B.*

The presence of $\neg$ in $\mathcal{L}$ enables one to express that a formula A is consistent, viz. is not true together with its negation $\sim A$. The straightforward way to express in **CLuNs** that A is *consistently true* is by the formula $A \wedge \neg(A \wedge \sim A)$, which is **CLuNs**-equivalent to $\neg\sim A$.[10] Similarly, the **CLuNs**-equivalent formulas $\sim A \wedge \neg(A \wedge \sim A)$ and $\neg A$ express that A is *consistently false*. That A is consistent is then expressed by, for example, $\neg\sim A \vee \neg A$.

It seems more convenient to introduce a new symbol for expressing consistent truth and falsehood. So let $!A =_{df} \neg\sim A$. Remark that $!\sim A$ is $\neg\sim\sim A$, which is **CLuNs**-equivalent to $\neg A$.

Fact 3 *Where $\circ \in \{\vee, \wedge\}$, $\vdash_{\mathbf{CLuNs}} !(A \circ B) \equiv (!A \circ !B)$.*

Let $f(A)$ be the result of replacing in the NNF of A every literal B by $!B$.

Theorem 3 *If $A \in \mathcal{W}^*$, then $\vdash_{\mathbf{CLuNs}} !A \equiv f(A)$.*

Proof. By an obvious induction on the complexity of A in view of Facts 1, 3 and Theorem 2 . ■

Theorem 4 *If $\Gamma \subseteq \mathcal{W}^{\not\sim}$, then Γ is* **CLuNs***-satisfiable.*

Proof. Consider a valuation v for which $v(\sigma) = v(\sim\sigma) = 1$ for all $\sigma \in \mathcal{S}$. An obvious induction on the complexity of A shows that v verifies all $A \in \mathcal{W}^{\not\sim}$. That v verifies all literals provides the basis. For the induction step: if v verifies A and B, then v verifies $\sim\sim A$, $A \supset B$, $A \wedge B$, $A \vee B$, $A \equiv B$, $\sim(A \supset B)$, $\sim(A \wedge B)$, $\sim(A \vee B)$, and $\sim(A \equiv B)$. ■

5 The Adaptive Logics

A (flat) adaptive logic[11] **AL** is defined by a triple: (i) a monotonic *lower limit logic* **LLL**, (ii) a *set of abnormalities* Ω, characterized by a (possibly restricted) logical form, and (iii) an *adaptive strategy* (specifying the meaning of "interpreting the premises as normally as possible").[12]

A *Dab-formula* is a disjunction of abnormalities. In any subsequent expression of the form $Dab(\Delta)$, Δ is a *finite* subset of Ω and $Dab(\Delta)$ is a disjunction of the members of Δ—in practice we shall identify $Dab(\Delta)$ with every disjunction of the members of Δ.

The dynamic proof theory of an adaptive logic is characterized by three (generic) deduction rules and a marking definition. Every line of an annotated

[10] In **CLuNs**, $\neg\sim A$ expresses that $\sim A$ is false, whereas $\sim\sim A$ merely expresses that A is true. Also, $\neg\sim A \vdash_{\mathbf{CLuNs}} \sim\sim A$ but not conversely. Similarly, $\neg A$ expresses that A is false, which entails that $\sim A$ is true.

[11] A recent survey of adaptive logics may be found in [8], an even more recent survey of inconsistency-adaptive logics in [16].

[12] Extending **LLL** with the requirement that no abnormality is logically possible results in a monotonic logic, which is called the *upper limit logic* **ULL**.

dynamic proof consists of a line number, a formula, a justification, and a *condition.* The condition is introduced by the rules; the marking definition acts upon it: whether a line is marked or not depends on its condition. Let Γ be the set of premises as before. We list the deduction rules in shorthand notation. Let

$$A \qquad \Delta$$

abbreviate that A occurs in the proof on the condition Δ.

PREM If $A \in \Gamma$:

$$\begin{array}{ll} \dots & \dots \\ \hline A & \emptyset \end{array}$$

RU If $A_1, \dots, A_n \vdash_{\mathbf{LLL}} B$:

$$\begin{array}{ll} A_1 & \Delta_1 \\ \vdots & \vdots \\ A_n & \Delta_n \\ \hline B & \Delta_1 \cup \dots \cup \Delta_n \end{array}$$

RC If $A_1, \dots, A_n \vdash_{\mathbf{LLL}} B \vee Dab(\Theta)$

$$\begin{array}{ll} A_1 & \Delta_1 \\ \vdots & \vdots \\ A_n & \Delta_n \\ \hline B & \Delta_1 \cup \dots \cup \Delta_n \cup \Theta \end{array}$$

Identify (in the following lemma and elsewhere) "$\vee Dab(\Delta)$" with the empty string iff $\Delta = \emptyset$.

Lemmas and theorems that occur in this section without proofs have been proved elsewhere. The easiest source is [12] in which all the proofs occur.

Lemma 1 *A is derivable on the condition Δ in a proof from Γ iff $\Gamma \vdash_{\mathbf{LLL}} A \vee Dab(\Delta)$.*

While the rules depend only on the lower limit logic and the set of abnormalities, the marking definition depends on the strategy—the specific definitions we shall need are mentioned below. It determines, at every stage of the proof, which lines are "in" and which lines are "out" at that stage. A formula is *derived* from Γ *at a stage* of the proof iff it is the formula of a line that is unmarked at that stage. As the proof proceeds, unmarked lines may be marked and vice versa. So, it is important that one defines a different, stable, kind of derivability:

Definition 3 *A is* finally derived *from Γ on line i of a proof at stage s iff (i) A is the formula of line i, (ii) line i is not marked at stage s, and (iii) any extension of the proof in which line i is marked may be further extended in such a way that line i is unmarked.*

This means that there is a (possibly infinite) proof in which line i is unmarked and that is *stable* with respect to line i (line i is unmarked in all extensions of the proof). The previous definition is more appealing, among other things because it has a nice game-theoretic interpretation: whenever an opponent is able to extend the proof in such a way that line i is marked, the proponent is able to extend it further in such a way that line i is unmarked.

Definition 4 $\Gamma \vdash_{\mathbf{AL}} A$ *(A is* finally **AL**-derivable *from* Γ*) iff A is finally derived on a line of a proof from* Γ.

The semantics of all adaptive logics is defined in the same way. The strategy *selects* one or more *sets* of **LLL**-models of Γ in view of the abnormalities verified by the models.[13]

Definition 5 $\Gamma \vDash_{\mathbf{AL}} A$ *(A is an* **AL**-semantic consequence *of* Γ*) iff A is verified by all members of a selected set of* **LLL***-models of* Γ.

It is provable in terms of the general characterization of an adaptive logic as a triple (see above) that $\Gamma \vdash_{\mathbf{AL}} A$ iff $\Gamma \vDash_{\mathbf{AL}} A$. The proof merely relies upon the following: (i) the lower limit logic **LLL** is monotonic and compact and is sound and complete with respect to its semantics, (ii) the set of abnormalities is characterized by a possibly restricted[14] logical form, and (iii) the properties of the strategy. The proofs for two of the strategies we need in this paper are outlined in [10]. There, a number of further properties of adaptive logics are proved in terms of the general characterization. In the present paper we need a third strategy and we shall prove enough about it to warrant soundness and completeness and some further properties.

We shall need three adaptive *strategies* for characterizing the signed systems: Reliability, Minimal Abnormality, and Normal Selections. We first introduce some technical stuff that we shall need in the sequel. $Dab(\Delta)$ is a *minimal Dab-consequence of* Γ iff $\Gamma \vdash_{\mathbf{LLL}} Dab(\Delta)$ and, for all $\Delta' \subset \Delta$, $\Gamma \nvdash_{\mathbf{LLL}} Dab(\Delta')$. At every stage of a proof, zero or more *Dab*-formulas are derived on the condition $\emptyset$. These will be called the *Dab-formulas of the stage.*[15] Some *Dab*-formulas of stage s are *minimal* (in the above sense). For the semantics we need:

Definition 6 *Where M is a* **LLL***-model,* $Ab(M) = \{A \in \Omega \mid M \models A\}$ *(the 'abnormal part' of M).*

Reliability This is the oldest strategy, introduced in [2] and [4] (which was written earlier), and studied thoroughly at the predicative level in [5]. The underlying idea is that all disjuncts of minimal *Dab*-consequences of Γ are considered as unreliable with respect to Γ. If one (provisionally) identifies the minimal *Dab*-formulas of a stage s of a proof from Γ with the minimal *Dab*-consequences of Γ, all disjuncts of the minimal *Dab*-formulas of stage s are considered as unreliable at that stage.

Where $Dab(\Delta_1)$, ..., $Dab(\Delta_n)$ are the minimal *Dab*-formulas of stage s of the proof, $U_s(\Gamma) = \Delta_1 \cup \ldots \cup \Delta_n$ is the set of unreliable formulas at stage s. In view of Definition 4, the description of the proofs is completed by:

[13]Adaptive logics in standard format select a single set of models, which simplifies the definition that follows in the text. Two of the adaptive logics mentioned below are in standard format, but we need a set of selected sets for the third one, which uses the Normal Selections strategy (see below in the text). It can be shown that the Normal Selections strategy is easily reduced to the Simple strategy (which delivers a logic in standard format) under a modal translation. We shall neglect this matter here.

[14]We shall see an example of such a restriction when we come to the adaptive logics that have **CLuNs** as their lower limit logic. Some requirements on the restriction are useful to warrant a nice upper limit logic, but are not important for the adaptive logic itself. Anyway, all such restrictions are fulfilled by the logics discussed in this paper.

[15]That $Dab(\Delta)$ occurs on a condition $\Theta \neq \emptyset$ in a proof from Γ does not warrant that $\Gamma \vdash_{\mathbf{LLL}} Dab(\Delta)$. It warrants that $\Gamma \vdash_{\mathbf{LLL}} Dab(\Delta \cup \Theta)$, but we shall only call $Dab(\Delta \cup \Theta)$ a *Dab*-formula of stage s iff, at stage s, $Dab(\Delta \cup \Theta)$ is actually derived on the condition $\emptyset$.

Definition 7 *Marking for Reliability: Line i is marked at stage s iff, where Θ is its condition, $\Theta \cap U_s(\Gamma) \neq \emptyset$.*

Where $Dab(\Delta_1),\ Dab(\Delta_2),\ \ldots$ are the minimal *Dab*-consequences of a premise set Γ,[16] $U(\Gamma) = \Delta_1 \cup \Delta_2 \cup \ldots$ is the set of formulas that are unreliable with respect to Γ. In view of Definition 5 the description of the semantics is completed by the following definitions:

Definition 8 *A* **LLL***-model M of Γ is* reliable *iff $Ab(M) \subseteq U(\Gamma)$.*

Definition 9 *The (sole)* selected set *of* **LLL***-models of Γ is the set of the reliable* **LLL***-models of Γ.*

Theorem 5 $\Gamma \vdash_{\mathbf{AL}^r} A$ *iff* $\Gamma \vDash_{\mathbf{AL}^r} A$. *(Soundness and Completeness)*

The following theorem provides the bridge between the adaptive logic $\mathbf{AL}^r$ and the lower limit logic **LLL**.

Theorem 6 $\Gamma \vdash_{\mathbf{AL}^r} A$ *iff there is a finite $\Delta \subset \Omega$ such that $\Gamma \vdash_{\mathbf{LLL}} A \vee Dab(\Delta)$ and $\Delta \cap U(\Gamma) = \emptyset$.*

Minimal Abnormality This strategy was first introduced in [3] and was studied thoroughly at the predicative level in [5]. The underlying idea is that only minimal abnormal models are selected. Let us first complete the semantics:

Definition 10 *A* **LLL***-model M of Γ is* minimal abnormal *iff there is no* **LLL***-model M' of Γ such that $Ab(M') \subset Ab(M)$.*

Definition 11 *The (sole)* selected set *of* **LLL***-models of Γ is the set of the minimal abnormal* **LLL***-models of Γ.*

Completing the description of the proofs is slightly more tiresome. The idea is that the insights provided by a proof at a stage s determine which 'derived' formulas are indeed considered as derivable at stage s. So one (provisionally) identifies the minimal *Dab*-consequences of Γ with the minimal *Dab*-formulas of stage s of the proof from Γ.

The way to do this is indicated by the following consideration. Let $\Phi^\circ(\Gamma)$ be the set of all sets that contain one disjunct out of each minimal *Dab*-consequence of Γ. Let $\Phi(\Gamma)$ contain those members of $\Phi^\circ(\Gamma)$ that are not proper supersets of other members of $\Phi^\circ(\Gamma)$.

Theorem 7 $\Phi(\Gamma) = \{Ab(M) \mid M$ *is a minimal abnormal model of* $\Gamma\}$.

We now apply this to a proof at a stage. Let $\Phi^\circ_s(\Gamma)$ be the set of all sets that contain one disjunct out of each minimal *Dab*-formula at stage s. Let $\Phi_s(\Gamma)$ contain those members of $\Phi^\circ_s(\Gamma)$ that are not proper supersets of other members of $\Phi^\circ_s(\Gamma)$.[17]

[16] The minimal *Dab*-consequences of Γ may be semantically defined in view of the soundness and completeness of **LLL** with respect to its semantics.

[17] The proofs become somewhat shorter if the definition is slightly complicated by letting $\Phi^\star_s(\Gamma)$ contain, for every $\varphi \in \Phi^\circ_s(\Gamma)$, $\mathrm{Cn}_{\mathbf{LLL}}(\varphi) \cap \Omega$, and letting $\Phi_s(\Gamma)$ contain those members of $\Phi^\star_s(\Gamma)$ that are not proper supersets of other members of $\Phi^\star_s(\Gamma)$.

Definition 12 *Marking for Minimal Abnormality: Line i is marked at stage s iff, where A is derived on the condition Δ at line i, (i) there is no $\varphi \in \Phi_s(\Gamma)$ such that $\varphi \cap \Delta = \emptyset$, or (ii) for some $\varphi \in \Phi_s(\Gamma)$, there is no line at which A is derived on a condition Θ for which $\varphi \cap \Theta = \emptyset$.*

This completes the description of the proofs in view of Definition 4. Compare the following two theorems to Theorems 5 and 6.

Theorem 8 $\Gamma \vdash_{\mathbf{AL}^m} A$ *iff* $\Gamma \vDash_{\mathbf{AL}^m} A$. *(Soundness and Completeness)*

Theorem 9 $\Gamma \vdash_{\mathbf{AL}^m} A$ *iff there are finite* $\Delta_1 \subset \Omega, \Delta_2 \subset \Omega, \ldots$ *such that, for every* $\varphi \in \Phi(\Gamma)$*, some* Δ_i *is such that* $\Gamma \vdash_{\mathbf{LLL}} A \vee Dab(\Delta_i)$ *and* $\Delta_i \cap \varphi = \emptyset$.

We shall need the following theorem and lemmas in the sequel. Theorem 10, first proved in [10] for a specific system, is sometimes called Stopperedness or Smoothness.

Theorem 10 *If a* **LLL***-model* M *of* Γ *is not a minimal abnormal model of* Γ*, then there is a minimal abnormal model* M' *of* Γ*, such that* $Ab(M') \subset Ab(M)$. *(Strong Reassurance for Minimal Abnormality)*

Lemma 2 *A* **LLL***-model* M *of* Γ *verifies* $Dab(\Delta)$ *iff* $\Delta \cap Ab(M) \neq \emptyset$.

Lemma 3 $\Gamma \vdash_{\mathbf{LLL}} Dab(\Delta)$ *iff* $\Gamma \vdash_{\mathbf{AL}^m} Dab(\Delta)$. *Equivalently:* $\Gamma \vdash_{\mathbf{LLL}} Dab(\Delta)$ *iff every minimal abnormal model of* Γ *verifies* $Dab(\Delta)$.

Lemma 4 $U(\Gamma) = \bigcup \Phi(\Gamma)$.

Normal Selections This strategy was first introduced in [7], where it was used for characterizing the Weak Rescher–Manor consequence relation—see also Section 9. We now understand the strategy better and are able to phrase its semantics more elegantly. The description of the proofs is completed by:

Definition 13 *Marking for Normal Selections: Line i is marked at stage s iff, where Δ is the condition of line i, $Dab(\Delta)$ has been derived on the condition $\emptyset$ at stage s.*

The semantics is completed by:

Definition 14 *A set* Ξ *of* **LLL***-models of* Γ *is a* selected set *iff, for some* $\varphi \in \Phi(\Gamma)$, $\Xi = \{M \mid M \models \Gamma; Ab(M) = \varphi\}$.

As this characterization of the semantics is new, we need to prove that final derivability is sound and complete with respect to the semantics.

Theorem 11 $\Gamma \vdash_{\mathbf{AL}^n} A$ *iff there is a* $\Delta \subset \Omega$ *such that* $\Gamma \vdash_{\mathbf{LLL}} A \vee Dab(\Delta)$ *and* $\Gamma \nvdash_{\mathbf{LLL}} Dab(\Delta)$.

Proof. Left–right. Suppose that $\Gamma \vdash_{\mathbf{AL}^n} A$, and hence, by Definition 4 that A is finally derived on some condition, say Δ, at line i of a stage s of a proof from Γ. By Lemma 1, it follows that $\Gamma \vdash_{\mathbf{LLL}} A \vee Dab(\Delta)$. Suppose next that $\Gamma \vdash_{\mathbf{LLL}} Dab(\Delta)$. It is then possible to extend the proof in such a way that it contains a line at which $Dab(\Delta)$ is derived on the condition $\emptyset$. In the extension, and in all extensions of the extension, line i is marked in view of Definition 13. But this contradicts that A is finally derived on condition Δ at line i of a stage s.

Right–left. Suppose that there is a $\Delta \subset \Omega$ such that $\Gamma \vdash_{\mathbf{LLL}} A \vee Dab(\Delta)$ and $\Gamma \nvdash_{\mathbf{LLL}} Dab(\Delta)$. By Lemma 1, there is a proof from Γ in which A is derived on the condition Δ at a line i. By the same Lemma $Dab(\Delta)$ cannot be derived on the condition $\emptyset$ in any extension of this proof. So, by Definitions 3, 4, and 13, $\Gamma \vdash_{\mathbf{AL}^n} A$. ■

Let $\Gamma^{\neg} = \{\neg A \mid A \in \Gamma\}$.

Theorem 12 $\Gamma \vDash_{\mathbf{AL}^n} A$ *iff there is a* $\Delta \subset \Omega$ *such that* $\Gamma \vDash_{\mathbf{LLL}} A \vee Dab(\Delta)$ *and* $\Gamma \nvDash_{\mathbf{LLL}} Dab(\Delta)$.

Proof. Left–right. Suppose that $\Gamma \vDash_{\mathbf{AL}^n} A$. By Definitions 5 and 14, there is a $\varphi \in \Phi(\Gamma)$ such that all members of $\{M \mid M \models \Gamma; Ab(M) = \varphi\}$ verify A. So all **LLL**-models of $\Gamma \cup (\Omega - \varphi)^{\neg}$ verify A, and, by the completeness of **LLL** with respect to its semantics, $\Gamma \cup (\Omega - \varphi)^{\neg} \vdash_{\mathbf{LLL}} A$. By the compactness of **LLL** there is a finite $\Gamma' \subset \Gamma$ and a finite $\varphi' \subset \Omega - \varphi$ such that $\Gamma' \cup \varphi'^{\neg} \vdash_{\mathbf{LLL}} A$. It follows that $\Gamma' \vdash_{\mathbf{LLL}} A \vee Dab(\varphi')$[18] and hence that $\Gamma \vdash_{\mathbf{LLL}} A \vee Dab(\varphi')$. Moreover, as $\varphi \cap \varphi' = \emptyset$, $\Gamma \nvdash_{\mathbf{LLL}} Dab(\varphi')$. But then $\Gamma \nvDash_{\mathbf{LLL}} Dab(\varphi')$ by the completeness of **LLL** with respect to the semantics.

Right–left. Suppose that there is a $\Delta \subset \Omega$ such that $\Gamma \vDash_{\mathbf{LLL}} A \vee Dab(\Delta)$ and $\Gamma \nvDash_{\mathbf{LLL}} Dab(\Delta)$. By Lemmas 3 and 2, there is a $\varphi \in \Phi(\Gamma)$ such that $\varphi \cap \Delta = \emptyset$. It follows that every member of $\{M \mid M \models \Gamma; Ab(M) = \varphi\}$ falsifies $Dab(\Delta)$ and verifies $A \vee Dab(\Delta)$, and hence that every member of $\{M \mid M \models \Gamma; Ab(M) = \varphi\}$ verifies A. So $\Gamma \vDash_{\mathbf{AL}^n} A$ by Definitions 5 and 14. ■

Theorems 11 and 12 give us:

Corollary 2 $\Gamma \vdash_{\mathbf{AL}^n} A$ *iff* $\Gamma \vDash_{\mathbf{AL}^n} A$. *(Soundness and Completeness)*

We shall need three adaptive logics, which share their lower limit logic and their set of abnormalities, viz. $\Omega_1 = \{A \wedge {\sim}A \mid A \in \mathcal{S}\}$.[19]

adaptive logic	lower limit logic	set of abnormalities	strategy
$\mathbf{CLuNs}^r$	**CLuNs**	Ω_1	Reliability
$\mathbf{CLuNs}^m$	**CLuNs**	Ω_1	Minimal Abnormality
$\mathbf{CLuNs}^n$	**CLuNs**	Ω_1	Normal Selections

To simplify the phraseology, we shall talk about a valuation as about a model. Thus we shall say that v verifies Γ iff $v(A) = 1$ for all $A \in \Gamma$, and we shall write $Ab(v)$ to denote $\{A \in \Omega_1 \mid v(A) = 1\}$.

[18] We suppose here, as is the case for all logics considered, that $\vee$ is classical disjunction. For a completely general formulation, it is always supposed that **LLL** contains all logical symbols of **CL** (or that these are added).

[19] In some other papers, $\mathbf{CLuNs}^r$ was called **ACLuNs1** and $\mathbf{CLuNs}^m$ was called **ACLuNs2**.

6 Characterization in Terms of Adaptive Logics

The signed systems (the six consequence relations) are defined for $\Gamma \subseteq \mathcal{W}^*$ and $A \in \mathcal{W}^*$. We shall prove that, under the same restriction:

$\Gamma \vdash_p A$ iff $\Gamma \vdash_{\mathbf{CLuNs}^r} !A$
$\Gamma \vdash_s A$ iff $\Gamma \vdash_{\mathbf{CLuNs}^m} !A$
$\Gamma \vdash_c A$ iff $\Gamma \vdash_{\mathbf{CLuNs}^n} !A$
$\Gamma \vdash_p^{\pm} A$ iff $\Gamma \vdash_{\mathbf{CLuNs}^r} A$
$\Gamma \vdash_s^{\pm} A$ iff $\Gamma \vdash_{\mathbf{CLuNs}^m} A$
$\Gamma \vdash_c^{\pm} A$ iff $\Gamma \vdash_{\mathbf{CLuNs}^n} A$

Let us begin by proving some properties of the signed systems.

Lemma 5 *If $\Delta \in \text{Ext}(\Gamma^{\pm})$, then $(\sigma^+ \leftrightarrow \sigma) \wedge (\sigma^- \leftrightarrow {\sim}\sigma) \in \Delta$ iff $\sigma^+ \leftrightarrow {\sim}\sigma^- \in \Delta$.*

Proof. Obvious in view of Definition 1. ■

Lemma 6 *If, for some $\Sigma \subseteq \mathcal{S}$, $\Delta = \text{Cn}^{\pm}_{\mathbf{CL}}(\Gamma^{\pm} \cup \{(\sigma^+ \leftrightarrow \sigma) \wedge (\sigma^- \leftrightarrow {\sim}\sigma) \mid \sigma \in \mathcal{S} - \Sigma\})$ is consistent, then ${\sim}(\sigma^+ \leftrightarrow {\sim}\sigma^-) \in \Delta$ iff $\sigma^+ \wedge \sigma^- \in \Delta$.*

Proof. Suppose that the antecedent is true. We have to prove an equivalence, the right–left direction of which is obvious.

For the left–right direction, suppose that ${\sim}(\sigma_1^+ \leftrightarrow {\sim}\sigma_1^-) \in \Delta$ and that $\sigma_1^+ \wedge \sigma_1^- \notin \Delta$. So there is a **CL**-valuation v that verifies Δ, whence $v(\sigma_1^+ \leftrightarrow {\sim}\sigma_1^-) = 0$, and for which $v(\sigma_1^+ \wedge \sigma_1^-) = 0$. It follows that $v(\sigma_1^+) = 0$, that $v(\sigma_1^-) = 0$, and that $\sigma_1 \in \Sigma$. Let v' be a **CL**-valuation that is exactly like v except in that $v'(\sigma_1^+) = 1$. As the members of $\Gamma^{\pm}$ are merely composed of (signed) sentential letters, conjunctions, disjunctions and parentheses, and v verifies $\Gamma^{\pm}$, so does v'. As $\sigma_1 \in \Sigma$ and v verifies $\{(\sigma^+ \leftrightarrow \sigma) \wedge (\sigma^- \leftrightarrow {\sim}\sigma) \mid \sigma \in \mathcal{S} - \Sigma\}$, so does v'. It follows that v' verifies Δ. But $v'(\sigma_1^+ \leftrightarrow {\sim}\sigma_1^-) = 1$, which contradicts ${\sim}(\sigma_1^+ \leftrightarrow {\sim}\sigma_1^-) \in \Delta$. ■

Lemma 7 *If $\Delta \in \text{Ext}(\Gamma^{\pm})$, then $\sigma^+ \vee \sigma^- \in \Delta$ for all $\sigma \in \mathcal{S}$.*

Proof. As $\Delta \in \text{Ext}(\Gamma^{\pm})$, there is a $\Sigma \subseteq \mathcal{S}$ such that (i) $\Delta = \text{Cn}^{\pm}_{\mathbf{CL}}(\Gamma^{\pm} \cup \{(\sigma^+ \leftrightarrow \sigma) \wedge (\sigma^- \leftrightarrow {\sim}\sigma) \mid \sigma \in \mathcal{S} - \Sigma\})$ is consistent and (ii) for all $\sigma \in \Sigma$, $\sigma^+ \wedge \sigma^- \in \Delta$ (by Definition 1 and Lemma 6). If $\sigma \in \mathcal{S} - \Sigma$, then $\sigma^+ \vee \sigma^- \in \Delta$ (because $\sigma^+ \vee {\sim}\sigma^+, {\sim}\sigma^+ \leftrightarrow \sigma^- \in \Delta$). If $\sigma \in \Sigma$, then $\sigma^+ \wedge \sigma^- \in \Delta$ and hence $\sigma^+ \vee \sigma^- \in \Delta$. ■

Lemma 8 *$\Delta \in \text{Ext}(\Gamma^{\pm})$ iff $\Delta = \text{Cn}^{\pm}_{\mathbf{CL}}(\Gamma^{\pm} \cup \{(\sigma^+ \leftrightarrow \sigma) \wedge (\sigma^- \leftrightarrow {\sim}\sigma) \mid \sigma^+ \wedge \sigma^- \notin \Delta\})$.*

Proof. For the left–right direction, suppose that $\Delta \in \text{Ext}(\Gamma^{\pm})$. In view of Definition 1, there is a $\Sigma \subseteq \mathcal{S}$ such that (i) $\Delta = \text{Cn}^{\pm}_{\mathbf{CL}}(\Gamma^{\pm} \cup \{(\sigma^+ \leftrightarrow \sigma) \wedge (\sigma^- \leftrightarrow {\sim}\sigma) \mid \sigma \in \mathcal{S} - \Sigma\})$ is consistent and (ii) for all $\sigma \in \Sigma$, ${\sim}(\sigma^+ \leftrightarrow {\sim}\sigma^-) \in \Delta$. By Lemma 6, $\Sigma = \{\sigma \mid \sigma^+ \wedge \sigma^- \in \Delta\}$.

For the right–left direction, suppose that $\Delta = \text{Cn}^{\pm}_{\mathbf{CL}}(\Gamma^{\pm} \cup \{(\sigma^+ \leftrightarrow \sigma) \wedge (\sigma^- \leftrightarrow {\sim}\sigma) \mid \sigma^+ \wedge \sigma^- \notin \Delta\})$. Let $\Sigma = \{\sigma \mid \sigma^+ \wedge \sigma^- \in \Delta\}$. If $\Delta = \text{Cn}^{\pm}_{\mathbf{CL}}(\Gamma^{\pm} \cup \{(\sigma^+ \leftrightarrow \sigma) \wedge (\sigma^- \leftrightarrow {\sim}\sigma) \mid \sigma \in \mathcal{S} - \Sigma\})$ were inconsistent, then $\sigma^+ \wedge \sigma^- \in \Delta$ for all $\sigma \in \mathcal{S}$, whence $\Delta = \text{Cn}^{\pm}_{\mathbf{CL}}(\Gamma^{\pm})$. But $\text{Cn}^{\pm}_{\mathbf{CL}}(\Gamma^{\pm})$ cannot be inconsistent as no negation occurs in $\Gamma^{\pm}$. By Lemma 6, $\sim(\sigma^+ \leftrightarrow \sim\sigma^-) \in \Delta$ for all $\sigma \in \Sigma$. But then $\Delta \in \mathrm{Ext}(\Gamma^{\pm})$ in view of Definition 1. ■

In order to establish the results from the beginning of this section, we define, for every **CL**-valuation v for $\mathcal{L}^{\pm}$, an abnormal part $Ab^{\pm}(v)$, we define certain **CL**-valuations for $\mathcal{L}^{\pm}$ as regular, and we define a relation C between **CL**-valuations for $\mathcal{L}^{\pm}$ and **CLuNs**-valuations for $\mathcal{L}$.

Definition 15 *Where v is a* **CL**-*valuation for $\mathcal{L}^{\pm}$, $Ab^{\pm}(v) = \{\sigma^+ \wedge \sigma^- \mid v(\sigma^+ \wedge \sigma^-) = 1\}$.*

Definition 16 *A* **CL**-*valuation for $\mathcal{L}^{\pm}$ is* regular *iff, for all $\sigma \in \mathcal{S}$, (i) $v(\sigma^+ \vee \sigma^-) = 1$ and (ii) if $v(\sigma^+ \wedge \sigma^-) = 0$, then $v(\sigma) = v(\sigma^+)$.*

Definition 17 *Where v is a* **CL**-*valuation for $\mathcal{L}^{\pm}$ and v_s is a* **CLuNs**-*valuation for $\mathcal{L}$, Cvv_s (v* corresponds to *v_s) iff, for all σ, $v(\sigma^+) = v_s(\sigma)$ and $v(\sigma^-) = v_s(\sim\sigma)$.*

Lemma 9 *If $\Delta \in \mathrm{Ext}(\Gamma^{\pm})$ and v is a* **CL**-*valuation for $\mathcal{L}^{\pm}$ that verifies Δ, then $\sigma^+ \wedge \sigma^- \in \Delta$ iff $\sigma^+ \wedge \sigma^- \in Ab^{\pm}(v)$.*

Proof. Suppose that the antecedent is true. As v verifies Δ, $\sigma^+ \wedge \sigma^- \in Ab^{\pm}(v)$ if $\sigma^+ \wedge \sigma^- \in \Delta$ by Definition 15 and Lemma 8.

If $\sigma^+ \wedge \sigma^- \notin \Delta$, then by Lemmas 5 and 8, $\sigma^+ \leftrightarrow \sim\sigma^- \in \Delta$. So $v(\sigma^+ \leftrightarrow \sim\sigma^-) = 1$, whence $v(\sigma^+ \wedge \sigma^-) = 0$. So $\sigma^+ \wedge \sigma^- \notin Ab^{\pm}(v)$ by Definition 15. ■

Lemma 10 *A* **CL**-*valuation v for $\mathcal{L}^{\pm}$ verifies a $\Delta \in \mathrm{Ext}(\Gamma^{\pm})$ iff v is regular, v verifies $\Gamma^{\pm}$, and $Ab^{\pm}(v) = \{\sigma^+ \wedge \sigma^- \mid \sigma^+ \wedge \sigma^- \in \Delta\}$.*

Proof. For the left–right direction, suppose that v is a **CL**-valuation v for $\mathcal{L}^{\pm}$ that verifies $\Delta \in \mathrm{Ext}(\Gamma^{\pm})$. As $v(\sigma^+ \vee \sigma^-) = 1$ (by Lemma 7) and $v(\sigma) = v(\sigma^+)$ if $v(\sigma^+ \wedge \sigma^-) = 0$ (by Lemma 8), v is regular. By Lemma 8 v verifies $\Gamma^{\pm}$. By Lemma 9 $Ab^{\pm}(v) = \{\sigma^+ \wedge \sigma^- \mid \sigma^+ \wedge \sigma^- \in \Delta\}$.

For the right–left direction, suppose that v is regular, that v verifies $\Gamma^{\pm}$, and that $Ab^{\pm}(v) = \{\sigma^+ \wedge \sigma^- \mid \sigma^+ \wedge \sigma^- \in \Delta\}$ for some $\Delta \in \mathrm{Ext}(\Gamma^{\pm})$. By Definitions 15 and 16, $v((\sigma^+ \leftrightarrow \sigma) \wedge (\sigma^- \leftrightarrow \sim\sigma)) = 1$ if $\sigma^+ \wedge \sigma^- \notin Ab^{\pm}(v)$, whence v verifies $\{(\sigma^+ \leftrightarrow \sigma) \wedge (\sigma^- \leftrightarrow \sim\sigma) \mid \sigma^+ \wedge \sigma^- \notin \Delta\}$. So, by Lemma 8, v verifies Δ. ■

Lemma 11 *For every regular* **CL**-*valuation v for $\mathcal{L}^{\pm}$ there is a* **CLuNs**-*valuation v_s for $\mathcal{L}$ such that Cvv_s, and for every* **CLuNs**-*valuation v_s for $\mathcal{L}$ there is a regular* **CL**-*valuation v for $\mathcal{L}^{\pm}$ such that Cvv_s.*

Proof. From Definitions 16 and 17. ■

If v is regular, Cvv_s establishes a straightforward correspondence between the v-value of *certain* formulas of $\mathcal{L}^{\pm}$ and the v_s-value of certain formulas of $\mathcal{L}$. Where $A \in \mathcal{W}^{\pm}$ does not contain any unsigned sentential letters, let $g(A)$ be the result of systematically replacing in A first $\sim$ by $\neg$, $\rightarrow$ by $\supset$, and $\leftrightarrow$ by $\equiv$, and next every σ^+ by σ and every σ^- by $\sim\sigma$. It is easily established (by an obvious induction on the complexity of A) that $v(A) = 1$ iff $v_s(g(A)) = 1$. This does not help us for establishing the desired result because of the weird role played by unsigned sentential letters in the signed systems. If $v(\sigma^+) \neq v(\sigma^-)$, then $v(\sigma) = v(\sigma^+)$; but if $v(\sigma^+) = v(\sigma^-) = 1$, then $v(\sigma)$ is completely independent of $v(\sigma^+)$ and $v(\sigma^-)$ in that it may be 0 as well as 1. Because of this, no total translation function between $\mathcal{L}^\pm$ and $\mathcal{L}$ corresponds to Cvv_s. However, if v is regular, then Cvv_s establishes a *different* correspondence between v and v_s:

Lemma 12 *If v is a regular* **CL***-valuation for $\mathcal{L}^\pm$, v_s is a* **CLuNs***-valuation for $\mathcal{L}$, Cvv_s, and $A \in \mathcal{W}^*$, then $v_s(A) = 1$ iff $v(A^\pm) = 1$.*

Proof. The proof proceeds by an obvious induction on the complexity of A. The basis is provided by Definitions 16 and 17. In view of Corollary 1, the induction step reduces to two cases, which are justified by (i) $v_s(A \vee B) = 1$ iff $v_s(A) = 1$ or $v_s(B) = 1$ iff (by the induction hypothesis) $v(A^\pm) = 1$ or $v(B^\pm) = 1$ iff $v((A \vee B)^\pm) = 1$, and (ii) $v_s(A \wedge B) = 1$ iff $v_s(A) = 1$ and $v_s(B) = 1$ iff (by the induction hypothesis) $v(A^\pm) = 1$ and $v(B^\pm) = 1$ iff $v((A \wedge B)^\pm) = 1$. ■

Lemma 13 *If v is a regular* **CL***-valuation for $\mathcal{L}^\pm$, v_s is a* **CLuNs***-valuation for $\mathcal{L}$, and Cvv_s, then $\sigma^+ \wedge \sigma^- \in Ab^\pm(v)$ iff $\sigma \wedge {\sim}\sigma \in Ab(v_s)$.*

Proof. Immediate in view of Lemma 12 and Definitions 6 and 15. ■

Lemma 14 *If $\Gamma \subseteq \mathcal{W}^*$, then, for all $\Sigma \subseteq \mathcal{S}$, $\mathrm{Cn}^\pm_{\mathbf{CL}}(\Gamma^\pm \cup \{(\sigma^+ \leftrightarrow \sigma) \wedge (\sigma^- \leftrightarrow {\sim}\sigma) \mid \sigma \in \mathcal{S} - \Sigma\}) \in \mathrm{Ext}(\Gamma^\pm)$ iff $\{\sigma \wedge {\sim}\sigma \mid \sigma \in \Sigma\} \in \Phi(\Gamma)$.*

Proof. Let $\Gamma \subseteq \mathcal{W}^*$, $\Sigma \subseteq \mathcal{S}$, $\Delta = \mathrm{Cn}^\pm_{\mathbf{CL}}(\Gamma^\pm \cup \{(\sigma^+ \leftrightarrow \sigma) \wedge (\sigma^- \leftrightarrow {\sim}\sigma) \mid \sigma \in \mathcal{S} - \Sigma\})$, and $\varphi = \{\sigma \wedge {\sim}\sigma \mid \sigma \in \Sigma\}$. We have to prove $\Delta \in \mathrm{Ext}(\Gamma^\pm)$ iff $\varphi \in \Phi(\Gamma)$.

For the left–right direction, suppose that $\Delta \in \mathrm{Ext}(\Gamma^\pm)$ but that $\varphi \notin \Phi(\Gamma)$. By Definition 1 and Lemma 6, Δ is consistent and $\sigma^+ \wedge \sigma^- \in \Delta$ iff $\sigma \in \Sigma$. Let v be a **CL**-valuation that verifies Δ. By Lemma 10, v is regular, v verifies $\Gamma^\pm$, and $Ab^\pm(v) = \{\sigma^+ \wedge \sigma^- \mid \sigma \in \Sigma\}$. So there is a **CLuNs**-valuation v_s such that Cvv_s (by Lemma 11), for which $Ab(v_s) = \{\sigma \wedge {\sim}\sigma \mid \sigma \in \Sigma\}$ (by Lemma 13), and that verifies Γ (by Lemma 12). As $\{\sigma \wedge {\sim}\sigma \mid \sigma \in \Sigma\} \notin \Phi(\Gamma)$, there is a **CLuNs**-valuation v'_s that verifies Γ and for which $Ab(v'_s) \subset Ab(v_s)$ (by Definition 10 and Theorem 7). But then there is a regular **CL**-valuation v' such that $Cv'v'_s$ (by Lemma 11). By Lemma 12, v' verifies $\Gamma^\pm$. By Lemma 13, $Ab^\pm(v') = \{\sigma^+ \wedge \sigma^- \mid \sigma \wedge {\sim}\sigma \in Ab(v'_s)\}$. So $Ab^\pm(v') \subset \{\sigma^+ \wedge \sigma^- \mid \sigma \in \Sigma\}$. By Definitions 15 and 16, $v'((\sigma^+ \leftrightarrow \sigma) \wedge (\sigma^- \leftrightarrow {\sim}\sigma)) = 1$ if $\sigma^+ \wedge \sigma^- \notin Ab^\pm(v')$, whence v' verifies $\{(\sigma^+ \leftrightarrow \sigma) \wedge (\sigma^- \leftrightarrow {\sim}\sigma) \mid \sigma \in \mathcal{S} - \Sigma\}$. But then v' verifies Δ, which contradicts $Ab^\pm(v') \subset \{\sigma^+ \wedge \sigma^- \mid \sigma \in \Sigma\}$ in view of Lemma 10.

For the right–left direction, suppose that $\varphi \in \Phi(\Gamma)$. So there is a **CLuNs**-valuation v_s for $\mathcal{L}$ for which $Ab(v_s) = \{\sigma \wedge {\sim}\sigma \mid \sigma \in \Sigma\}$ and that verifies Γ (by Definition 10 and Theorem 7). It follows that there is a regular **CL**-valuation v for $\mathcal{L}^\pm$ such that Cvv_s (by Lemma 11), for which $Ab^\pm(v) = \{\sigma^+ \wedge \sigma^- \mid \sigma \in \Sigma\}$ (by Lemma 13), and that verifies $\Gamma^\pm$ (by Lemma 12). If $\sigma \in \mathcal{S} - \Sigma$, then $\sigma^+ \wedge \sigma^- \notin Ab^\pm(v)$, whence $v(\sigma^+) \neq v(\sigma^-)$ (by Definitions 15 and 16). Moreover, $v(\sigma) = v(\sigma^+)$ for all $\sigma \in \mathcal{S} - \Sigma$ by Definition 16. So, for all $\sigma \in \mathcal{S} - \Sigma$, $v((\sigma^+ \leftrightarrow \sigma) \wedge (\sigma^- \leftrightarrow {\sim}\sigma)) = 1$. It follows that v verifies Δ. But then Δ is consistent, whence $\sigma^+ \wedge \sigma^- \notin \Delta$ iff $\sigma \in \mathcal{S} - \Sigma$. So $\Delta = \mathrm{Cn}^\pm_{\mathbf{CL}}(\Gamma^\pm \cup \{(\sigma^+ \leftrightarrow \sigma) \wedge (\sigma^- \leftrightarrow {\sim}\sigma) \mid \sigma^+ \wedge \sigma^- \notin \Delta\})$, whence $\Delta \in \mathrm{Ext}(\Gamma^\pm)$ by Lemma 8. ■

Lemma 15 *If $\Gamma \subseteq \mathcal{W}^*$ and $A \in \mathcal{W}^*$ and $\Delta \in \mathrm{Ext}(\Gamma^\pm)$, then $A^\pm \in \Delta$ iff $v_s(A) = 1$ for all* **CLuNs***-valuations that verify Γ and for which $Ab(v_s) = \{\sigma \wedge {\sim}\sigma \mid \sigma^+ \wedge \sigma^- \in \Delta\}$.*

Proof. Supposing that the antecedent is true, we have to prove an equivalence.

For the left–right direction, suppose that $v_s(A) = 0$ for a **CLuNs**-valuation v_s that verifies Γ, for which $Ab(v_s) = \{\sigma \wedge {\sim}\sigma \mid \sigma^+ \wedge \sigma^- \in \Delta\}$. By Lemma 11, there is a regular **CL**-valuation v for which Cvv_s. By Lemma 12, v verifies $\Gamma^\pm$ and $v(A^\pm) = 0$. By Lemma 13, $Ab^\pm(v) = \{\sigma^+ \wedge \sigma^- \mid \sigma^+ \wedge \sigma^- \in \Delta\}$. So, by Lemma 10, v verifies Δ, whence $A^\pm \notin \Delta$.

For the right–left direction, suppose that $A^\pm \notin \Delta$. So there is a **CL**-valuation v that verifies Δ and for which $v(A^\pm) = 0$. By Lemma 10 v is regular and $Ab^\pm(v) = \{\sigma^+ \wedge \sigma^- \mid \sigma^+ \wedge \sigma^- \in \Delta\}$. By Lemma 11, there is a **CLuNs**-valuation v_s such that Cvv_s. By Lemma 12, v_s verifies Γ and $v_s(A) = 0$. By Lemma 13, $Ab(v_s) = \{\sigma \wedge {\sim}\sigma \mid \sigma^+ \wedge \sigma^- \in \Delta\}$. ■

Lemma 16 $(\sigma^+ \leftrightarrow \sigma) \wedge (\sigma^- \leftrightarrow {\sim}\sigma) \in \mathrm{nor}(\Gamma^\pm)$ *iff* $\sigma \wedge {\sim}\sigma \notin U(\Gamma)$.

Proof. If $\Delta \in \mathrm{Ext}(\Gamma^\pm)$, then, by the definition of $\mathrm{nor}(\Delta)$ and by Lemma 8, $(\sigma^+ \leftrightarrow \sigma) \wedge (\sigma^- \leftrightarrow {\sim}\sigma) \in \mathrm{nor}(\Delta)$ iff $\sigma^+ \wedge \sigma^- \notin \Delta$. So, by the definition of $\mathrm{nor}(\Gamma^\pm)$,

$$(\sigma^+ \leftrightarrow \sigma) \wedge (\sigma^- \leftrightarrow {\sim}\sigma) \in \mathrm{nor}(\Gamma^\pm) \tag{1}$$

iff there is no $\Delta \in \mathrm{Ext}(\Gamma^\pm)$ for which $\sigma^+ \wedge \sigma^- \in \Delta$. It follows by Lemma 14 that (1) iff there is no $\varphi \in \Phi(\Gamma)$ such that $\sigma \wedge {\sim}\sigma \in \varphi$. So, by Lemma 4, (1) iff $\sigma \wedge {\sim}\sigma \notin U(\Gamma)$. ■

Lemma 17 *If no unsigned sentential letters occur in* $A \in \mathcal{W}^\pm$ *and* $v(A) = 0$ *for a* **CL**-*valuation* v *that verifies* $\Gamma^\pm \cup \mathrm{nor}(\Gamma^\pm) \cup \mathcal{T}$, *then* $v'(A) = 0$ *for a regular* **CL**-*valuation* v' *that verifies* $\Gamma^\pm \cup \mathrm{nor}(\Gamma^\pm) \cup \mathcal{T}$.

Proof. Suppose that the antecedent is true. Let v' be exactly as v except in that $v'(\sigma) = v'(\sigma^+) = v(\sigma^+)$ whenever $v(\sigma^+) \neq v(\sigma^-)$. As no unsigned sentential letter occurs in $\Gamma^\pm$, in $\mathcal{T}$, or in A, v' verifies $\Gamma^\pm$ and $\mathcal{T}$ and $v'(A) = 0$. As $v(\sigma) = v(\sigma^+)$ whenever $(\sigma^+ \leftrightarrow \sigma) \wedge (\sigma^- \leftrightarrow {\sim}\sigma) \in \mathrm{nor}(\Gamma^\pm)$, v' verifies $\mathrm{nor}(\Gamma^\pm)$. ■

Let $A^{\sim\mp}$ be the result of replacing in $A^\pm$ every sentential letter σ^+ by the literal ${\sim}\sigma^-$ and every sentential letter σ^- by the literal ${\sim}\sigma^+$. The proof of Lemma 18 is obvious and left to the reader.

Lemma 18 *Where* $A \in \mathcal{W}^*$, ${\sim}({\sim}A)^\pm$ *is* **CL**-*equivalent to* $A^{\sim\mp}$.

Lemma 19 *If* v *is a regular* **CL**-*valuation for* $\mathcal{L}^\pm$, v_s *a* **CLuNs**-*valuation for* $\mathcal{L}$, Cvv_s, *and* $A \in \mathcal{W}^*$, *then* $v_s(!A) = 1$ *iff* $v({\sim}({\sim}A)^\pm) = 1$.

Proof. Suppose that the antecedent is true. $v_s(!A) = 1$ iff $v_s(A) = 1$ and $v_s({\sim}A) = 0$. In view of the **CLuNs**-semantics, $v_s(A) = 1$ and $v_s({\sim}A) = 0$ iff $v_s({\sim}A) = 0$. By Lemma 12, $v_s({\sim}A) = 0$ iff $v(({\sim}A)^\pm) = 0$. So $v_s(!A) = 1$ iff $v({\sim}({\sim}A)^\pm) = 1$. ■

Remark that ${\sim}({\sim}\sigma)^\pm$ is ${\sim}\sigma^-$ and that ${\sim}({\sim}{\sim}\sigma)^\pm$ is ${\sim}\sigma^+$.

Lemma 20 *If* $\Gamma \subseteq \mathcal{W}^*$, $\dagger\sigma_1 \vee \ldots \vee \dagger\sigma_n \in \mathcal{W}^*$ *(in which each* $\dagger$ *is either* ${\sim}$ *or nothing),* $\Sigma \subseteq \mathcal{S}$, *and* $\Delta = \mathrm{Cn}^\pm_{\mathbf{CL}}(\Gamma^\pm \cup \{(\sigma^+ \leftrightarrow \sigma) \wedge (\sigma^- \leftrightarrow {\sim}\sigma) \mid \sigma \in \mathcal{S} - \Sigma\})$ *or* $\Delta = \mathrm{Cn}^\pm_{\mathbf{CL}}(\Gamma^\pm \cup \{(\sigma^+ \leftrightarrow \sigma) \wedge (\sigma^- \leftrightarrow {\sim}\sigma) \mid \sigma \in \mathcal{S} - \Sigma\} \cup \mathcal{T})$, *then* $\dagger\sigma_1 \vee \ldots \vee \dagger\sigma_n \in \Delta$ *iff* ${\sim}({\sim} \dagger \sigma_1)^\pm \vee \ldots \vee {\sim}({\sim} \dagger \sigma_n)^\pm \in \Delta$.

Proof. Supposing that the antecedent is true, we have to prove an equivalence.

For the left–right direction, suppose that

$$\dagger\sigma_1 \vee \ldots \vee \dagger\sigma_n \tag{2}$$

is a member of Δ. *Case 1*: (2) is a minimal disjunction in Δ (viz. if any disjunct of (2) is deleted, the result is not a member of Δ). It follows that there is a **CL**-valuation that verifies Δ and that assigns the value 0 to all but one disjuncts of (2). So, if there is a $\sigma_i \in \{\sigma_1, \ldots, \sigma_n\} \cap \Sigma$, then, as σ_i does not occur unsigned in $\Gamma^{\pm}$ or in $\mathcal{T}$, some **CL**-valuation verifies Δ and assigns the value 0 to all disjuncts of (2),[20] which is impossible. It follows that $\sigma_1, \ldots, \sigma_n \in \mathcal{S} - \Sigma$ whence $\sim(\sim \dagger \sigma_1)^{\pm} \vee \ldots \vee \sim(\sim \dagger \sigma_n)^{\pm} \in \Delta$. *Case 2*: (2) is not a minimal disjunction in Δ. Then Case 1 obtains for a selection of disjuncts of (2), and, by Addition, $\sim(\sim \dagger \sigma_1)^{\pm} \vee \ldots \vee \sim(\sim \dagger \sigma_n)^{\pm} \in \Delta$.

The proof of the right–left direction is wholly analogous to the proof of the left–right direction. ■

Lemma 21 *If $\Gamma \subseteq \mathcal{W}^*$, $A \in \mathcal{W}^*$ and, for some $\Sigma \subseteq \mathcal{S}$, $\Delta = \mathrm{Cn}^{\pm}_{\mathbf{CL}}(\Gamma^{\pm} \cup \{(\sigma^+ \leftrightarrow \sigma) \wedge (\sigma^- \leftrightarrow \sim\sigma) \mid \sigma \in \mathcal{S} - \Sigma\})$ or $\Delta = \mathrm{Cn}^{\pm}_{\mathbf{CL}}(\Gamma^{\pm} \cup \{(\sigma^+ \leftrightarrow \sigma) \wedge (\sigma^- \leftrightarrow \sim\sigma) \mid \sigma \in \mathcal{S} - \Sigma\} \cup \mathcal{T})$, then $A \in \Delta$ iff $\sim(\sim A)^{\pm} \in \Delta$.*

Proof. Suppose that the antecedent is true. Let B be the conjunctive normal form of A, whence A is **CL**-equivalent to B and $\sim(\sim A)^{\pm}$ is **CL**-equivalent to $\sim(\sim B)^{\pm}$. Let $\dagger\sigma_1 \vee \ldots \vee \dagger\sigma_n$ be as in Lemma 20. $A \in \Delta$ iff $\dagger\sigma_1 \vee \ldots \vee \dagger\sigma_n \in \Delta$ for every conjunct $\dagger\sigma_1 \vee \ldots \vee \dagger\sigma_n$ of B. By Lemma 20, the consequent holds true iff $\sim(\sim\dagger\sigma_1)^{\pm} \vee \ldots \vee \sim(\sim\dagger\sigma_n)^{\pm} \in \Delta$ for every conjunct $\sim(\sim\dagger\sigma_1)^{\pm} \vee \ldots \vee \sim(\sim\dagger\sigma_n)^{\pm}$ of $B^{\sim\mp}$, and this consequent holds true iff $B^{\sim\mp} \in \Delta$. By Lemma 18, $B^{\sim\mp}$ is **CL**-equivalent to $\sim(\sim B)^{\mp}$, and this is equivalent to $\sim(\sim A)^{\pm}$. So $A \in \Delta$ iff $\sim(\sim A)^{\pm} \in \Delta$. ■

Lemma 22 *If $\Gamma \subseteq \mathcal{W}^*$, $A \in \mathcal{W}^*$, and $\Delta \in \mathrm{Ext}(\Gamma^{\pm})$, then $A \in \Delta$ iff $v_s(!A) = 1$ for all* **CLuNs***-valuations that verify Γ and for which $Ab(v_s) = \{\sigma \wedge \sim\sigma \mid \sigma^+ \wedge \sigma^- \in \Delta\}$.*

Proof. Let $\Gamma \subseteq \mathcal{W}^*$, $A \in \mathcal{W}^*$, and $\Delta \in \mathrm{Ext}(\Gamma^{\pm})$.

For the left–right direction, suppose that $v_s(!A) = 0$ for a **CLuNs**-valuation v_s that verifies Γ and for which $Ab(v_s) = \{\sigma \wedge \sim\sigma \mid \sigma^+ \wedge \sigma^- \in \Delta\}$. By Lemma 11, there is a regular **CL**-valuation such that Cvv_s. By Lemma 12, v verifies $\Gamma^{\pm}$, by Lemma 19, $v(\sim(\sim A)^{\pm}) = 0$, and by Lemma 13, $Ab^{\pm}(v) = \{\sigma^+ \wedge \sigma^- \mid \sigma^+ \wedge \sigma^- \in \Delta\}$. So, by Lemma 10, v verifies Δ, whence $\sim(\sim A)^{\pm} \notin \Delta$. But then $A \notin \Delta$ by Lemma 21.

For the right–left direction, suppose that $A \notin \Delta$, whence $\sim(\sim A)^{\pm} \notin \Delta$ by Lemma 21. So there is a **CL**-valuation v that verifies Δ and for which $v(\sim(\sim A)^{\pm}) = 0$. By Lemma 10 v is regular and $Ab^{\pm}(v) = \{\sigma^+ \wedge \sigma^- \mid \sigma^+ \wedge \sigma^- \in \Delta\}$. By Lemma 11, there is a **CLuNs**-valuation v_s such that Cvv_s. By Lemma 12, v_s verifies Γ, by Lemma 19, $v_s(!A) = 0$, and by Lemma 13, $Ab(v_s) = \{\sigma \wedge \sim\sigma \mid \sigma^+ \wedge \sigma^- \in \Delta\}$. ■

[20]We have seen that there is a **CL**-valuation v for which $v(\sigma_i) = 1$ whereas $v(\sigma_j) = 0$ for all $\sigma_j \in \{\sigma_1, \ldots, \sigma_n\} - \{\sigma_i\}$. So, where v' is exactly like v except that $v'(\sigma_i) = 0$, v' verifies all members of Δ and falsifies all disjuncts of (2).

Theorem 13 *Where* $\Gamma \subseteq \mathcal{W}^*$ *and* $A \in \mathcal{W}^*$, $\Gamma \vdash_s^{\pm} A$ *iff* $\Gamma \vdash_{\mathbf{CLuNs}^m} A$ *and* $\Gamma \vdash_s A$ *iff* $\Gamma \vdash_{\mathbf{CLuNs}^m} !A$.

Proof. Let $\Gamma \subseteq \mathcal{W}^*$ and $A \in \mathcal{W}^*$.

For the left–right direction, suppose that $\Gamma \nvdash_{\mathbf{CLuNs}^m} A$ (respectively $\Gamma \nvdash_{\mathbf{CLuNs}^m} !A$). So, by Definitions 5, 10, and 11 and Theorems 7 and 8, $v_s(A) = 0$ (respectively $v_s(!A) = 0$) for some **CLuNs**-valuation that verifies Γ and for which $Ab(v_s) \in \Phi(\Gamma)$. By Definition 1 and Lemmas 6 and 14, there is a $\Delta \in \mathrm{Ext}(\Gamma^{\pm})$ for which $\sigma^+ \wedge \sigma^- \in \Delta$ iff $\sigma \wedge {\sim}\sigma \in Ab(v_s)$. By Lemma 15 (respectively Lemma 22), $A^{\pm} \notin \Delta$ (respectively $A \notin \Delta$). So, by Definitions 1 and 2, $\Gamma \nvdash_s^{\pm} A$ (respectively $\Gamma \nvdash_s A$).

For the right–left direction, suppose that $\Gamma \nvdash_s^{\pm} A$ (respectively $\Gamma \nvdash_s A$). By Definition 2, there is a $\Delta \in \mathrm{Ext}(\Gamma^{\pm})$ for which $A^{\pm} \notin \Delta$ (respectively $A \notin \Delta$). By Lemma 15 (respectively Lemma 22), $v_s(A) = 0$ (respectively $v_s(!A) = 0$) for some **CLuNs**-valuation v_s that verifies Γ and for which $Ab(v_s) = \{\sigma \wedge {\sim}\sigma \mid \sigma^+ \wedge \sigma^- \in \Delta\}$. By Definition 1 and Lemmas 6 and 14, $Ab(v_s) \in \Phi(\Gamma)$, whence v_s is a minimal abnormal valuation that verifies Γ. So, by Definitions 5, 10, and 11 and Theorems 7 and 8, $\Gamma \nvdash_{\mathbf{CLuNs}^m} A$ (respectively $\Gamma \nvdash_{\mathbf{CLuNs}^m} !A$). ■

Theorem 14 *Where* $\Gamma \subseteq \mathcal{W}^*$ *and* $A \in \mathcal{W}^*$, $\Gamma \vdash_c^{\pm} A$ *iff* $\Gamma \vdash_{\mathbf{CLuNs}^n} A$ *and* $\Gamma \vdash_c A$ *iff* $\Gamma \vdash_{\mathbf{CLuNs}^n} !A$.

Proof. Let $\Gamma \subseteq \mathcal{W}^*$ and $A \in \mathcal{W}^*$.

For the left–right direction, suppose (i) that $\Gamma \vdash_c^{\pm} A$ (respectively $\Gamma \vdash_c A$) and (ii) that $\Gamma \nvdash_{\mathbf{CLuNs}^n} A$ (respectively $\Gamma \nvdash_{\mathbf{CLuNs}^n} !A$). By Definition 2 (i) entails that $A^{\pm} \in \Delta$ (respectively $A \in \Delta$) for some $\Delta \in \mathrm{Ext}(\Gamma^{\pm})$. By Definition 1 and Lemmas 6 and 14, $\{\sigma \wedge {\sim}\sigma \mid \sigma^+ \wedge \sigma^- \in \Delta\} \in \Phi(\Gamma)$. In view of Definitions 5 and 14, Theorem 7, and Corollary 2, (ii) and $\{\sigma \wedge {\sim}\sigma \mid \sigma^+ \wedge \sigma^- \in \Delta\} \in \Phi(\Gamma)$ jointly entail that, $v_s(A) = 0$ (respectively $v_s(!A) = 0$) for some **CLuNs**-valuation that verifies $\Gamma^{\pm}$ and for which $Ab(v_s) = \{\sigma \wedge {\sim}\sigma \mid \sigma^+ \wedge \sigma^- \in \Delta\}$. But then, by Lemma 15 (respectively Lemma 22), $A^{\pm} \notin \Delta$ (respectively $A \notin \Delta$), which is impossible.

For the right–left direction, suppose that $\Gamma \nvdash_c^{\pm} A$ (respectively $\Gamma \nvdash_c A$). So, By Definition 2, $A^{\pm} \notin \Delta$ (respectively $A \notin \Delta$) holds for all $\Delta \in \mathrm{Ext}(\Gamma^{\pm})$. By Lemma 15 (respectively 22), it holds for all $\Delta \in \mathrm{Ext}(\Gamma^{\pm})$ that $v_s(A) = 0$ (respectively $v_s(!A) = 0$) for some **CLuNs**-valuation v_s that verifies Γ and for which $Ab(v_s) = \{\sigma \wedge {\sim}\sigma \mid \sigma^+ \wedge \sigma^- \in \Delta\}$. By Definition 1 and Lemmas 6 and 14, it holds for all $\varphi \in \Phi(\Gamma)$ that $v_s(A) = 0$ (respectively $v_s(!A) = 0$) for some **CLuNs**-valuation v_s that verifies Γ and for which $Ab(v_s) = \varphi$. But then $\Gamma \nvdash_{\mathbf{CLuNs}^n} A$ (respectively $\Gamma \nvdash_{\mathbf{CLuNs}^n} !A$) by Definitions 5 and 14 and Corollary 2. ■

Theorem 15 *Where* $\Gamma \subseteq \mathcal{W}^*$ *and* $A \in \mathcal{W}^*$, $\Gamma \vdash_p^{\pm} A$ *iff* $\Gamma \vdash_{\mathbf{CLuNs}^r} A$ *and* $\Gamma \vdash_p A$ *iff* $\Gamma \vdash_{\mathbf{CLuNs}^r} !A$.

Proof. Let $\Gamma \subseteq \mathcal{W}^*$ and $A \in \mathcal{W}^*$.

For the left–right direction, suppose that $\Gamma \nvdash_{\mathbf{CLuNs}^r} A$ (respectively $\Gamma \nvdash_{\mathbf{CLuNs}^r} !A$). By Theorem 5 and Definitions 5, 8 and 9, $v_s(A) = 0$ (respectively $v_s(!A) = 0$) for some **CLuNs**-valuation v_s that verifies Γ and for which $Ab(v_s) \subseteq U(\Gamma)$. In view of Lemma 11, there is a regular **CL**-valuation v such that Cvv_s. As

v is regular, v verifies $\mathcal{T}$. By Lemma 12, v verifies $\Gamma^{\pm}$. By Lemma 12 (respectively Lemma 19), $v(A^{\pm}) = 0$ (respectively $v({\sim}({\sim}A)^{\pm}) = 0$). By Lemma 13, if $\sigma \wedge {\sim}\sigma \notin U(\Gamma)$, then $\sigma^+ \wedge \sigma^- \notin Ab^{\pm}(v)$ and hence $v(\sigma^+) \neq v(\sigma^-)$. As v is regular, $v((\sigma^+ \leftrightarrow \sigma) \wedge (\sigma^- \leftrightarrow {\sim}\sigma)) = 1$ whenever $\sigma \wedge {\sim}\sigma \notin U(\Gamma)$ (by Definition 15). So, by Lemma 16, $v((\sigma^+ \leftrightarrow \sigma) \wedge (\sigma^- \leftrightarrow {\sim}\sigma)) = 1$ whenever $(\sigma^+ \leftrightarrow \sigma) \wedge (\sigma^- \leftrightarrow {\sim}\sigma) \in \mathrm{nor}(\Gamma^{\pm})$. By the Soundness of **CL** with respect to its semantics, $\Gamma \nvdash_p^{\pm} A$ (respectively $\Gamma^{\pm} \cup \mathrm{nor}(\Gamma^{\pm}) \cup \mathcal{T} \nvdash_{\mathbf{CL}} {\sim}({\sim}A)^{\pm}$ and , by Lemma 21, $\Gamma^{\pm} \cup \mathrm{nor}(\Gamma^{\pm}) \cup \mathcal{T} \nvdash_{\mathbf{CL}} A$). But then $\Gamma \nvdash_p^{\pm} A$ (respectively $\Gamma \nvdash_p A$) in view of Definition 2.

For the right–left direction, suppose that $\Gamma \nvdash_p^{\pm} A$ (respectively $\Gamma \nvdash_p A$). By Definition 2, $\Gamma^{\pm} \cup \mathrm{nor}(\Gamma^{\pm}) \cup \mathcal{T} \nvdash_{\mathbf{CL}} A^{\pm}$ (respectively $\Gamma^{\pm} \cup \mathrm{nor}(\Gamma^{\pm}) \cup \mathcal{T} \nvdash_{\mathbf{CL}} A$ whence, by Lemma 21, $\Gamma^{\pm} \cup \mathrm{nor}(\Gamma^{\pm}) \cup \mathcal{T} \nvdash_{\mathbf{CL}} {\sim}({\sim}A)^{\pm}$). So there is a **CL**-valuation v that verifies $\Gamma^{\pm} \cup \mathrm{nor}(\Gamma^{\pm}) \cup \mathcal{T}$ and for which $v(A^{\pm}) = 0$ (respectively $v({\sim}({\sim}A)^{\pm}) = 0$, whence $v(A^{\sim\mp}) = 0$ by Lemma 18). By Lemma 17, there is a regular **CL**-valuation v that verifies $\Gamma^{\pm} \cup \mathrm{nor}(\Gamma^{\pm}) \cup \mathcal{T}$ and for which $v(A^{\pm}) = 0$ (respectively $v(A^{\sim\mp}) = 0$, whence $v({\sim}({\sim}A)^{\pm}) = 0$ in view of Lemma 18). In view of the **CL**-semantics, $\sigma^+ \wedge \sigma^- \notin Ab^{\pm}(v)$ if $(\sigma^+ \leftrightarrow \sigma) \wedge (\sigma^- \leftrightarrow {\sim}\sigma) \in \mathrm{nor}(\Gamma)$. So, by Lemma 16, $\sigma^+ \wedge \sigma^- \notin Ab^{\pm}(v)$ if $\sigma \wedge {\sim}\sigma \notin U(\Gamma)$. As v is regular, there is a **CLuNs**-valuation v_s such that Cvv_s. By Lemma 12, v_s verifies Γ. By Lemma 12 (respectively Lemma 19), $v_s(A) = 0$ (respectively $v_s(!A) = 0$). By Lemma 13, $\sigma \wedge {\sim}\sigma \notin Ab(v_s)$ if $\sigma \wedge {\sim}\sigma \notin U(\Gamma)$, whence $Ab(v_s) \subseteq U(\Gamma)$. But then, by Definitions 5, 8, and 9 and Theorem 5, $\Gamma \nvdash_{\mathbf{CLuNs}^r} A$ (respectively $\Gamma \nvdash_{\mathbf{CLuNs}^r} !A$). ■

7 The Original Sceptical Consequence Relations

In Section 3 we have deviated, for the prudent signed and unsigned consequence relations, from the original definitions, which go as follows (we use a different font for the subscripted p):

Definition 18 *Where $\Gamma \subseteq \mathcal{W}^*$ and $A \in \mathcal{W}^*$,*

prudent unsigned consequence*:* $\Gamma \vdash_{\mathsf{p}} A$ *iff* $A \in \mathrm{Cn}^{\pm}_{\mathbf{CL}}(\Gamma^{\pm} \cup \mathrm{nor}(\Gamma^{\pm}))$

prudent signed consequence*:* $\Gamma \vdash^{\pm}_{\mathsf{p}} A$ *iff* $A^{\pm} \in \mathrm{Cn}^{\pm}_{\mathbf{CL}}(\Gamma^{\pm} \cup \mathrm{nor}(\Gamma^{\pm}))$

We deviated for different reasons. The first, pragmatic, reason is that the deviation allows for a more systematic characterization of the consequence relations in Section 6. A second, philosophical, reason is that, where the problem is to handle inconsistencies, it is odd that the prudent consequence relations do not (seem to) require negation-completeness (either A or ${\sim}A$ is true), the more so as the credulous and skeptical consequence relations are negation-complete. A third reason is related to the prudent signed consequence relation only and requires some more explanation.

Astonishing as it may seem, the change in the definition of the prudent unsigned consequence relation does not affect its consequence set:

Theorem 16 *Under the linguistic restrictions imposed by the definitions on Γ and A, $\Gamma \vdash_{\mathsf{p}} A$ iff $\Gamma \vdash_p A$.*

Proof. The left–right direction is obvious in view of the definitions. For the right–left direction, suppose that some **CL**-valuation v verifies $\Gamma^{\pm} \cup \mathrm{nor}(\Gamma^{\pm})$

but that $v(A) = 0$ (whence $\Gamma \nvdash_{\mathsf{p}} A$) and that, for one or more σ, $v(\sigma^+) = v(\sigma^-) = 0$ (whence v does not verify $\mathcal{T}$). Let v' be exactly as v except in that $v(\sigma^+) = v'(\sigma^-) = 1$ whenever $v(\sigma^+) = v(\sigma^-) = 0$. As v verifies $\Gamma^{\pm}$, so does v' (for the reasons explained in the proof of Lemma 6). As v verifies nor($\Gamma^{\pm}$), so does v' (because v verifies nor($\Gamma^{\pm}$) and σ cannot occur in any member of nor($\Gamma^{\pm}$) if $v(\sigma^+) = v(\sigma^-) = 0$). As $v(A) = 0$, $v'(A) = 0$ (because σ does not occur in $A \in \mathcal{W}^{\pm}$). But v' verifies $\mathcal{T}$. So $\Gamma \nvdash_p A$. ■

However $\Gamma \vdash^{\pm}_{p} A$ does *not* entail $\Gamma \vdash^{\pm}_{\mathsf{p}} A$. Of course the original prudent signed consequence relation can be reconstructed in terms of an adaptive logic, viz. in terms of an adaptive logic that has **CLoNs** as its lower limit logic.[21] **CLoNs** is obtained by removing axiom A$\sim$ from the axiom system for **CLuNs**; its semantics by removing clause C4 from the **CLuNs**-semantics.

The adaptive logic **CLoNs**^{r1} is defined by the lower limit logic **CLoNs**, the set of abnormalities Ω_1 and the Reliability strategy. Where $\Gamma \subseteq \mathcal{W}^*$ and $A \in \mathcal{W}^*$:

$\Gamma \vdash^{\pm}_{\mathsf{p}} A$ iff $\Gamma \vdash_{\mathbf{CLoNs}^{r1}} A$

The proof that the characterization is correct is wholly analogous to the proofs (for the signed relations) from the previous section, except that "regular **CL**-valuation" should be replaced by "quasi-regular **CL**-valuation," where a **CL**-valuation for $\mathcal{L}^{\pm}$ is quasi-regular iff, for all $\sigma \in \mathcal{S}$, $v(\sigma) = v(\sigma^+)$ whenever $v(\sigma^+ \wedge \sigma^-) = 0$.

That the original prudent signed consequence relation is characterized by a different adaptive logic than the five other original consequence relations, reveals that it handles premise sets differently. An immediate consequence, for example, is that, for the original prudent signed consequence relation, premise sets are invariant under **CLoNs**-transformations only (if $\mathrm{Cn}_{\mathbf{CLoNs}}(\Gamma_1) = \mathrm{Cn}_{\mathbf{CLoNs}}(\Gamma_2)$ then $\Gamma_1 \vdash^{\pm}_{\mathsf{p}} A$ iff $\Gamma_2 \vdash^{\pm}_{\mathsf{p}} A$), whereas all other original consequence relations are invariant under **CLuNs**-transformations.

This is easily illustrated by the following example. Let $\Gamma_1 = \{p \vee q, p \vee {\sim}q, {\sim}p \vee q, {\sim}p \vee {\sim}q\}$ and $\Gamma_2 = \Gamma_1 \cup \{p \vee {\sim}p\}$. Γ_1 and Γ_2 have exactly the same sets of consequences for all discussed consequence relations except for the original prudent signed consequence relation. Indeed, $\Gamma_1 \nvdash^{\pm}_{\mathsf{p}} p \vee {\sim}p$ (because $\Gamma^{\pm}_1 \cup \mathrm{nor}(\Gamma^{\pm}_1) \nvdash_{\mathbf{CL}} p^+ \vee p^-$), whereas $\Gamma_2 \vdash^{\pm}_{\mathsf{p}} p \vee {\sim}p$ (because $p^+ \vee p^- \in \Gamma_2$). This also shows that the modified prudent signed consequence relation from Section 3 is more coherent with the five other consequence relations than the original one. Indeed, $\Gamma_1 \vdash^{\pm}_{p} p \vee {\sim}p$ (because $p^+ \vee p^- \in \mathcal{T}$).[22]

The authors of [18] did not remark this anomaly. Worse, their Theorem 4.1 states that, if $\Gamma \vdash_{\mathsf{p}} A$, then $\Gamma \vdash^{\pm}_{\mathsf{p}} A$. To see that this is a mistake: $\Gamma_1 \vdash_{\mathsf{p}} p \vee {\sim}p$ but $\Gamma_1 \nvdash^{\pm}_{\mathsf{p}} p \vee {\sim}p$. When one looks at the 'proof' of the theorem (p. 209 sub (1)), one readily sees what went wrong. The authors claim (modified to our use of symbols) in relation to an unsigned consequence A that "for every (unsigned) occurrence of σ or ${\sim}\sigma$ in A, there is an equivalence $\sigma \leftrightarrow \sigma^+$ or ${\sim}\sigma \leftrightarrow \sigma^-$ in the

[21] **CLoNs** allows for both gluts and gaps with respect to negation, and the "s" refers to the fact that it (its propositional version) was first presented by Schütte in [22].

[22] Remark, however, that $\Gamma_1 \vdash^{\pm}_{\mathsf{p}} (p \vee {\sim}p) \vee (q \vee {\sim}q)$. In general, it can be proved that, for every $\Gamma \subseteq \mathcal{W}^*$ and for every $\sigma_1 \in \mathcal{S}$, either $\Gamma \vdash^{\pm}_{\mathsf{p}} \sigma_1 \vee {\sim}\sigma_1$ or there is a set $\{\sigma_2, \ldots, \sigma_n\}$ such that $\Gamma \vdash^{\pm}_{\mathsf{p}} (\sigma_1 \vee {\sim}\sigma_1) \vee \ldots \vee (\sigma_n \vee {\sim}\sigma_n)$ whereas $\Gamma \nvdash^{\pm}_{\mathsf{p}} \sigma_i \vee {\sim}\sigma_i$ whenever $i \in \{1, \ldots, n\}$—another bizarre property of the original prudent signed consequence relation.

underlying theory" (the underlying theories are mentioned in the consequent of the equivalences in Definitions 2 and 18). But this is simply false. First of all $\{p, q, \sim q\} \vdash_i p \vee q$ (where $i \in \{s, \mathsf{p}, c\}$) holds true notwithstanding the absence of $q \leftrightarrow q^+$, because $\{p, q, \sim q\} \vdash_i p$. Next, all **CL**-theorems (in $\mathcal{W}^*$) are members of all unsigned consequence relations, because the latter are closed under **CL**. Put differently, even in the absence of $(\sigma \leftrightarrow \sigma^+) \wedge (\sim\sigma \leftrightarrow \sigma^-)$ all **CL**-valuations have to assign a truth-value to σ, whence they all verify all **CL**-theorems in $\mathcal{W}^*$.

Of course we could easily characterize variants for all six consequence relations in terms of $\mathbf{CLoNs}^{r1}$, $\mathbf{CLoNs}^{m1}$, and $\mathbf{CLoNs}^{n1}$ respectively, but this would be somewhat funny, as it is not clear how the variants are defined in terms of extensions of $\Gamma^{\pm}$ and as the authors of [18] did not realize the singular properties of the prudent signed consequence relation. Moreover, we do not like the whole approach. If there are reasons to allow for both gluts and gaps with respect to negation, and next to interpret the premises as normally as possible, one should not only rule out negation gluts whenever possible, but also negation gaps. So, in that case, we would advertise the adaptive logics $\mathbf{CLoNs}^r$, $\mathbf{CLoNs}^m$, and $\mathbf{CLoNs}^n$, defined from the lower limit logic **CLoNs**, the set of abnormalities $\Omega_2 = \{A \wedge \sim A \mid A \in \mathcal{S}\} \cup \{\neg A \wedge \neg\sim A \mid A \in \mathcal{S}\}$, and the strategies Reliability, Minimal Abnormality, and Normal Selections respectively.

The previous paragraph involves an implicit criticism of the signed systems approach. First, the transformation to $\Gamma^{\pm}$ allows for both gluts and gaps—both $v(\sigma^+) = v(\sigma^-) = 1$ and $v(\sigma^+) = v(\sigma^-) = 0$ are possible—which is odd if one merely wants to handle inconsistencies. Next, for five of the six consequence relations negation gaps ($v(\sigma^+) = v(\sigma^-) = 0$) are ruled out indirectly, viz. as a *side effect* (and not a transparent one) of minimizing negation gluts. No wonder something went wrong somewhere.

8 Linguistic Extensions of the Logics

Until now it was always presupposed that all premises belong to $\mathcal{W}^*$. The reason for this is that, in [18] and [19], implication is handled as a disjunction: $A \rightarrow B$ is handled exactly as $\sim A \vee B$.

Once the signed systems have been characterized in terms of adaptive logics, there is no reason for not allowing the detachable implication $\supset$ to occur in the premises. The main effect of this move is as expected, viz. that the implication is detachable independently of the occurrence of inconsistencies. This has not only effects for the signed consequence relations, but also, as we shall illustrate, for the unsigned ones.

Remember that $\mathcal{W}^{\not\neg}$ comprises all members of $\mathcal{W}$ in which $\neg$ does not occur. It is easily provable that every finite $\Gamma \subset \mathcal{W}^{\not\neg}$ has **CLuNs**-models and hence has a non-trivial **CLuNs**-consequence set. So there is no technical justification for not allowing $\supset$ in premises. That such premises can only be handled in the signed systems at the expense of further complications (which we shall not describe here) only illustrates the restrictions imposed by signed systems.

Example 1. $\Gamma_1 = \{p, \sim p, p \rightarrow q\}$, $\Gamma_2 = \{p, \sim p, p \supset q\}$. For the three adaptive logics considered (let $\mathbf{CLuNs}^x$ refer to all three of them), $\Gamma_1 \vdash_{\mathbf{CLuNs}^x} !A$ only holds if A is a **CL**-theorem. This is not the case for Γ_2, as, for example, $\Gamma_2 \vdash_{\mathbf{CLuNs}^x} !q$. As $p \rightarrow q$ is an obvious consequence of $\sim p$, $\mathrm{Cn}^x_{\mathbf{CLuNs}}(\Gamma_1) =$

$\mathrm{Cn}^x_{\mathbf{CLuNs}}(\{p, \sim p\})$. In view of the fact that Modus Ponens is **CLuNs**-valid for $\supset$, $\mathrm{Cn}^x_{\mathbf{CLuNs}}(\Gamma_2) = \mathrm{Cn}^x_{\mathbf{CLuNs}}(\{p, \sim p, q\})$.

Unlike Modus Ponens, Modus Tollens is not **CLuNs**-valid. From (i) A is false or B is true and (ii) $\sim B$ is false does not follow that $\sim A$ is true. So $\supset$ is asymmetric, as its semantic interpretation clearly reveals. The effect of this is illustrated by the following example.

Example 2. $\Gamma_1 = \{p, \sim q, p \to q, \sim p \vee r, q \vee s\}$, $\Gamma_2 = \{p, \sim q, p \supset q, \sim p \vee r, q \vee s\}$. Γ_1 has two extensions, one in which p is consistent and q inconsistent, and one in which p is inconsistent and q consistent. Γ_2 to the contrary has only one extension, in which q is inconsistent. As a result, $\Gamma_1 \vdash_{\mathbf{CLuNs}^m} !(p \vee \sim q) \wedge !(r \vee s)$, whereas $\Gamma_2 \vdash_{\mathbf{CLuNs}^m} !p \wedge !r$.

Remark that $\Gamma_2 \nvdash_{\mathbf{CLuNs}^m} !(p \supset q)$ because $\Gamma_2 \vdash_{\mathbf{CLuNs}} p \wedge \sim q$.

Most realistic applications of the signed systems will require that they are extended to the predicative level. To do so, however, is far from obvious. One might attach the signs to primitive formulas, rather than to letters, but then one should allow for quantification over signed formulas, which turns the signs into operators rather than into notational symbols that introduce new sentential letters—for example one should ensure that $\forall x(Px^- \vee Qx^+) \vdash_{\mathbf{CL}} Pa^- \vee Qa^+$; and special provisions are required to ensure $a = b^+, a = c^- \vdash_{\mathbf{CL}} b = c^-$ whereas $a = b^-, a = c^- \nvdash_{\mathbf{CL}} b = c^-$. Further trouble arises from the predicative equivalent of $(\sigma^+ \leftrightarrow \sigma) \wedge (\sigma^- \leftrightarrow \sim\sigma)$, in which σ should apparently be replaced by closed primitive formulas only.

No such trouble arises for the adaptive logics. The language and sets of formulas are extended as expected. The predicative versions of $\mathbf{CLuNs}^r$, $\mathbf{CLuNs}^m$, and $\mathbf{CLuNs}^n$ are simply obtained by having the full (predicative) logic **CLuNs** (see [13] for its syntax and semantics) as the lower limit logic, retaining the strategy, and redefining $\Omega_1 = \{\exists(A \wedge \sim A) \mid A \in \mathcal{F}^p\}$, in which $\mathcal{F}^p$ is the set of (closed or open) primitive formulas and $\exists A$ is the existential closure of A (A preceded by an existential quantifier over every free variable that occurs in it).

9 In Conclusion

We have modified the prudent signed consequence relation and offered good reasons for doing so. Remark that this modification is a result of our characterization in terms of adaptive logics. This holds even for the philosophical reasons because the oddities of the prudent signed consequence relation were discovered by attempting to find their adaptive characterization.

We have spelled out a number of advantages of the adaptive characterization in Section 1 and trust that these advantages became clear during the preceding sections. As for the transparency that was improved by the adaptive characterization, remark that, for all adaptive logics, the consequence sets are closed under the lower limit logic, and the premise sets are invariant under transformations by the lower limit logic. By the latter we mean that, if $\mathrm{Cn}_{\mathbf{LLL}}(\Gamma_1) = \mathrm{Cn}_{\mathbf{LLL}}(\Gamma_2)$ then $\mathrm{Cn}_{\mathbf{AL}}(\Gamma_1) = \mathrm{Cn}_{\mathbf{AL}}(\Gamma_2)$. As for further properties that are available (for all adaptive logics in standard format), we mention only a few general ones (they hold for all three strategies where not indicated otherwise):

Strong Reassurance for Minimal Abnormality (Theorem 10) and Reliability.

Fixed Point: $\mathrm{Cn}_{\mathbf{AL}}(\Gamma) = \mathrm{Cn}_{\mathbf{AL}}(\mathrm{Cn}_{\mathbf{AL}}(\Gamma))$.

Immunity: for all $\Delta \subseteq \Omega$, $Dab(\Delta) \in \mathrm{Cn}_{\mathbf{AL}}(\Gamma)$ iff $Dab(\Delta) \in \mathrm{Cn}_{\mathbf{LLL}}(\Gamma)$.

Cautious Cut: if $\Delta \subseteq \mathrm{Cn}_{\mathbf{AL}}(\Gamma)$ and $A \in \mathrm{Cn}_{\mathbf{AL}}(\Gamma \cup \Delta)$, then $A \in \mathrm{Cn}_{\mathbf{AL}}(\Gamma)$.

Cautious Monotonicity:
if $\Delta \subseteq \mathrm{Cn}_{\mathbf{AL}}(\Gamma)$, and $A \in \mathrm{Cn}_{\mathbf{AL}}(\Gamma)$, then $A \in \mathrm{Cn}_{\mathbf{AL}}(\Gamma \cup \Delta)$.

Relation between the adaptive logics:
general: $\mathrm{Cn}_{\mathbf{LLL}}(\Gamma) \subseteq \mathrm{Cn}_{\mathbf{AL}^r}(\Gamma) \subseteq \mathrm{Cn}_{\mathbf{AL}^m}(\Gamma) \subseteq \mathrm{Cn}_{\mathbf{AL}^n}(\Gamma) \subseteq \mathrm{Cn}_{\mathbf{ULL}}(\Gamma)$
Γ normal: $\mathrm{Cn}_{\mathbf{LLL}}(\Gamma) \subset \mathrm{Cn}_{\mathbf{AL}^r}(\Gamma) = \mathrm{Cn}_{\mathbf{AL}^m}(\Gamma) = \mathrm{Cn}_{\mathbf{AL}^n}(\Gamma) = \mathrm{Cn}_{\mathbf{ULL}}(\Gamma)$
Γ abnormal: $\mathrm{Cn}_{\mathbf{LLL}}(\Gamma) \subseteq \mathrm{Cn}_{\mathbf{AL}^r}(\Gamma) \subseteq \mathrm{Cn}_{\mathbf{AL}^m}(\Gamma) \subseteq \mathrm{Cn}_{\mathbf{AL}^n}(\Gamma) \subset \mathrm{Cn}_{\mathbf{ULL}}(\Gamma)$

plus a specification of the conditions under which $\subseteq$ can be specified to $\subset$ in the above.

We also claimed that the adaptive characterization gives the logics a place in a unified framework, which facilitates the comparison with other inconsistency-handling logics. To illustrate this, compare the lower half of the list at the beginning of Section 6 with the following one for the (flat) Rescher–Manor consequence relations (see [20], [21], [17]). Let $\mathcal{W}^{\not\sim}$ comprise the members of $\mathcal{W}$ in which $\sim$ does not occur and let $\Gamma^{\sim\neg} = \{\sim\neg A \mid A \in \Gamma\}$. It was proven in [7] that, where $\Gamma \subseteq \mathcal{W}^{\not\sim}$ and $A \in \mathcal{W}^{\not\sim}$:

$\Gamma \vdash_{Free} A$ iff $\Gamma^{\sim\neg} \vdash_{\mathbf{CLuN}^r} A$.
$\Gamma \vdash_{Strong} A$ iff $\Gamma^{\sim\neg} \vdash_{\mathbf{CLuN}^m} A$.
$\Gamma \vdash_{Weak} A$ iff $\Gamma^{\sim\neg} \vdash_{\mathbf{CLuN}^n} A$

In these definitions, the adaptive logics $\mathbf{CLuN}^x$ are defined by the following triples: lower limit logic: the (predicative) paraconsistent logic **CLuN** from [5] (see [1] for the propositional version); set of abnormalities $\Omega = \{\exists(A \wedge \sim A) \mid A \in \mathcal{F}\}$, in which $\mathcal{F}$ is the set of all (open and closed) formulas; strategy: Reliability, Minimal Abnormality, and Normal Selections respectively. Incidentally, the adaptive approach at once offers an interesting set of alternative logics by further varying on lower limit logic, possibly combining it with the $\sim\neg$ transformation of the premises, with the !-requirement on the consequences, with modal transformations as in [9], etc.—the strategies do not seem to offer much further variations.

Let us finally offer some comments on the logics studied in this paper. For mathematical applications, inconsistency-adaptive logics that have **CLuNs** as their lower limit logic may be useful. For inconsistencies that occur in empirical theories, it was argued, for example in [6], that inconsistency-adaptive logics that have **CLuN** as their lower limit logic are usually superior. So we suggest that the adequacy of specific inconsistency-handling mechanisms should be given due attention. We also have some doubt about the usefulness of selecting the 'consistent part' (the consequences $!A$ for which consistency is adaptively provable) of an inconsistent theory. If the intention is to replace an inconsistent theory by a consistent improvement of it, the consistent part of the theory is clearly too poor. One has to locate the problems, viz. the inconsistencies, and next set out to eliminate them in a justified way, which will require non-logical work triggered by the specific inconsistencies. Maybe consistent parts are useful for some application contexts, but then the adequacy of a specific method for obtaining them should be carefully studied.

We would like to submit, as a general conclusion, that the adaptive approach is promising for unifying and systematically elaborating the study of inconsistency-handling mechanisms.[23]

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