A Procedural Criterion for Final Derivability in Inconsistency-Adaptive Logics*

Diderik Batens
Centre for Logic and Philosophy of Science
Universiteit Gent, Belgium
Diderik.Batens@ugent.be

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Abstract

This paper concerns a (prospective) goal directed proof procedure for the propositional fragment of the inconsistency-adaptive logic **ACLuN1**. At the propositional level, the procedure forms an algorithm for final derivability. If extended to the predicative level, it provides a *criterion* for final derivability. This is essential in view of the absence of a positive test. The procedure may be generalized to all flat adaptive logics.

1 The Problem

Inference relations for which there is no positive test abound in both everyday and scientific reasoning processes.[1] Adaptive logics are a means for characterizing such inference relations. The characterization has a specific metalinguistic standard format. This format provides the logic with a semantics and with a proof theory, and warrants soundness, completeness, and a set of properties of the logic.[2] The first adaptive logics were inconsistency-adaptive. The articulation of other adaptive logics provided increasing insight into the underlying mechanisms and required that adaptive logics were systematized in a new way. This systematization is presented in [16] and will be followed here.

This paper is concerned with a specific problem in adaptive logics. I describe the problem in the sequel of this section. A substantiated motivation for adaptive logics has been presented in many other papers ([5], [6], [7], [8], [13], [38], etc.); repeating it here would leave no room for the results I want to present. The first two sentences of this section summarize the motivation. I

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[1] A positive test is a systematic procedure that, for every set of premises Γ and for every conclusion A, leads after finitely many steps to a "yes" if A is a consequence of Γ. Remark that the consequence relation defined by classical logic is undecidable, but that there is a positive test for it—see [26] for such matters.

[2] Only part of these results are written up, viz. in [17].

shall, however, do my utmost to offer the reader a good idea of the functioning of adaptive logics.

An especially important feature of adaptive logics is their dynamic proof theory. This proof theory is intended for *explicating* actual reasoning—see [34] for a historical example—a task that cannot be accomplished by definitions, semantic systems, and other more abstract characterizations.

The dynamics of the proof theory results from the absence of a positive test. More often than not, the dynamics is double. The *external* dynamics is well known: as new premises become available, consequences derived from the earlier premise set may be withdrawn. In other words, the external dynamics results from the non-monotonic character of the consequence relation—the fact that, for some Γ, Δ and A, $\Gamma \vdash A$ but $\Gamma \cup \Delta \nvdash A$. The *internal* dynamics is very different from the external one. Even if the premise set is constant, certain formulas are considered as derived at some stage of the proof, but are considered as not derived at a later stage. For any consequence relation, insight into the premises is only gained by deriving consequences from them. In the absence of a positive test, this results in the internal dynamics. The external dynamics always entails an internal dynamics. The converse, however, does not hold. The Weak consequence relation from [44] and [45]—see [23] and [24] for an extensive study of such consequence relations—is monotonic. Nevertheless, its proof theory necessarily displays an internal dynamics because there is no positive test for it. Also, some logics for which there is a positive test, may nevertheless be characterized in a nice way in terms of a dynamic proof theory—see [15] on the pure calculus of the **R**-implication from [2].

Dynamic proofs differ in two main respects from usual proofs. The first difference concerns annotated versions. Apart from (i) a line number, (ii) a formula, (iii) the line numbers of the formulas from which the formula is derived, and (iv) the rule by which the formula is derived (the latter two are the justification of the line), dynamic proofs also contain (v) a *condition.* Intuitively, this is a set of formulas that are supposed to be false, or, to be more precise, formulas the truth of which is not required by the premises.

The second main difference is that, apart from the deduction rules that allow one to add lines to the proof, there is a *marking definition.* The underlying idea is as follows. As the proof proceeds, more formulas are derived from the premises. In view of these formulas, some conditions may turn out not to hold. The lines at which such conditions occur are *marked.* Formulas derived at marked lines are taken not to be derived from the premises. In other words, they are considered as 'out'. One way to understand the procedure is as follows. As the proof proceeds, one's insight into the premises improves. More particularly, some of the conditions that were introduced earlier may turn out not to hold.

For any stage of the proof, the marking definition settles which lines are marked and which lines are unmarked. This leads to a precise definition of *derivability at a stage.* Notwithstanding the precise character of this notion, we also want a more stable form of derivability, which is called *final derivability.* The latter does not depend on the stage of the proof; nor does it depend on the way in which a specific proof from a set of premises proceeds. It is an abstract and stable relation between a set of premises and a conclusion. A different way for putting this is that final derivability refers to a stage of the proof at which the mark (or its absence) of a line has become stable. Final derivability should be sound and strongly complete with respect to the semantics. For any adaptive logic **AL**, A should be finally derivable from Γ ($\Gamma \vdash_{\mathbf{AL}} A$) if and only if A is a semantic consequence of Γ ($\Gamma \vDash_{\mathbf{AL}} A$).

Consider a dynamic proof from a set of premises. At any point in time, the proof will be finite. It will reveal what is derivable from the premises at that stage of the proof. But obviously we are interested in final derivability. Whence the question: what does a proof at a stage reveal about final derivability? As there is no positive test for the consequence relation, there is no algorithm for final derivability. So, there are at best certain *criteria* to decide, for specific A and Γ, whether A is finally derivable from Γ.

What if no criterion enables one to conclude from the proof whether certain formulas are or are not finally derivable from the premise set? The answer or rather the answers to this question are presented in [10]. Roughly, they go as follows. First, there is a characteristic semantics for derivability at a stage. Next, it can be shown that, as the dynamic proof proceeds, the insight into the premises provided by the proof never decreases and may increase.[3] In other words, derivability at a stage provides an estimate for final derivability, and, as the proof proceeds, this estimate may become better, and never becomes worse. In view of all this, derivability at a stage gives one exactly what one might expect, viz. a fallible but *sensible* estimate of final derivability.[4] At any stage of the proof, one has to decide (obviously on the basis of pragmatic considerations) whether one will continue the proof or rely on present insights.

Needless to say, one should apply a criterion for final derivability whenever one can. This motivated the search for such criteria—see [10], [19] and [20]. Unfortunately, most of these criteria are complex and only transparent for people that are well acquainted with dynamic proofs. Recently, it turned out that a specific kind of goal directed proof offers a way out in this respect. The idea is not to formulate a criterion, but rather to specify a specific proof procedure that functions as a criterion. The proof procedure is applied to $\Gamma \vdash_{\mathbf{AL}} A$. Whenever the proof procedure stops, it is possible to conclude from the resulting proof whether or not $\Gamma \vdash_{\mathbf{AL}} A$. Preparatory work on the propositional fragment of **CL** (classical logic) is presented in [21] and some first results on the proof procedure for inconsistency-adaptive logics are presented in this paper.

The present paper is restricted to the propositional level. So, all references to logical systems concern the propositional fragments only. At this level the proof procedure forms an *algorithm* for final derivability: if the proof procedure is applied to $A_1, \ldots, A_n \vdash B$, it always terminates after finitely many steps—see Theorem 4. If, at the last stage of the proof, B is derived at an unmarked line, then B is finally derivable from $A_1, \ldots, A_n$; if B is not derived at an unmarked line, it is not finally derivable from $A_1, \ldots, A_n$. However, the proof procedure may be extended to the predicative level and there provides a criterion for final derivability if it terminates. The main interest of the procedure lies there.

The results presented in subsequent sections are not only interesting because they form an important tool for adaptive logics. It has been shown for a number of logics and logical mechanisms that they are characterized by an adaptive logic. Moreover, this characterization led for several systems to an interesting

[3]More particularly, this insight increases if informative steps are added to the proof, where "informative step" is clearly definable—see [10].

[4]This estimate is defined in terms of the proof theory, and the latter explicates actual reasoning. So, the estimate should not be confused with approximations that may be obtained by certain computational procedures.

strengthening or variant. Among the finished results are [14], [18], [22] and [50] for the consequence relations from [45], [23] and [24]; [36] and [35] for [51]; [40] for [1] and other logic based approaches to abduction (see [43]); [39] for the notion of empirical progress from [31]; [37] for [41] and [28]; [30] and [9] for default reasoning and circumscription respectively (see [3], [27] and [33]); [49] and [48] for prioritized consequence relations. Work in progress concerns default reasoning and the signed systems from [25]. For all logics and logical mechanisms that can be characterized by an adaptive logic in standard format, the results of the present paper can be extended in such a way that those logics and logical mechanisms are provided, next to a semantics and a provably sound syntactic characterization, with criteria for (final) derivability (and with a decision method at the propositional level).

In Section 2, I briefly present the inconsistency-adaptive logic **ACLuN1** and its dynamic proof theory. In Section 3, the goal-directed proof procedure is applied to **CL**. This will make the matter easily understood by everyone. The proof procedure for the adaptive logic **ACLuN1** is spelled out in Section 4.

2 The Inconsistency-Adaptive Logic ACLuN1

The central difference between paraconsistent logics and inconsistency-adaptive logics can easily be described in proof theoretic terms. In a (monotonic) paraconsistent logic some deduction *rules* of **CL** are invalid; in an inconsistency-adaptive logic, some *applications* of deduction rules of **CL** are invalid.

The original application context that led to inconsistency-adaptive logics—see [8]—is still one of the most clarifying ones. Suppose that a theory T was intended as consistent and was formulated with **CL** as its underlying logic. Suppose next that T turns out to be inconsistent. Of course, one will want to replace T by some consistent improvement T'. Typically, one does not just throw away T, restarting from scratch. One *reasons* from T in order to locate the inconsistency or inconsistencies and in order to locate constraints for the replacement T'. Needless to say, logic alone is not sufficient to find the justified replacement T'.[5] However, logic is able to *locate* the inconsistencies in T. It can provide one with an interpretation of T that is 'as consistent as possible'. Let me phrase this in intuitive terms. At points where T is inconsistent, some deduction rules of **CL** cannot apply—if they did, the resulting interpretation of T would be trivial in that it would make *every* sentence of the language a theorem of T. But where T is consistent, all deduction rules of **CL** should apply.

An extremely simple propositional example will clarify the matter. Consider the theory T that is characterized by the premise set $\{p, \sim p \vee r, q, \sim q \vee s, \sim p\}$. From these premises, r should not be derived by Disjunctive Syllogism. Indeed, $\sim p \vee r$ is just an obvious weakening of $\sim p$. If one were to derive r from the premises, then, by the same reasoning, one should derive $\sim r$ from p and $\sim p \vee \sim r$, which also is an obvious weakening of $\sim p$. However, if one interprets the premises as consistently as possible, one should derive s from them, viz. by

[5] If T is a mathematical theory, more conceptual analysis will be required. The different set theories that originated from Frege's are a good example of this. If T is an empirical theory, new factual data (observations and outcomes of experiments) may be required and the theory needs to be reorganized. Usually the conceptual schema will be changed as a result of the specific problem-solving process that removes the inconsistency—an extremely interesting study in this respect is [29].

Disjunctive Syllogism from q and $\sim q \vee s$. Indeed, while the premises require p to behave inconsistently (require $p \wedge \sim p$ to be true), they do not require q to behave inconsistently (they do not require $q \wedge \sim q$ to be true).

As the matter is central, let me phrase it differently. The theory T from the previous paragraph turns out to be inconsistent. As it was intended to be consistent, it should be interpreted as consistently as possible. Given that T is inconsistent, one will move 'down' to a paraconsistent logic—a logic that allows for inconsistencies. If a formula turns out to be inconsistent on the paraconsistent reading of T, one cannot apply certain rules of **CL** to it. Thus, even on the paraconsistent interpretation of T, $p \wedge \sim p$ is true. But consider $p \wedge (\sim p \vee r)$. Given the meaning of conjunction and disjunction, this formula entails $(p \wedge \sim p) \vee r$. According to **CL**, $p \wedge \sim p$ cannot be true, and hence r is true. However, the premises state that $p \wedge \sim p$ is true. So, if one wants to reason sensibly *from* these premises, one cannot rely on the **CL**-presupposition that $p \wedge \sim p$ is bound to be false. However, where the paraconsistent reading of T does not require that a specific formula A behaves inconsistently, one may retain the **CL**-presupposition that A is consistent, and hence apply **CL**-rules where they are validated by *this* presupposition. Thus T affirms $q \wedge (\sim q \vee s)$, which entails $(q \wedge \sim q) \vee s$. As T does not require $q \wedge \sim q$ to be true, it should be taken to be false and one should conclude to s.

The intuitive statements from the two preceding paragraphs are given a precise and coherent formulation by inconsistency-adaptive logics.

An adaptive logic is characterized by the following triple:[6]

(i) a monotonic *lower limit logic*,
(ii) a *set of abnormalities* (characterized by a logical form),[7] and
(iii) an *adaptive strategy* (specifying the meaning of "interpreting the premises as normally as possible").

Extending the lower limit logic with the requirement that no abnormality is logically possible results in a monotonic logic, which is called the *upper limit logic*.

Let me illustrate this by the specific inconsistency-adaptive logic **ACLuN1**. In this paper, I shall only consider the propositional level of the logic and I shall consider no other strategy than Reliability.

The *lower limit logic* of **ACLuN1** is **CLuN**. This monotonic paraconsistent logic is just like **CL**, except in that it allows for gluts with respect to negation—whence the name **CLuN**. Axiomatically, **CLuN** is obtained by extending full positive propositional logic with the axiom schema $A \vee \sim A$—see [11] for a study of the full logics **CLuN** and **ACLuN1**, including the semantics. **CLuN** isolates inconsistencies. Indeed, Double Negation, de Morgan rules, and all similar negation reducing rules are *not* validated by **CLuN**. As a result, complex contradictions do not reduce to truth functions of simpler contradictions.[8] There

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[6]In this paper I consider only *flat* adaptive logics. Other adaptive logics are the prioritized ones, which are defined as specific combinations of flat adaptive logics—see [16].

[7]In my view, it is philosophically important that all formulas of a certain *logical form* are abnormalities, and hence are taken to be false until and unless proven otherwise. Some flat adaptive logics are described and studied as formula-preferential systems in [32]—see also—[4]. Ω is then any set of formulas. It is not clear whether this may be generalized to all adaptive logics, but, by a somewhat nasty trick, all formula-preferential systems can be shown to be characterized by an adaptive logic.

[8]For example, $(p \wedge q) \wedge \sim(p \wedge q) \nvdash_{\mathbf{CLuN}} (p \wedge \sim p) \vee (q \wedge \sim q)$ and $\sim p \wedge \sim\sim p \nvdash_{\mathbf{CLuN}} p \wedge \sim p$. Of course, one still has $(p \wedge \sim p) \wedge \sim(p \wedge \sim p) \vdash_{\mathbf{CLuN}} p \wedge \sim p$.

are several versions of **CLuN**. In this paper I consider a version for which classical negation is present in the language—I shall discuss this convention below.

The *set of abnormalities*, Ω, comprises all formulas of the form $A \wedge {\sim}A$. Extending **CLuN** with the axiom schema $(A \wedge {\sim}A) \supset B$ results in the *upper limit logic*, which is **CL**.

Finally, we come to the adaptive *strategy.* Below I shall often need to refer to *disjunctions of abnormalities*, which I shall call *Dab*-formulas. From now on an expression of the form $Dab(\Delta)$ will refer to a disjunction of abnormalities; in other words, $Dab(\Delta)$ is the disjunction of the members of Δ, which is a finite subset of Ω.[9] $Dab(\Delta)$ will be called a *minimal Dab-consequence of* Γ iff $\Gamma \vdash_{\mathbf{CLuN}} Dab(\Delta)$ and there is no $\Delta' \subset \Delta$ for which $\Gamma \vdash_{\mathbf{CLuN}} Dab(\Delta')$. If $Dab(\Delta)$ is a minimal *Dab*-consequence of Γ and Δ is not a singleton, the premises require some member of Δ to be true, but do not specify which member is true. In view of this possibility, one needs to introduce an adaptive strategy. One wants to interpret the premises "as normally as possible" (which for the present Ω means "as consistently as possible"), but this phrase is ambiguous. As indicated in (iii), an adaptive strategy disambiguates the phrase.

The *Reliability strategy* from [8][10] is the oldest known strategy, and the one that is simplest from a proof theoretic point of view. I shall not consider any other strategies in this paper. Let $U(\Gamma) = \{A \mid A \in \Delta$ for some minimal *Dab*-consequence $Dab(\Delta)$ of $\Gamma\}$ be the set of formulas that are *unreliable* with respect to Γ. Below, I shall define $\Gamma \vdash_{\mathbf{ACLuN1}} A$, which will be read as "A is finally **ACLuN1**-derivable from Γ". The following Theorem, proved as Theorem 4.3 of [11], says in plain words that A is **ACLuN1**-derivable from Γ iff there is a Δ such that $A \vee Dab(\Delta)$ is **CLuN**-derivable from Γ and no member of Δ is unreliable with respect to Γ.

Theorem 1 $\Gamma \vdash_{\mathbf{ACLuN1}} A$ *iff there is a* $\Delta \subseteq \Omega$ *such that* $\Gamma \vdash_{\mathbf{CLuN}} A \vee Dab(\Delta)$ *and* $\Delta \cap U(\Gamma) = \emptyset$.

The dynamic proof theory of any (flat) adaptive logic is characterized by three (generic) rules, except of course that the rules RU and RC should refer to the right lower limit logic. Let Γ be the set of premises as before. I now list the official deduction rules.[11] Immediately thereafter I shall mention a shorthand notation that most people will find more transparent.

PREM If $A \in \Gamma$, one may add a line comprising the following elements: (i) an appropriate line number, (ii) A, (iii) $-$, (iv) PREM, and (v) $\emptyset$.

RU If $A_1, \ldots, A_n \vdash_{\mathbf{CLuN}} B$ and each of $A_1, \ldots, A_n$ occurs in the proof, say at lines $i_1, \ldots, i_n$ that have conditions $\Delta_1, \ldots, \Delta_n$ respectively, then one may add a line comprising the following elements: (i) an appropriate line number, (ii) B, (iii) $i_1, \ldots, i_n$, (iv) RU, and (v) $\Delta_1 \cup \ldots \cup \Delta_n$.

[9] It can be shown that $\Gamma \vdash_{\mathbf{CL}} \bot$ iff there is a finite $\Delta \subset \Omega$ such that $\Gamma \vdash_{\mathbf{CLuN}} Dab(\Delta)$. So, both expressions may be taken to define that Γ is inconsistent.

[10] This is the oldest paper on adaptive logics, but it appeared in a book that took ten years to come out.

[11] Only RC introduces non-empty conditions. In other words, as long as RC is not applied, the condition of every line is $\emptyset$.

RC If $A_1, \ldots, A_n \vdash_{\mathbf{CLuN}} B \vee Dab(\Theta)$ and each of $A_1, \ldots, A_n$ occurs in the proof say at lines $i_1, \ldots, i_n$ that have conditions $\Delta_1, \ldots, \Delta_n$ respectively, then one may add a line comprising the following elements: (i) an appropriate line number, (ii) B, (iii) $i_1, \ldots, i_n$, (iv) RC, and (v) $\Delta_1 \cup \ldots \cup \Delta_n \cup \Theta$.

Where "A^Δ" abbreviates that A occurs in the proof on the condition Δ, the rules may be phrased more transparently as follows:

PREM If $A \in \Gamma$:
$$\frac{\ldots}{A^{\emptyset}}$$

RU If $A_1, \ldots, A_n \vdash_{\mathbf{CLuN}} B$:
$$\frac{\begin{array}{l} {A_1}^{\Delta_1} \\ \ldots \\ {A_n}^{\Delta_n} \end{array}}{B^{\Delta_1 \cup \ldots \cup \Delta_n}}$$

RC If $A_1, \ldots, A_n \vdash_{\mathbf{CLuN}} B \vee Dab(\Theta)$:
$$\frac{\begin{array}{l} {A_1}^{\Delta_1} \\ \ldots \\ {A_n}^{\Delta_n} \end{array}}{B^{\Delta_1 \cup \ldots \cup \Delta_n \cup \Theta}}$$

While the deduction rules enable one to add lines to the proof, the marking definition, which depends on the strategy, determines which lines are "in" and which lines are "out". For the Reliability strategy, we first need to define the set $U_s(\Gamma)$ of formulas that are unreliable at a stage s of a proof. Let $Dab(\Delta)$ be a *minimal Dab-formula* at stage s of the proof iff, at that stage, $Dab(\Delta)$ has been derived on the condition $\emptyset$ *and* there is no $\Delta' \subset \Delta$ for which $Dab(\Delta')$ has been derived on the condition $\emptyset$.[12] Let $U_s(\Gamma) =_{df} \{A \mid A \in \Delta$ for some minimal *Dab*-formula $Dab(\Delta)$ at stage s of the proof$\}$.

Definition 1 *Where Δ is the condition of line i, line i is marked at stage s iff $\Delta \cap U_s(\Gamma) \neq \emptyset$. (Marking definition for Reliability)*

Lines that are unmarked at one stage may be marked at the next, and vice versa. Finally, I list the definitions that concern final derivability—the definitions are identical for all adaptive logics.

Definition 2 *A is* finally derived *from Γ at line i of a proof at stage s iff A is derived at line i, line i is not marked at stage s, and any extension of the proof in which line i is marked may be further extended in such a way that line i is unmarked.*

Definition 3 $\Gamma \vdash_{\mathbf{ACLuN1}} A$ *(A is* finally **ACLuN1**-derivable *from Γ) iff A is finally derived at a line of a proof from Γ.*

[12] The minimal *Dab*-formulas that occur in a proof at a stage should not be confused with minimal *Dab*-consequences of the set of premises. At a stage s, a new minimal *Dab*-formula may be derived, and the effect may be that a *Dab*-formula that was minimal at stage $s-1$ is not minimal at stage s. Whether some formula is a minimal *Dab*-consequence of the premises is obviously independent of the stage of a proof from those premises.

Remark that by "a proof" I mean (here and elsewhere) a sequence of lines that is obtained by applying certain instructions. In the present context, this means that each line in the sequence is obtained by applying a deduction rule and that the marking definition was applied. Here is a very simple dynamic proof.

1	$(p \wedge q) \wedge t$	–	PREM	$\emptyset$
2	${\sim}p \vee r$	–	PREM	$\emptyset$
3	${\sim}q \vee s$	–	PREM	$\emptyset$
4	${\sim}p \vee {\sim}q$	–	PREM	$\emptyset$
5	$t \supset {\sim}p$	–	PREM	$\emptyset$
6	r	1, 2	RC	$\{p \wedge {\sim}p\}$ $\boxed{9}$
7	s	1, 3	RC	$\{q \wedge {\sim}q\}$
8	$(p \wedge {\sim}p) \vee (q \wedge {\sim}q)$	1, 4	RU	$\emptyset$
9	$p \wedge {\sim}p$	1, 5	RU	$\emptyset$

Up to stage 7 of the proof, all lines are unmarked. At stage 8, lines 6 and 7 are marked because $U_8(\Gamma) = \{p \wedge {\sim}p, q \wedge {\sim}q\}$. At stage 9, only line 6 is marked because $U_9(\Gamma) = \{p \wedge {\sim}p\}$. It is easily seen that, if 1–5 are the only premises, then the marks will remain unchanged in all extensions of the proof. So, r is not a final consequence of Γ whereas s is a final consequence of Γ.

The convention on classical negation. As promised, I now discuss the convention that the language contains classical negation, which will be written as "$\neg$" (or that the language contains $\perp$ together with $\neg A =_{df} A \supset \perp$). In a sense then, **CLuN** is an extension of **CL**. It has the full inferential power of **CL**, $\neg$ functioning as the **CL**-negation, and moreover contains the paraconsistent negation $\sim$. In the original application context, mentioned in the second paragraph of this section, the premises belong to the $\neg$-free (and $\perp$-free) fragment of the language. Of course different application contexts are possible, but even in the original application context the presence of $\neg$ is useful: it greatly simplifies metatheoretic proofs and technical matters in general, and in no way hampers the limitations imposed by the application context.[13] As will appear in Section 4, the presence of $\neg$ also greatly simplifies the prospective procedure that will serve as a criterion for final derivability.

3 Prospective Proofs for Classical Logic

In this section I merely present an example: a prospective proof for $p \supset (q \wedge s), \neg(q \vee r) \vdash_{\mathbf{CL}} \neg p$.[14] As the proof is simple, I skip the rules as well as the heuristic instructions—these are spelled out in [21]—and merely offer some comments.

The first step introduces the main goal:

[13] By present lights, it is harmless as well as useful, for all adaptive logics, to extend the language and the lower limit logic in such a way that all classical connectives belong to the lower limit logic. This holds even if these connectives do not occur in the premises or in the conclusions a user is interested in—see [12] for an example.

[14] The rules are listed in the next section. In order to avoid useless complications, I shall write classical negation as $\neg$ even in the context of **CL**.

1	$[\neg p]\, \neg p$		GOAL	

This step, which expresses the truism that $\neg p$ can be obtained on the condition that $\neg p$ can be obtained, is meant to remind one that one is looking for the formula that occurs in the condition, viz. $\neg p$. The purpose served by a condition in prospective proofs is very different from the one in dynamic proofs—it is 'prospective' rather than 'defeasible'. In view of the condition of line 1, one introduces a premise from which $\neg p$ may be obtained, and next analyses the premise:

2	$p \supset (q \wedge s)$		PREM
3	$[\neg(q \wedge s)]\, \neg p$	2	$\supset$E

The prospective condition for 2 is empty for obvious reasons, and I shall write such conditions invisibly. Line 3 illustrates a formula analysing rule: in view of 2, one would have $\neg p$ if one had $\neg(q \wedge s)$. As $\neg(q \wedge s)$ cannot be obtained by analysing a premise, one applies a condition analysing rule to $\neg(q \wedge s)$:

4	$[\neg q]\, \neg p$	3	C$\neg\wedge$E

The following steps require no comment:

5	$\neg(q \vee r)$		PREM
6	$\neg q$	5	$\neg\vee$E
7	$\neg p$	4, 6	Trans

As the main goal is obtained on the empty condition at line 7, the proof is completed.

It is easily seen that, in a proof for $\Gamma \vdash_{\mathbf{CL}} A$, a formula B is derivable on the condition Δ—with some notational abuse: $[\Delta]\, B$ is derivable—just in case $\Gamma \cup \Delta \vdash_{\mathbf{CL}} B$.

Some lines are marked in goal-directed proofs for **CL**. Unlike what was the case in the previous section, these marks indicate that one should not try to derive the members of the condition of marked lines. More details are presented in the next section, where these marks will be called D-marks because they relate to derivability—A-marks will relate to the adaptive character of the logic.

4 Prospective Proofs for ACLuN1

Prospective proofs for **ACLuN1** have lines that contain *two* conditions:

i	$[\Delta]\, A$	...	...	Θ

A will be called the *formula* of the line. The prospective condition, Δ, is called the *D-condition.* As in prospective proofs for **CL**, it contains the formulas that one needs to derive in order to obtain A. The adaptive condition, Θ, will be called the *A-condition.* It contains the abnormalities that should not belong to $U(\Gamma)$ in order for A to be derivable from the premises. The occurrence of the above line i in an **ACLuN1**-proof from Γ warrants that $\Gamma \cup \Delta \vdash_{\mathbf{CLuN}} A \vee Dab(\Theta)$—see Theorem 5. In order to show that $\Gamma \vdash_{\mathbf{ACLuN1}} G$ one needs a line like the displayed one at which $A = G$, $\Delta = \emptyset$, and $\Theta \cap U(\Gamma) = \emptyset$.

To facilitate the exposition, I shall write $[\Delta]\, A^{\Theta}$ to denote that A has been derived on the D-condition Δ and on the A-condition Θ, and I shall write A^{Θ}

when Δ is known to be empty. Before describing the procedure, I shall present the rules and some required definitions. Let $*A$ denote the 'complement' of A, viz. B if A is $\neg B$ and $\neg A$ otherwise.[15]

The following rules introduce premises or start new phases or subphases of the proof. A-Goal and X-Goal are identical but are used in different contexts.

Prem	If $A \in \Gamma$, introduce $A^{\emptyset}$.
Goal	Introduce $[G]\, G^{\emptyset}$.
A-Goal	Introduce $[Dab(\Delta)]\, Dab(\Delta)^{\emptyset}$.
X-Goal	Introduce $[Dab(\Delta)]\, Dab(\Delta)^{\emptyset}$.
EFQ	If $A \in \Gamma$, introduce $[*A]\, G^{\emptyset}$.

We have seen that **CLuN** contains all of **CL**. The formula analysing rules and the condition analysing rules for **CL** may be summarized by distinguishing $\mathfrak{a}$-formulas from $\mathfrak{b}$-formulas (varying on a theme from [47]). To each formula two other formulas are assigned according to the following table:

$\mathfrak{a}$	$\mathfrak{a}_1$	$\mathfrak{a}_2$		$\mathfrak{b}$	$\mathfrak{b}_1$	$\mathfrak{b}_2$
$A \wedge B$	A	B		$\neg(A \wedge B)$	$*A$	$*B$
$A \equiv B$	$A \supset B$	$B \supset A$		$\neg(A \equiv B)$	$\neg(A \supset B)$	$\neg(B \supset A)$
$\neg(A \vee B)$	$*A$	$*B$		$A \vee B$	A	B
$\neg(A \supset B)$	A	$*B$		$A \supset B$	$*A$	B
$\neg\neg A$	A	A				

The formula analysing rules for $\mathfrak{a}$-formulas and $\mathfrak{b}$-formulas are respectively:[16]

$$\frac{[\Delta]\, \mathfrak{a}^{\Theta}}{[\Delta]\, \mathfrak{a}_1{}^{\Theta} \quad [\Delta]\, \mathfrak{a}_2{}^{\Theta}} \qquad \frac{[\Delta]\, \mathfrak{b}^{\Theta}}{[\Delta \cup \{*\mathfrak{b}_2\}]\, \mathfrak{b}_1{}^{\Theta} \quad [\Delta \cup \{*\mathfrak{b}_1\}]\, \mathfrak{b}_2{}^{\Theta}}$$

For **ACLuN1** we moreover need:

$$\sim\text{E} \quad \frac{[\Delta]\, {\sim}A^{\Theta}}{[\Delta]\, *A^{\Theta \cup \{A \wedge \sim A\}}} \qquad \neg{\sim}\text{E} \quad \frac{[\Delta]\, \neg{\sim}A^{\Theta}}{[\Delta]\, A^{\Theta}}$$

The rule $\sim$E expresses that ${\sim}A$ entails $\neg A$ on the condition that $A \wedge {\sim}A$ is false (because then A is false and hence $\neg A$ is true). $\neg{\sim}$E states that A is true whenever ${\sim}A$ is false; the converse obviously does not hold.

The condition analysing rules for $\mathfrak{a}$-formulas and $\mathfrak{b}$-formulas are respectively:

$$\frac{[\Delta \cup \{\mathfrak{a}\}]\, A^{\Theta}}{[\Delta \cup \{\mathfrak{a}_1, \mathfrak{a}_2\}]\, A^{\Theta}} \qquad \frac{[\Delta \cup \{\mathfrak{b}\}]\, A^{\Theta}}{[\Delta \cup \{\mathfrak{b}_1\}]\, A^{\Theta} \quad [\Delta \cup \{\mathfrak{b}_2\}]\, A^{\Theta}}$$

For **ACLuN1** we moreover need:

$$\text{C}{\sim}\text{E} \quad \frac{[\Delta \cup \{{\sim}B\}]\, A^{\Theta}}{[\Delta \cup \{*B\}]\, A^{\Theta}} \qquad \text{C}\neg{\sim}\text{E} \quad \frac{[\Delta \cup \{\neg{\sim}B\}]\, A^{\Theta}}{[\Delta \cup \{B\}]\, A^{\Theta \cup \{B \wedge \sim B\}}}$$

[15]Sometimes a double complement will be needed. Remark that $**\neg p$ is $\neg p$ but that $**\neg\neg p$ is p, just like $**p$.

[16]The rule to the left actually summarizes two rules: both $[\Delta]\, \mathfrak{a}_1{}^{\Theta}$ and $[\Delta]\, \mathfrak{a}_2{}^{\Theta}$ may be derived from $[\Delta]\, \mathfrak{a}^{\Theta}$; similarly for the rule to the right and for the condition analysing rule to the right below.

For C∼E: if one can obtain $\neg A$, then one can obtain $\sim A$; for C¬∼E: if one can obtain B, and $B \wedge \sim B$ is reliable, then one can obtain $\neg\sim B$.

To obtain a complete system one needs Trans and EM. Moreover, the derivable rule EM0 and the permissible rule IC simplify the proof procedure.

$$\text{Trans}\quad \frac{\begin{array}{l}[\Delta \cup \{B\}]\, A^{\Theta} \\ [\Delta']\, B^{\Theta'}\end{array}}{[\Delta \cup \Delta']\, A^{\Theta \cup \Theta'}} \qquad \text{EM}\quad \frac{\begin{array}{l}[\Delta \cup \{B\}]\, A^{\Theta} \\ [\Delta' \cup \{\neg B\}]\, A^{\Theta'}\end{array}}{[\Delta \cup \Delta']\, A^{\Theta \cup \Theta'}}$$

$$\text{EM0}\quad \frac{[\Delta \cup \{\neg A\}]\, A^{\Theta}}{[\Delta]\, A^{\Theta}} \qquad \text{IC}\quad \frac{[\Delta]\, Dab(\Lambda \cup \Lambda')^{\Theta \cup \Lambda'}}{[\Delta]\, Dab(\Lambda \cup \Lambda')^{\Theta}}$$

That A is a *positive part* of another formula is recursively defined by the following clauses:[17]

1. $\text{pp}(A, A)$.
2. $\text{pp}(A, \neg\sim A)$.
3. $\text{pp}(*A, \sim A)$.
4. If $\text{pp}(A, \mathfrak{a}_1)$ or $\text{pp}(A, \mathfrak{a}_2)$, then $\text{pp}(A, \mathfrak{a})$.
5. If $\text{pp}(A, \mathfrak{b}_1)$ or $\text{pp}(A, \mathfrak{b}_2)$, then $\text{pp}(A, \mathfrak{b})$.
6. If $\text{pp}(A, B)$ and $\text{pp}(B, C)$, then $\text{pp}(A, C)$.

A-marking (marking in view of the A-conditions, providing from the adaptive character of the logic) is taken over by the procedure below. D-marking (marking in view of D-conditions) is governed by the following definition.

Definition 4 *Where $[\Delta]\, A^{\Theta}$ is derived at line i, line i is D-marked iff one of the following conditions is fulfilled:*

1. *line i is not an application of a goal rule and $A \in \Delta$,*
2. *for some $\Delta' \subset \Delta$ and $\Theta' \subseteq \Theta$, $[\Delta']\, A^{\Theta'}$ occurs in the proof,*
3. *no application of EFQ occurs at a line preceding i and $B, \neg B \in \Delta$ for some B,*
4. *no application of EFQ occurs at a line preceding i and, for some $B \in \Delta$, $\neg B^{\emptyset}$ occurs in the proof.*

If 1 is the case, the condition is circular; if 2 is the case, some (set theoretically) weaker condition is sufficient to obtain A. In the other two cases, line i indicates a search path that can only be successful if the premises are ¬-inconsistent. Although it is not necessary to mark such search paths, it turns out more efficient to postpone them to phase 1B—see below.

[17] Unlike what is done in [46] and [21], I do not introduce negative parts because this complicates the predicative case. Clause 6 is only required in view of clauses 2 and 3.

The procedure. Several variants are possible. To save some space I describe a variant that leaves much choice to the person who constructs the proof and hence may lead to rather inefficient proofs, but nevertheless warrants that all steps are sensible with respect to the aim. I shall disregard infinite Γ.

A prospective **ACLuN1**-proof for $A_1, \ldots, A_n \vdash G$ will consist of three phases. In the first phase, one tries to obtain G^{Θ} for some Θ. If this succeeds and $\Theta \neq \emptyset$, one moves to phase 2 and tries to obtain $Dab(\Theta)^{\Lambda}$ for some Λ. If this succeeds and $\Lambda \neq \emptyset$, one moves on to phase 3 and tries to obtain $Dab(\Lambda)^{\emptyset}$. If a phase terminates, one returns to the previous one. In phase 1, there are two subphases: phase 1 starts with subphase 1A, and only if no other step is possible one applies EFQ, which starts subphase 1B.

Each phase starts by applying a goal rule. In a phase, the members of the D-conditions of *unmarked* lines of the phase are called the *targets*. The following *restrictions* are important. Premises are introduced and formulas analysed iff a target is a positive part of the formula of the added line. Condition analysing rules are only applied to targets. A formula analysing rule is never applied to a formula that does not have a premise in its path—analysing a goal is provably a useless complication. Once $[\Delta]\, A^{\Theta}$ occurs in the proof, one never adds another line with that same formula, D-condition and A-condition (even if the justification of the line is different). Finally, EFQ is only applied in subphase 1B.[18] The restrictions are important because they define when the procedure terminates (in a phase)—as suggested before, introducing more restrictions may lead to more efficient proofs.

Let us consider the three phases and the conclusions that may be drawn from them. During phases 2 and 3, a line may be A-marked (marked in view of its A-condition). A phase terminates if no lines can be added in view of current targets.

Phase 1. Phase 1 starts with $[G]\, G$, justified by the Goal rule.
Subphase 1A. Aim: to derive G^{Θ} for some Θ. There are three possibilities:
(1.1) $G^{\emptyset}$ is derived. Then $\Gamma \vdash_{\mathbf{ACLuN1}} G$.
(1.2) G^{Θ} is derived, say at line i: the procedure moves to phase 2 and later returns to phase 1, at which point there are two possibilities:
(1.2.1) line i is not A-marked: $\Gamma \vdash_{\mathbf{ACLuN1}} G$.
(1.2.2) line i is A-marked: go on (aim: derive $G^{\Theta'}$ for some $\Theta' \not\supseteq \Theta$).
(1.3) The procedure terminates and G^{Θ} is not derived at an unmarked line for any Θ: move to subphase 1B.
Subphase 1B. Aim: to derive $G^{\emptyset}$ by applications of EFQ as well as well of the other **CLuN**-rules.[19] If $G^{\emptyset}$ is derived, $\Gamma \vdash_{\mathbf{ACLuN1}} G$; otherwise $\Gamma \nvdash_{\mathbf{ACLuN1}} G$.

Phase 2. G^{Θ} was derived in phase 1 for some Θ, say at line i. Phase 2 starts by applying A-Goal in order to add $[Dab(\Theta)]\, Dab(\Theta)^{\emptyset}$. Aim: to derive $Dab(\Theta)^{\Lambda}$ for some $\Lambda \subseteq \Omega$. There are three possibilities:
(2.1) $Dab(\Theta)^{\emptyset}$ is derived: line i is A-marked; the procedure returns to phase 1.

[18] It can be shown that, if $\neg$ does not occur in the premises—see the second paragraph of Section 2—then the premises cannot be $\neg$-inconsistent and hence phase 1B is useless. I nevertheless include it here for the sake of completeness.

[19] The **CLuN**-rules are all rules except for $\sim$E and C$\neg\sim$E, which introduce an A-condition, IC, which modifies the A-condition, and A-Goal and X-Goal which start from an A-condition.

(2.2) $Dab(\Theta)^{\Lambda}$ is derived for some $\Lambda \neq \emptyset$, say at line j. The procedure moves to phase 3 and later returns to phase 2, at which point there are two possibilities:

(2.2.1) line j is A-marked: go on (aim: derive $Dab(\Theta)^{\Lambda'}$ for some $\Lambda' \not\supseteq \Lambda$).

(2.2.2) line j is not A-marked: line i is A-marked; the procedure returns to phase 1.

(2.3) Phase 2 terminates, $Dab(\Theta)^{\Lambda}$ not being derived at an unmarked line for any Λ: line i is not A-marked and the procedure returns to phase 1.

Phase 3. G^{Θ} was derived in phase 1 for some Θ, say at line i, and $Dab(\Theta)^{\Lambda}$ was derived in phase 2 for some Λ, say at line j. Phase 3 starts by applying X-Goal in order to add $[Dab(\Lambda)]\, Dab(\Lambda)^{\emptyset}$. Aim: to derive $Dab(\Lambda)^{\emptyset}$ by the **CLuN**-rules (see footnote 19), whence all lines of phase 3 have the A-condition $\emptyset$. There are two possibilities:

(3.1) $Dab(\Lambda)^{\emptyset}$ is derived: line j is A-marked; the procedure returns to phase 2.

(3.2) Phase 3 stops without $Dab(\Lambda)^{\emptyset}$ being derived: line j is not A-marked; the procedure returns to phase 2.

Some fine tuning. Before moving to some examples, I shall present some comments that concern the procedure as well as some comments that pertain to the efficiency of the proofs.

EFQ is never applied in phase 2 or 3. This is justified by the following consideration. EFQ can only be successfully applied in a proof for $\Gamma \vdash G$ if Γ is $\neg$-inconsistent. In that case, $G^{\emptyset}$ is derivable from the premises and will be derived in phase 1B. Deriving any *Dab*-formula from Γ by applying EFQ (possibly combined with other rules) is a useless detour.

Moreover, EFQ is only applied in phase 1 at points where no other rule can be applied and, from that point on—that is, in subphase 1B—one adds only lines with an empty A-condition to the proof, and hence never moves on to phase 2. The reason for this is obvious: if the main goal can only be obtained by EFQ, then it is derivable by the lower limit logic, viz. **CLuN**, and hence there is no point in deriving it on some A-condition.

I now describe an apparently rather efficient variant of the procedure; it is nearly identical for the three phases. Let me start with some general instructions. No line is added to the proof if it would at once be marked. At each stage, one first tries to apply EM0, EM and Trans provided this leads to a line being marked. IC is applied whenever possible.

If this does not lead to the aim of the phase, one proceeds in a strictly goal directed way. More particularly, one considers the first formula in the last unmarked condition (of the current phase) as the sole target. If the target cannot be obtained from the premises, then obtaining the other members of the same condition is useless anyway. If no step is possible in view of the target—this means that the target is a dead end—one considers the first formula in the next-to-last unmarked condition of the current phase as the target, and so on.

If it is possible to act in view of the target, one applies the rules in the following order—remember what was said about positive parts. First, one tries to apply a formula analysing rule to a formula that occurs at an unmarked line. Next, one tries to introduce a premise. If all this does not enable one to derive the target and eliminate it from the D-condition by transitivity, one applies a condition analysing rule to the target.

If the goal of the current phase cannot be obtained by strictly goal directed moves (the ones described above), all members of conditions of unmarked lines of the current phase are considered as targets, and one applies all rules, including Trans and EM, whenever this enables one to obtain the goal (of the current phase) on a new condition. One returns to strictly goal directed moves as soon as possible.

Only if all this fails, and the current phase is phase 1, one applies EFQ and, as said before, from there on only adds lines with an empty A-condition.

Some examples. Let us start with two simple examples. Consider first a prospective proof for $\sim p \vee r, p \wedge \sim q, q \vdash_{\mathbf{ACLuN1}} r$:

1	$[r]\, r$		Goal	$\emptyset$	
2	$\sim p \vee r$		Prem	$\emptyset$	
3	$[\neg\sim p]\, r$	2	$\vee$E	$\emptyset$	
4	$[p]\, r$	3	C$\neg\sim$E	$\{p \wedge \sim p\}$	D^7
5	$p \wedge \sim q$		Prem	$\emptyset$	
6	p	5	$\wedge$E	$\emptyset$	
7	r	4, 6	Trans	$\{p \wedge \sim p\}$	
8	$[p \wedge \sim p]\, p \wedge \sim p$		A-Goal	$\emptyset$	
9	$[p, \sim p]\, p \wedge \sim p$	8	C$\wedge$E	$\emptyset$	
10	$[\sim p]\, p \wedge \sim p$	6, 9	Trans	$\emptyset$	
11	$[\neg r]\, \sim p$	2	$\vee$E	$\emptyset$	
12	$[\neg p]\, p \wedge \sim p$	10	C$\sim$E	$\emptyset$	
13	$[\neg r]\, \neg p$	11	$\sim$E	$\{p \wedge \sim p\}$	
14	$[\neg r]\, p \wedge \sim p$	12, 13	Trans	$\{p \wedge \sim p\}$	
15	$[\neg r]\, p \wedge \sim p$	14	IC	$\emptyset$	

The proof is successful: at line 7 r is derived on the empty D-condition and on the A-condition $\{p \wedge \sim p\}$, whence line 4 is D-marked. In phase 2 $p \wedge \sim p$ turns out not to be derivable on any A-condition. So line 7 is unmarked and $\sim p \vee r, p \wedge \sim q, q \vdash_{\mathbf{ACLuN1}} r$. Incidently, the premise set is $\sim$-consistent, in which case no line is ever A-marked.

Next, consider a prospective proof for $\sim p, p \vee q, p \vdash_{\mathbf{ACLuN1}} q$:

1	$[q]\, q$		Goal	$\emptyset$	
2	$p \vee q$		Prem	$\emptyset$	
3	$[\neg p]\, q$	2	$\vee$E	$\emptyset$	
4	$\sim p$		Prem	$\emptyset$	
5	$\neg p$	4	$\sim$E	$\{p \wedge \sim p\}$	
6	q	3, 5	Trans	$\{p \wedge \sim p\}$	A^{12}
7	$[p \wedge \sim p]\, p \wedge \sim p$		A-Goal	$\emptyset$	D^{12}
8	$[p, \sim p]\, p \wedge \sim p$	7	C$\wedge$E	$\emptyset$	D^{12}
9	$[p]\, p \wedge \sim p$	4, 8	Trans	$\emptyset$	D^{12}
10	$[\neg q]\, p$	2	$\vee$E	$\emptyset$	D^{11}
11	p		Prem	$\emptyset$	
12	$p \wedge \sim p$	9, 11	Trans	$\emptyset$	
13	$[\neg(p \vee q)]\, q$		EFQ	$\emptyset$	
14	$[\neg\sim p]\, q$		EFQ	$\emptyset$	

After $q^{\{p \wedge \sim p\}}$ is derived at line 6, $p \wedge \sim p^{\emptyset}$ turns out to be derivable (line 12). The procedure then sets out to derive q in a different way, which fails. C$\neg\vee$E is not applied to the condition of line 13 because the resulting line would at once be marked in view of line 3. C$\neg\sim$E cannot be applied to the condition of line 14 because this would introduce a non-empty A-condition. $[\neg p]\, q$ is not derived by EFQ in view of line 10. So the proof terminates and $\sim p, p \vee q, p \nvdash_{\mathbf{ACLuN1}} q$.

Finally, a prospective proof for $p, \sim p \vee s, r \supset t, \sim p \vee q, \sim q \vdash_{\mathbf{ACLuN1}} s$:

1	$[s]\, s$		Goal	$\emptyset$	
2	$\sim p \vee s$		Prem	$\emptyset$	
3	$[\neg\sim p]\, s$	2	$\vee$E	$\emptyset$	
4	$[p]\, s$	3	C$\neg\sim$E	$\{p \wedge \sim p\}$	
5	p		Prem	$\emptyset$	
6	s	4, 5	Trans	$\{p \wedge \sim p\}$	A^{20}
7	$[p \wedge \sim p]\, p \wedge \sim p$		A-Goal	$\emptyset$	
8	$[p, \sim p]\, p \wedge \sim p$	7	C$\wedge$E	$\emptyset$	D^{9}
9	$[\sim p]\, p \wedge \sim p$	8, 5	Trans	$\emptyset$	
10	$[\neg s] \sim p$	2	$\vee$E	$\emptyset$	
11	$\sim p \vee q$		Prem	$\emptyset$	
12	$[\neg q] \sim p$	11	$\vee$E	$\emptyset$	
13	$\sim q$		Prem	$\emptyset$	
14	$\neg q$	13	$\sim$E	$\{q \wedge \sim q\}$	
15	$\sim p$	12, 14	Trans	$\{q \wedge \sim q\}$	
16	$p \wedge \sim p$	9, 15	Trans	$\{q \wedge \sim q\}$	
17	$[q \wedge \sim q]\, q \wedge \sim q$		X-Goal	$\emptyset$	
18	$[q, \sim q]\, q \wedge \sim q$	17	C$\wedge$E	$\emptyset$	
19	$[q]\, q \wedge \sim q$	13, 18	Trans	$\emptyset$	
20	$[\neg\sim p]\, q$	11	$\vee$E	$\emptyset$	

Here phase 3 stops. Line 16 is not A-marked and the procedure returns to phase 2; there line 6 is A-marked and the procedure returns to phase 1. There the procedure will continue, aiming at deriving s^{Θ} in phase 1 for some $\Theta \not\supseteq \{p \wedge \sim p\}$, but this will fail (and is bound to fail as the only open road is by EFQ whereas the premises are $\neg$-consistent). So $p, \sim p \vee s, r \supset t, \sim p \vee q, \sim q \nvdash_{\mathbf{ACLuN1}} s$

A computer program that implements the procedure can be downloaded from `http://logica.ugent.be/dirk/`—the above proofs are produced by it. The data file that goes with the program contains a set of instructive example exercises.

Metatheoretic matters. The procedure is an algorithm for $\Gamma \vdash_{\mathbf{ACLuN1}} A$ as it is discussed here, viz. at the propositional level and for finite Γ.

In order to facilitate the proof of some lemma's, I now mention the **CLuN**-semantics.[20] Let $\mathcal{S}$ be the set of sentential letters, $\mathcal{W}$ the set of formulas, and $\mathcal{N}$ the set of formulas of the form $\sim A$. The semantics proceeds in terms of an assignment, $v : \mathcal{S} \cup \mathcal{N} \mapsto \{0, 1\}$, and a valuation, $v_M : \mathcal{W} \mapsto \{0, 1\}$, determined by a model $M = \langle v \rangle$. The valuation is defined as follows:

C1 Where $A \in \mathcal{S}$, $v_M(A) = v(A)$.
C2 $v_M(\neg A) = 1$ iff $v_M(A) = 0$.
C3 $v_M(\sim A) = 1$ iff $v_M(A) = 0$ or $v(\sim A) = 1$.

[20] The semantics for **ACLuN1** can be found, e.g., in [11].

C4 $v_M(\mathfrak{a}) = 1$ iff $v_M(\mathfrak{a}_1) = v_M(\mathfrak{a}_2) = 1$.
C5 $v_M(\mathfrak{b}) = 1$ iff $v_M(\mathfrak{b}_1) = 1$ or $v_M(\mathfrak{b}_2) = 1$.

M verifies A iff $v_M(A) = 1$. M is a model of Γ iff it verifies all members of Γ. A is valid iff all models verify A. $\Gamma \vDash_{\mathbf{CLuN}} A$ iff all models of Γ verify A. The semantics is obviously equivalent to a more standard one (not mentioning $\mathfrak{a}$ and $\mathfrak{b}$. That **CLuN** is sound and complete with respect to the semantics is proved in [11] (for the predicative version, and without $\neg$, which however is easily modified).

Remember that prospective proofs for **CLuN** are defined by the rules for **ACLuN1** save $\sim$E, C$\neg\sim$E, IC, A-Goal and X-Goal—these modify the A-condition, or rely on it, whereas there is no A-condition in prospective proofs for **CLuN**. So prospective proofs for **CLuN** are defined by the rules for **CL** plus $\neg\sim$E and C$\sim$E (but without the A-condition). The proof of Theorem 2 is a simpler variant of that for Theorem 4 below.

Theorem 2 *If Γ is finite, every prospective proof for $\Gamma \vdash_{\mathbf{CLuN}} A$ terminates.*

Theorem 3 *If a prospective proof for $\Gamma \vdash_{\mathbf{CLuN}} G$ stops with G being derived, then $\Gamma \vdash_{\mathbf{CLuN}} G$. If a prospective proof for $\Gamma \vdash_{\mathbf{CLuN}} G$ stops without G being derived, then $\Gamma \nvdash_{\mathbf{CLuN}} G$.*

Corollary 1 *The prospective proof procedure for* **CLuN** *is a decision method for* **CLuN***-derivability.*

The proof of Theorem 3 requires five pages, but is nearly identical to the corresponding proof for **CL**, which is available in [21, pp. 126–131]. The only changes to that proof concern those for accommodating $\neg\sim$E and C$\sim$E, and the changes are completely obvious. The difficult bit is obviously with the second statement in the theorem. This requires a ticking-off method as well as a demonstration that a model of Γ that falsifies G may be constructed from the set of lines of the form $[\Delta]\, G$ that are neither marked nor ticked off. Central to that demonstration is that, if the proof is terminated without G being derived, then some **CLuN**-model falsifies a member of every such Δ and falsifies G, and every such model is a model of Γ. The proof method is new and deserves being republished, but I need the allowed space for the rest of the metatheory.

Lemma 1 $\mathrm{pp}(A, B)$ *iff* $\mathrm{pp}(*A, *B)$.

Proof. The left–right direction is proved by an induction on the length of the recursion. For the basis, we have to consider the cases where $\mathrm{pp}(A, B)$ holds because of the first three clauses:
Clause 1: then B is A and $*A$ is $*B$, whence $\mathrm{pp}(*A, *B)$ by clause 1.
Clause 2: $\mathrm{pp}(*A, *B)$ is warranted by clause 3 and the fact that $**A$ is A.
Clause 3: $\mathrm{pp}(*A, *B)$ is warranted by clause 2 and the fact that $**A$ is $\sim A$.

For the recursive steps, we have to consider:
Clause 4.1: if A is $\neg\neg B$, then both $*A$ and $*B$ are $\neg B$, whence $\mathrm{pp}(*A, *B)$ is warranted by clause 1.
Clause 4.2: if B is an $\mathfrak{a}$-formula that is not of the form $\neg\neg C$, then $*B$ is the $\mathfrak{b}$-formula of the same row of the table on page 10; it is easily seen from that table that clause 5 warrants $\mathrm{pp}(*A, *B)$.
Clause 5: if B is an $\mathfrak{b}$-formula, then $*B$ is the $\mathfrak{a}$-formula of the same row of

the table on page 10; it is easily seen from that table that clause 4 warrants $pp(*A, *B)$.

Clause 6: $pp(A, B)$ holds because there is a C such that $pp(A, C)$ and $pp(C, B)$; by the induction hypothesis $pp(*A, *C)$ and $pp(*C, *B)$ and hence $pp(*A, *B)$ by clause 6.

The proof of the right–left direction is similar. ■

Let us extend the positive part function as follows: $pp(A, \Gamma)$ iff $pp(A, B)$ for some $B \in \Gamma$.

Lemma 2 *If* $[B_1, \ldots, B_n]\, A^{\{C_1 \wedge \sim C_1, \ldots, C_m \wedge \sim C_m\}}$ *is derived in a prospective proof for* $\Gamma \vdash_{\mathbf{ACLuN1}} G$ *in a phase starting with* $[G']\, G'$,[21] *then* A *is* G' *or* $pp(A, \Gamma)$, $pp(*B_i, \Gamma \cup \{*G'\})$ *for every* B_i *(*$1 \leq i \leq n$*), and* $pp(\sim C_i, \Gamma \cup \{*G'\})$ *for every* C_i *(*$1 \leq i \leq m$*).*

Proof. By an induction on the length of the prospective proof. The basis is formed by an application of the rules Goal, A-Goal or X-Goal, leading to $[G']\, G'$. Obviously A is G' and $pp(*B_i, *G')$. For the induction step, I only consider a few cases.

Rule Prem: $pp(A, \Gamma)$.

Rule EFQ: A is G' and $pp(*B, \Gamma)$.

Formula analysing rules for $\mathfrak{a}$*-formulas:* where the rule is applied to a formula A', $pp(A', \Gamma)$ by the induction hypothesis and the restriction (see the heading "The procedure") that a formula analysing rule can only be applied to an A' that has a premise in its path. As A is $\mathfrak{a}_1$ or $\mathfrak{a}_2$, and $pp(\mathfrak{a}, \Gamma)$ by the induction hypothesis, $pp(A, \mathfrak{a})$, and hence $pp(A, \Gamma)$.

Formula analysing rules for $\mathfrak{b}$*-formulas:* where the rule is applied to a formula A', $pp(A', \Gamma)$ by the induction hypothesis and the restriction (see the heading "The procedure"). If A is $\mathfrak{b}_1$, $pp(A, A')$ and hence $pp(A, \Gamma)$; moreover the newly introduced B_i is $*\mathfrak{b}_2$, $pp(\mathfrak{b}_2, \Gamma)$ and hence $pp(*B_i, \Gamma)$. Similarly if A is $\mathfrak{b}_2$.

Rule $\sim$E: where the rule is applied to a formula A', say $\sim D$, $pp(A', \Gamma)$. As A is $*D$, $pp(A, A')$, and hence $pp(A, \Gamma)$. The newly introduced $C_i \wedge \sim C_i$ is $D \wedge \sim D$ and hence $pp(\sim C_i, \Gamma)$.

Rule C$\neg\sim$E: the newly introduced B_i is obtained from $\neg\sim B_i$ and $pp(*\neg\sim B_i, \Gamma \cup \{*G'\})$ by the induction hypothesis; $*\neg\sim B_i$ is $\sim B_i$ and $pp(*B_i, \sim B_i)$. Hence $pp(*B_i, \Gamma \cup \{*G'\})$. The newly introduced $C_i \wedge \sim C_i$ is $B_i \wedge \sim B_i$ and $pp(\sim C_i, \Gamma \cup \{*G'\})$.

The other cases are left to the reader.[22] ■

Theorem 4 *If* Γ *is finite, every prospective proof for* $\Gamma \vdash_{\mathbf{ACLuN1}} G$ *terminates.*

Proof. Every line of a prospective proof for $\Gamma \vdash_{\mathbf{ACLuN1}} A$ has the form $[B_1, \ldots, B_n]\, A^{\{C_1 \wedge \sim C_1, \ldots, C_m \wedge \sim C_m\}}$ ($n \geq 0$ and $m \geq 0$). I shall show that A, $B_1, \ldots, B_n$ and $\{C_1 \wedge \sim C_1, \ldots, C_m \wedge \sim C_m\}$ belong to specific sets that are defined by Γ and G.

In view of on Lemma 2, each of the following holds true:

(i) $B_i \in \{D \mid pp(*D, \Gamma \cup \{*G'\})\}$ for every B_i ($1 \leq i \leq n$).

[21] So G' is either G or the *Dab*-formula introduced by A-Goal or the *Dab*-formula introduced by X-Goal.

[22] Condition analysing rules: $pp(*\mathfrak{a}_1, *\mathfrak{a})$ in view of Lemma 1 and $pp(\mathfrak{a}_1, \mathfrak{a})$.

(ii) $C_i \wedge {\sim}C_i \in \{D \wedge {\sim}D \mid \mathrm{pp}({\sim}D, \Gamma \cup \{*G'\})\}$ for every C_i $(1 \leq i \leq m)$. A is G or $\mathrm{pp}(A, \Gamma)$ or A is G' and $[G']G'$ is introduced by the rule A-Goal or by the rule X-Goal.
(iii) If G' is introduced by A-Goal, it has the form $E \wedge {\sim}E$ and is a member of the condition of a line from phase 1, whence $G' \in \{D \wedge {\sim}D \mid \mathrm{pp}({\sim}D, \Gamma \cup \{*G\})\}$.
(iv) If G' is introduced by X-Goal, it has the form $E \wedge {\sim}E$ and is a member of the condition of a line from phase 2, whence $G' \in \{D \wedge {\sim}D \mid \mathrm{pp}({\sim}D, \Gamma \cup \{\neg(F \wedge {\sim}F) \mid \mathrm{pp}({\sim}F, \Gamma \cup \{*G\})\})\}$.

If Γ is finite, all sets mentioned in the previous paragraph are finite. It follows that there are only finitely many 'triples' $[\Delta]\, A^{\Theta}$.

In view of the restrictions mentioned *sub* "The procedure", a 'triple' $[\Delta]\, A^{\Theta}$ can be derived at most once[23] in the same prospective proof, whence every prospective proof for $\Gamma \vdash_{\mathbf{ACLuN1}} G$ terminates. ■

Theorem 5 *If $[\Delta]\, A^{\Theta}$ is derived at a line in a prospective proof from Γ, then $\Gamma \cup \Delta \vdash_{\mathbf{CLuN}} A \vee Dab(\Theta)$.*

Proof. By an obvious induction on the length of the prospective proof. Basis: an application of the Goal rule is justified by $\Gamma \cup \{G\} \vdash_{\mathbf{CLuN}} G$. For the induction step, every rule has to be considered. I consider only one case as an example.

Case ${\sim}$E. Suppose that $[\Delta]\, {\sim}A^{\Theta}$ occurs in the proof. By the induction hypothesis

$$\Gamma \cup \Delta \vdash_{\mathbf{CLuN}} {\sim}A \vee Dab(\Theta)\,.$$

This holds iff

$$\Gamma \cup \Delta \vdash_{\mathbf{CLuN}} ({\sim}A \vee (A \wedge \neg A)) \vee Dab(\Theta)\,,$$

which entails

$$\Gamma \cup \Delta \vdash_{\mathbf{CLuN}} (\neg A \vee (A \wedge {\sim}A)) \vee Dab(\Theta)\,,$$

which holds iff

$$\Gamma \cup \Delta \vdash_{\mathbf{CLuN}} \neg A \vee Dab(\Theta \cup \{A \wedge {\sim}A\})\,,$$

which justifies that $[\Delta]\, \neg A^{\Theta \cup \{A \wedge \sim A\}}$ is added to the prospective proof. ■

In proof of the following lemma, we need the *depth* at which a subformula A of B is nested in B. Let $\dagger$ be a variable for $\neg$ and ${\sim}$ and let $\sharp$ be a variable for the binary connectives. The function $d(A, B)$ is defined by the following clauses: (i) if A is not a subformula of B, then $d(A, B) = 0$, (ii) $d(A, A) = 1$, (iii) if $d(A, B) > 0$, then $d(A, \dagger B) = d(A, B) + 1$, (iv) if $d(A, B) + d(A, C) > 0$, then $d(A, B \sharp C) = d(A, B) + d(A, C) + 2$.

Lemma 3 *If $M = \langle v \rangle$ is a* **CLuN***-model, $v_M(A) = v_M({\sim}A) = 1$, and hence $v({\sim}A) = 1$, and the* **CLuN***-model $M' = \langle v' \rangle$ is exactly as M except that $v'({\sim}A) = 0$, then*

(*) *If $v_{M'}(B) < v_M(B)$, then* $\mathrm{pp}({\sim}A, B)$*; if $v_{M'}(B) > v_M(B)$, then* $\mathrm{pp}(\neg{\sim}A, B)$.

[23]Most such triples cannot be derived in a prospective proof for $\Gamma \vdash_{\mathbf{ACLuN1}}$, but the point is that no others can.

Proof. Suppose that the antecedent is true. I shall prove (*) by an induction on the depth of $\sim A$ in B. From (*) follows that $v_{M'}(B) = v_M(B)$ if (but *not* only if) neither $\text{pp}(\sim A, B)$ nor $\text{pp}(\neg\sim A, B)$.

The basis is formed by the cases where $d(\sim A, B) \leq 1$. If $d(\sim A, B) = 0$, then $v_{M'}(B) = v_M(B)$; if $d(\sim A, B) = 1$, then B is $\sim A$. In both cases (*) holds. For the induction step we have eight cases.

Case 1: B is an $\mathfrak{a}$-formula and $v_{M'}(B) < v_M(B)$; it follows that $v_{M'}(\mathfrak{a}_1) < v_M(\mathfrak{a}_1)$ or that $v_{M'}(\mathfrak{a}_2) < v_M(\mathfrak{a}_2)$; if $v_{M'}(\mathfrak{a}_1) < v_M(\mathfrak{a}_1)$, then $\text{pp}(\sim A, \mathfrak{a}_1)$ by the induction hypothesis, and hence $\text{pp}(\sim A, B)$; similarly if $v_{M'}(\mathfrak{a}_2) < v_M(\mathfrak{a}_2)$.

The proof of cases 2–4 is analogous to that of case 1: case 2: B is an $\mathfrak{a}$-formula and $v_{M'}(B) > v_M(B)$; case 3: B is an $\mathfrak{b}$-formula and $v_{M'}(B) < v_M(B)$; case 4: B is an $\mathfrak{b}$-formula and $v_{M'}(B) > v_M(B)$.

Case 5: B is $\neg C$ and $v_{M'}(B) < v_M(B)$; then $v_{M'}(C) > v_M(C)$; by the induction hypothesis $\text{pp}(\neg\sim A, C)$; hence $\text{pp}(\sim A, B)$.

Case 6: B is $\neg C$ and $v_{M'}(B) > v_M(B)$; analogous to case 5.

Case 7: B is $\sim C$, where C is not A, and $v_{M'}(B) < v_M(B)$; as $v'(\sim C) = v(\sim C)$, it follows that $v_{M'}(C) > v_M(C)$; by the induction hypothesis $\text{pp}(\neg\sim A, C)$; hence $\text{pp}(\sim A, B)$.

Case 8: B is $\sim C$, where C is not A, and $v_{M'}(B) > v_M(B)$; analogous to case 7. ■

Theorem 6 *If* $\Gamma \vdash_{\mathbf{CLuN}} Dab(\Delta)$ *and* $\Gamma \nvdash_{\mathbf{CLuN}} Dab(\Delta')$ *for every* $\Delta' \subset \Delta$, *then* $\text{pp}(\sim A, \Gamma)$ *for every* $A \wedge \sim A \in \Delta$.

Proof. Suppose that the antecedent is true, that $A \wedge \sim A \in \Delta$, and that $\text{pp}(\sim A, \Gamma)$ is false.

Let $Dab(\Delta)$ be $(A \wedge \sim A) \vee (B_1 \wedge \sim B_1) \vee \ldots \vee (B_n \wedge \sim B_n)$. Remark that $\Gamma \cup \{\neg(B_1 \wedge \sim B_1), \ldots, \neg(B_n \wedge \sim B_n\} \vdash_{\mathbf{ACLuN1}} A \wedge \sim A$ whence $\text{pp}(\sim A, \Gamma \cup \{\neg(B_1 \wedge \sim B_1), \ldots, \neg(B_n \wedge \sim B_n)\})$ by Theorem 6.

As $\Gamma \nvdash_{\mathbf{ACLuN1}} Dab(\Delta')$ for every $\Delta' \subset \Delta$, there is a **CLuN**-model of Γ that verifies $A \wedge \sim A$ and falsifies every $B_i \wedge \sim B_i$ $(1 \leq i \leq n)$. More precisely, let $M = \langle v \rangle$ be such a model for which $v(\sim B_i) = 0$ for every B_i. There obviously is such a model: if $v_M(B_i) = 1$, then $v_M(\sim B_i) = 0$ whence $v(\sim B_i) = 0$; if $v_M(B_i) = 0$, then $v_M(\sim B_i) = 1$ even if $v(\sim B_i) = 0$.

Let $M' = \langle v' \rangle$ be exactly as M except that $v'(\sim A) = 0$. Unlike M, M' falsifies $A \wedge \sim A$. As $\text{pp}(\sim A, \Gamma)$ is false by the main supposition, $v_{M'}(C) \geq v_M(C)$ for all $C \in \Gamma$ in view of Lemma 3. Hence M' is a model of Γ. Moreover, as $v'(\sim B_i) = 0$ for every B_i $(1 \leq i \leq n)$, M' falsifies every $B_i \wedge \sim B_i$. It follows that M' is a model of Γ that falsifies $Dab(\Delta)$, which contradicts the main supposition. ■

Theorem 7 *If* $\Gamma \vdash_{\mathbf{CLuN}} G \vee (A_1 \wedge \sim A_1) \vee \ldots \vee (A_n \wedge \sim A_n)$ *and* $\Gamma \nvdash_{\mathbf{CLuN}} \Delta$ *for every* $\Delta \subset \{G, A_1 \wedge \sim A_1, \ldots, A_n \wedge \sim A_n\}$, *then,* $G^{\{A_1 \wedge \sim A_1, \ldots, A_n \wedge \sim A_n\}}$ *is derivable in every prospective proof for* $\Gamma \vdash_{\mathbf{ACLuN1}} G$.

As the proof of the theorem requires many pages, I can only present an outline. There are some provable things for which I offer no demonstration, for example, that the order in which certain rules are applied has no effect on whether the goal is derivable in the proof. I start by outlining the proof of the following lemma:

Lemma 4 *If* $[B, C_1, \ldots, C_n]\, A$ *can be derived in a prospective* **CLuN**-*proof from* Γ*, and* A *is not the Goal, then* $[*A, C_1, \ldots, C_n] * B$ *can be derived in a prospective* **CLuN**-*proof from* Γ *in which* $*B$ *is a target.*

The proof proceeds by an induction on the number of applications of Trans and EM that lead to $[B, C_1, \ldots, C_n]\, A$.

First the basis. If A is not the goal of the proof, then $\text{pp}(A, \Gamma)$ by Lemma 2 (more precisely, by the corresponding lemma for **CLuN**). So $[B, C_1, \ldots, C_n]\, A$ was obtained from at least one premise. $[B, C_1, \ldots, C_n]\, A$ can be obtained from a single premise D by formula analysing rules and condition analysing rules (and possibly by EM0) iff D has a 'conjunctive normal form' of which $\bigvee\{*B, *C_1, \ldots, *C_n, A\}$ is a conjunct.[24] But then $[B, C_1, \ldots, C_n]\, A$ can be obtained from a premise iff $[*A, C_1, \ldots, C_n] * B$ can be obtained from the same premise.

For the induction step, consider first Trans.

Case 1: $[\{B\} \cup \Delta \cup \Theta]\, A$ is obtained by Trans from $[\{B, E\} \cup \Delta]\, A$ and $[\Theta]\, E$. By the induction hypothesis $[\{*A, E\} \cup \Delta] * B$ can be obtained. From this and $[\Theta]\, E$ follows $[\{*A\} \cup \Delta \cup \Theta] * B$ by Trans.

Case 2: $[\{B\} \cup \Delta \cup \Theta]\, A$ is obtained by Trans from $[\{E\} \cup \Delta]\, A$ and $[\{B\} \cup \Theta]\, E$. By the induction hypothesis, both $[\{*A\} \cup \Delta] * E$ and $[\{*E\} \cup \Theta] * B$ can be obtained, and from these follows $[\{*A\} \cup \Delta \cup \Theta] * B$ by Trans.

Next consider EM.

Case 1: $[\{B\} \cup \Delta \cup \Theta]\, A$ is obtained by EM from $[\{B, C\} \cup \Delta]\, A$ and $[\{*C\} \cup \Theta]\, A$. By the induction hypothesis, both $[\{*A, C\} \cup \Delta] * B$ and $[\{*A\} \cup \Theta]\, C$ can be obtained. From this follows $[\{*A\} \cup \Delta \cup \Theta] * B$ by Trans.

Case 2: $[\{B\} \cup \Delta \cup \Theta]\, A$ is obtained by EM from $[\{C\} \cup \Delta]\, A$ and $[\{B, *C\} \cup \Theta]\, A$. Completely analogous to case 1.

I now outline the proof of the theorem. Let a line i of a prospective proof be a *descendant* of a line j iff i is j or j belongs to the path of i.

Suppose that the antecedent of Theorem 7 is true. Let X abbreviate $G \vee (A_1 \wedge {\sim} A_1) \vee \ldots \vee (A_n \wedge {\sim} A_n)$. The supposition entails that Γ is $\neg$-consistent—it it were not, $\Gamma \vdash_{\mathbf{CLuN}} G$ would obtain, which contradicts the supposition. So we can safely disregard EFQ in what follows.

The first line in the **CLuN**-proof for $\Gamma \vdash_{\mathbf{CLuN}} X$ contains the following formula-with-condition:

$$[X]\, X\,. \tag{1}$$

In view of the presupposition, the thus started proof is bound to end with a line at which X is derived.

Let us neglect what was said under the heading "Some fine tuning" and apply the procedure in its crude (more permissive) form. This enables us to apply C$\vee$E to the condition of (1) and to the condition of the resulting lines until we obtain a proof that contains $n+1$ lines on which are derived:

$$[G]\, X$$
$$[A_1 \wedge {\sim} A_1]\, X$$
$$\ldots$$

[24]It is not required that the disjuncts of the conjuncts are literals, whence I use the quotation marks; however, the 'conjunctive normal form' must be obtained by the standard transformations, in as far as they are **CLuN**-valid, for obtaining a CNF.

$$[A_n \wedge {\sim}A_n]\, X\, .$$

Let these lines be called the *basic lines* of proof, and let the present stage of the proof be called the *divorced stage*.

Fact 1 X can be derived in such a way that every descendant of a line of the divorced stage is a descendant of a basic line.[25]

In other words, X, which can be derived in the proof, can be derived from the basic lines. To see this, suppose that, due to the precise location of the parentheses in X, $[(A_1 \wedge {\sim}A_1) \vee (A_2 \wedge {\sim}A_2)]\, X$ also occurs in the proof at this point, viz. at line i, that $\mathrm{pp}((A_1 \wedge {\sim}A_1) \vee (A_2 \wedge {\sim}A_2), \Gamma)$, and that $[\Delta]\, X$ is a descendant of line i. It is easily seen that $\mathrm{pp}(A_1 \wedge {\sim}A_1, \Gamma)$, and that the proof leading from line i to $[\Delta]\, X$ can be transformed in such a way that $[\Delta \cup \{\neg(A_2 \wedge {\sim}A_2)\}]\, X$ is a descendant of $[A_1 \wedge {\sim}A_1]\, X$. Incidentally, $[\Delta]\, X$ can be obtained from $[A_2 \wedge {\sim}A_2]\, X$ and $[\Delta \cup \{\neg(A_2 \wedge {\sim}A_2)\}]\, X$ by EM.

Let us return to the divorced stage and extend it in such a way that (i) every descendant of a line of the divorced stage is a descendant of a single basic line, and that, with this restriction, (ii) every descendant of every basic line is derived. This will lead to a set of descendants of $[G]\, X$, say $[\Delta^0_1]\, X, \ldots [\Delta^0_{m_0}]\, X$, to a set of descendants of $[A_1 \wedge {\sim}A_1]\, X$, say $[\Delta^1_1]\, X, \ldots [\Delta^1_{m_1}]\, X$, and so on up to a set of descendants of $[A_n \wedge {\sim}A_n]\, X$, say $[\Delta^n_1]\, X, \ldots [\Delta^n_{m_n}]\, X$. Let us call the thus obtained stage, the *isolationist stage* of the proof. The formulas-with-condition of the form $[\Delta^i_j]\, X$ will be called the *resolution lines*.

Fact 2 The $[\ldots]\, G$ that can be derived in a prospective proof for $\Gamma \vdash_{\mathbf{CLuN}} G$, are exactly $[\Delta^0_1]\, G, \ldots, [\Delta^0_{m_0}]\, G$. The $[\ldots]\, A_1 \wedge {\sim}A_1$ that can be derived in a prospective proof for $\Gamma \vdash_{\mathbf{CLuN}} A_1 \wedge {\sim}A_1$ are exactly $[\Delta^1_1]\, A_1 \wedge {\sim}A_1, \ldots, [\Delta^1_{m_1}]\, A_1 \wedge {\sim}A_1$. And so on for the other $A_i \wedge {\sim}A_i$.

Fact 3 For every Δ^i_j ($1 \leq i \leq m$ and $1 \leq j \leq m_i$), $\Delta^i_j \neq \emptyset$.

If some Δ^0_j were empty, X would have been derived at the isolationist stage. Hence, in view of Fact 2, G would be derivable in the prospective proof for $\Gamma \vdash_{\mathbf{CLuN}} G$. But then, in view of Theorem 3, $\Gamma \vdash_{\mathbf{CLuN}} G$, which contradicts the main supposition.

Fact 4 X can be derived by applications of EM and Trans from the resolution lines.

To see this, remark that EM and Trans are the only rules that have two 'premises'. As these rules do not introduce any new conditions, their results cannot lead to further applications of Prem, of formula analysing rules, or of condition analysing rules.

Let us extend the isolationist stage with applications of EM and Trans until X is derived in the proof. This stage of the proof will be called the *final stage*.

Fact 5 The line at which X is derived at the final stage is a descendant of every basic line.

Indeed, if this line was only a descendant of some basic lines, then the disjunction of the conditions of those lines would be **CLuN**-derivable from Γ, which contradicts the main supposition.

Of course not all resolution lines are required to obtain X by applications of Trans and EM at the final stage. Let a *sufficient selection* of resolution lines

[25]Premise lines and their descendants are not descendants of any lines of the divorced stage.

be a set of resolution lines that is sufficient to obtain X by EM and Trans. In view of Fact 5:

Fact 6 Every selection of resolution lines that is sufficient to obtain X contains a descendant of each basic line.

Fact 7 X can be derived from the resolution lines by applications of EM alone.

Consider a sufficient selection of resolution lines and suppose that Trans can be applied to $[\Delta^i_j]\, X$ and $[\Delta^k_l]\, X$. It follows that $X \in \Delta^i_j$ or $X \in \Delta^k_l$; suppose $X \in \Delta^k_l$. The application of Trans then leads to $[\Delta^i_j \cup \Delta^k_l - \{X\}]\, X$. However, this line is D-marked in view of the presence of $[\Delta^i_j]\, X$. So no useful line for deriving X can be obtained from applying Trans to resolution lines or to lines obtained from two resolution lines by EM.

Let a sufficient selection of resolution lines be *clean* iff EM cannot be applied to two lines of the selection that are descendants of the same basic line.

Fact 8 There is a clean sufficient selection of resolution lines.

Indeed, all $[\Delta^k_l]\, X$ that are descendants of the same basic line were derived at the isolationist stage. If lines i and j are descendants of the same basic line, and line k is obtained by EM from i and j, then k occurs in the isolationist stage and hence can be selected instead of i and j.

Fact 9 For every $[\Delta^0_i]\, X$ in a clean sufficient selection, and for every $B \in \Delta^0_i$, there is a $[\Delta^j_k]\, X$ in the sufficient selection for which $j \neq 0$ and $*B \in \Delta^j_k$.

Consider a clean sufficient selection. Every member of the condition of every line in the selection has ultimately to be eliminated by EM. So if $B \in \Delta^j_i$ for some $[\Delta^j_i]\, X$ in the selection, then $*B \in \Delta^k_l$ for some $[\Delta^k_l]\, X$ in the selection, and $k \neq j$ in view of Fact 8. So if $B \in \Delta^0_i$, then $*B \in \Delta^k_l$ for some $k \neq 0$.

Fact 10 In a prospective proof for $\Gamma \vdash_{\mathbf{ACLuN1}} G$, $[\Delta^0_1]\, G^\emptyset,\ \ldots,\ [\Delta^0_{m_0}]\, G^\emptyset$ can all be derived. (See Fact 2.)

I now come to the final part of the proof. The thus far described prospective proof for $\Gamma \vdash_{\mathbf{CLuN}} G \vee (A_1 \wedge {\sim}A_1) \vee \ldots \vee (A_n \wedge {\sim}A_n)$ will henceforth be called *the* **CLuN**-proof. I shall derive conclusions from it for a prospective proof for $\Gamma \vdash_{\mathbf{ACLuN1}} G$, henceforth called *the* **ACLuN1**-proof.

Let the following formulas

$$[\Delta^0_i \cup \{B\}]\, X \tag{2}$$

$$[\Delta^j_k \cup \{*B\}]\, X \tag{3}$$

belong to a clean sufficient selection of resolution lines and $B \notin \Delta^0_i$ and $*B \notin \Delta^j_k$. So $[\Delta^0_i \cup \{B\}]\, X$ is a descendant of the basic line $[G]\, X$, $j \neq 0$ by Fact 9, and hence $[\Delta^j_k \cup \{*B\}]\, X$ is a descendant of the basic line $[A_j \wedge {\sim}A_j]\, X$. Applying EM to them results in

$$[\Delta^0_i \cup \Delta^j_k]\, X\,. \tag{4}$$

As (2) occurs in the **CLuN**-proof,

$$[\Delta^0_i \cup \{B\}]\, G^\emptyset\,, \tag{5}$$

can be derived in the **ACLuN1**-proof in view of fact 10.

Case 1. $\mathrm{pp}(A_j \wedge {\sim}A_j, \Gamma)$. In the **CLuN**-proof, the premise of which $A_j \wedge {\sim}A_j$ is a positive part was introduced, and the subsequent moves can be (if necessary) so reorganized that $[\Delta^j_k \cup \{*B\}]\, A_j \wedge {\sim}A_j$ is obtained, whence (3) results by Trans from this and the basic line $[A_j \wedge {\sim}A_j]\, X$.

As $[\Delta^j_k \cup \{*B\}]\, A_j \wedge {\sim}A_j$ can be derived in the **CLuN**-proof, Lemma 4 warrants that

$$[\Delta^j_k \cup \{\neg(A_j \wedge {\sim}A_j)\}]\, B^{\emptyset}$$

can be derived in the **ACLuN1**-proof, in which B is a target—see (5). Hence one can also derive the following two

$$[\Delta^j_k \cup \{\neg A_j\}]\, B^{\emptyset} \tag{6}$$

$$[\Delta^j_k \cup \{\neg{\sim}A_j\}]\, B^{\emptyset} \tag{7}$$

by C$\vee$E. From (7) follows (8) by C$\neg{\sim}$E; from (8) and (6) follows (9) by EM; from (9) and (5) follows (10) by Trans—compare (10) to (4).

$$[\Delta^j_k \cup \{A_j\}]\, B^{\{A_j \wedge {\sim}A_j\}} \tag{8}$$

$$[\Delta^j_k]\, B^{\{A_j \wedge {\sim}A_j\}} \tag{9}$$

$$[\Delta^0_i \cup \Delta^j_k]\, G^{\{A_j \wedge {\sim}A_j\}} \tag{10}$$

Case 2. Not $\mathrm{pp}(A_j \wedge {\sim}A_j, \Gamma)$. Hence C$\wedge$E was applied in the **CLuN**-proof to the resolution line $[A_j \wedge {\sim}A_j]\, X$, resulting in $[A_j, {\sim}A_j]\, X$. If not $\mathrm{pp}(A_j, \Gamma)$, then further condition analysing rules were applied to A_j. In this case the lines may be reorganized in such a way that a line

$$[C_1, \ldots, C_m, {\sim}A_j]\, X \tag{11}$$

has been obtained (together with similar lines) by condition analysing rules from $[A_j, {\sim}A_j]\, X$, that $m+1$ lines, viz.

$$[\Theta_1]\, C_1,\ \ldots,\ [\Theta_m]\, C_m,\ [\Theta_0]\, {\sim}A_j \tag{12}$$

were (together with similar lines) obtained in view of the targets of (11), and that (3), viz. $[\Delta^j_k \cup \{*B\}]\, X$, is obtained by Trans from the lines in (12); in other words $\Delta^j_k \cup \{*B\} = \Theta_1 \cup \ldots \cup \Theta_m \cup \Theta_0$.

Let us now move to the **ACLuN1**-proof. Every line mentioned in (12) is derivable in it, but now with $\emptyset$ as its A-condition. There are two subcases.

Subcase 2.1: $*B \in \Theta_0$. Then one can obtain $[\Theta_0 - \{*B\} \cup \{\neg{\sim}A_j\}]\, B^{\emptyset}$ in view of Lemma 4 and the fact that B is a target—see (5) (and compare with case 1). From there one can obtain $[\Theta_0 - \{*B\} \cup \{A_j\}]\, B^{\{A_j \wedge {\sim}A_j\}}$. Condition analysing rules will be applied to A_j, and lead (among others) to $[\Theta_0 - \{*B\} \cup \{C_1, \ldots, C_m\}]\, B^{\{A_j \wedge {\sim}A_j\}}$—compare with (11). Relying on all but the last item of (12), now each with the A-condition $\emptyset$, one obtains $[\Theta_0 - \{*B\} \cup \Theta_1 \cup \ldots \cup \Theta_m]\, B^{\{A_j \wedge {\sim}A_j\}}$ by m applications of Trans as in the **CLuN**-proof. In other words, as $\Delta^j_k \cup \{*B\} = \Theta_1 \cup \ldots \cup \Theta_m \cup \Theta_0$, one obtains $[\Delta^j_k]\, B^{\{A_j \wedge {\sim}A_j\}}$. This corresponds to obtaining (9) in case 1. From there we proceed as in case 1.

Subcase 2.2: $*B \notin \Theta_0$, and hence $*B \in \Theta_1 \cup \ldots \cup \Theta_m$. One basically proceeds as in the previous case, except that one selects an arbitrary $D \in \Theta_0$ and derives $[\Theta_0 - \{D\} \cup \{\neg{\sim}A_j\}]\, *D^{\emptyset}$; indeed, $*D$ is bound to be a target because $*D$ is bound to occur in the condition of another resolution line of the clean sufficient selection in order for D to be eliminated by EM. From there one obtains $[\Theta_0 - \{D\} \cup \{A\}]\, *D^{\{A_j \wedge \sim A_j\}}$. To A_j analysing rules are applied as in subcase 2.1, and, relying on all but all but the last item of (12), now each with the A-condition $\emptyset$, one obtains $[\Theta_0 - \{D\} \cup \Theta_1 \cup \ldots \cup \Theta_m]\, *D^{\{A_j \wedge \sim A_j\}}$ by m applications of Trans as in the **CLuN**-proof. As $*B \in \Theta_1 \cup \ldots \cup \Theta_m$ and B is a target in view of (5), one next obtains $[\Theta_0 \cup \Theta_1 \cup \ldots \cup \Theta_m - \{*B\}]\, B^{\{A_j \wedge \sim A_j\}}$, which is $[\Delta^j_k]\, B^{\{A_j \wedge \sim A_j\}}$ as in subcase 2.1. From there we proceed as in case 1.

Both case 1 and case 2 lead to a similar situation. In the **CLuN**-proof we eliminated a resolution line, and hence at most all resolution lines that are descendants of a specific basic line, in the example $[A_j \wedge \sim A_j]\, X$, and this led to (4). That this can be done in the **CLuN**-proof warrants that, in the **ACLuN1**-proof, (10) can be obtained and its A-condition contains the abnormality that occurred in the D-condition of that basic line, in the example $A_j \wedge \sim A_j$.

Moreover we arrived at a *new* 'clean sufficient selection', in which (10) replaces (2) and (3). In this selection, (10) is a descendant of basic line $[G]\, X$; it easily seen that, for every $B \in \Delta^0_i \cup \Delta^j_k$, there is a $[\Delta^{j'}_{k'}]\, X$ in the new sufficient selection for which $*B \in \Delta^{j'}_{k'}$ and $j' \neq 0$ (compare this to Fact 9).

By repeating the reasoning for all basic lines, $G^{\{A_1 \wedge \sim A_1, \ldots, A_n \wedge \sim A_n\}}$ is obtained in the **ACLuN1**-proof.

A possible worry removed. Cases 1–2 are not the only ones *if* the clean minimal selection contains a resolution line that is a descendant of a basic line $[A_j \wedge \sim A_j]\, X$ and was obtained by applying C$\sim$E to $[A_j, \sim A_j]\, X$, resulting in $[A_j, \neg A_j]\, X$. However, this is impossible: if it were the case, then

$$\begin{aligned}\Gamma \vdash_{\mathbf{CLuN}} & \; G \vee (A_1 \wedge \sim A_1) \vee \ldots \\ & \vee (A_{j-1} \wedge \sim A_{j-1}) \vee (A_j \wedge \neg A_j) \vee (A_{j-1} \wedge \sim A_{j-1}) \vee \ldots \vee (A_n \wedge \sim A_n)\end{aligned}$$

and hence also

$$\begin{aligned}\Gamma \vdash_{\mathbf{CLuN}} & \; G \vee (A_1 \wedge \sim A_1) \vee \ldots \\ & \vee (A_{j-1} \wedge \sim A_{j-1}) \vee (A_{j-1} \wedge \sim A_{j-1}) \vee \ldots \vee (A_n \wedge \sim A_n)\,.\end{aligned}$$

which contradicts the main supposition. So this ends the outline of the proof of the Theorem 7.

Theorem 8 *For all* finite Γ *and for all G, the procedure forms a decision method for* $\Gamma \vdash_{\mathbf{ACLuN1}} G$.

Proof. In view of Theorem 4, every started phase terminates, and the procedure terminates. So I only have to show that the conclusions drawn during the different phases of the procedure are correct. Let us proceed backwards.

Phase 3. For some Θ, G^{Θ} was derived at line i in phase 1, for some Λ, $Dab(\Theta)^{\Lambda}$ was derived at line j in phase 2, $[Dab(\Lambda)]\, Dab(\Lambda)^{\emptyset}$ was introduced by X-Goal, and one tries to obtain $Dab(\Lambda)^{\emptyset}$. In view of Theorem 7, $Dab(\Lambda)^{\emptyset}$ will be derived iff it is derivable.[26] There were two possibilities:

(3.1) $Dab(\Lambda)^{\emptyset}$ is derived. It follows that $\Gamma \vdash_{\mathbf{CLuN}} Dab(\Lambda)$ and hence that $\Lambda \cap U(\Gamma) \neq \emptyset$, whence line j is justly A-marked.

[26]Even Theorem 3 warrants this, as the A-conditions are uniformly empty in phase 3.

(3.2) Phase 3 stops without $Dab(\Lambda)^{\emptyset}$ being derived. For all one knows, it is possible that $\Lambda \cap U(\Gamma) = \emptyset$. So line j should not be A-marked.

Phase 2. For some Θ, G^{Θ} was derived at line i in phase 1, $[Dab(\Theta)]\, Dab(\Theta)^{\emptyset}$ was introduced by A-Goal, and one tries to obtain $Dab(\Theta)^{\Lambda}$ for some Λ. There were three possibilities:
(2.1) $Dab(\Theta)^{\emptyset}$ is derived. Line i is justly A-marked (analogous to (3.1)).
(2.2) For some $\Lambda \neq \emptyset$, $Dab(\Theta)^{\Lambda}$ is derived at line j. After moving to phase 3:
(2.2.1) Line j is A-marked. As $\Lambda \cap U(\Gamma) \neq \emptyset$, it is compatible with all one knows that $\Theta \cap U(\Gamma) = \emptyset$. So line i is not A-marked. One continues in phase 2, trying to derive $Dab(\Theta)^{\Lambda'}$ for some $\Lambda' \not\supseteq \Lambda$.[27]
(2.2.2) Line j is not A-marked. So $\Gamma \vdash_{\mathbf{CLuN}} Dab(\Theta \cup \Lambda)$ and $\Gamma \nvdash_{\mathbf{CLuN}} Dab(\Lambda)$, whence $\Theta \cap U(\Gamma) \neq \emptyset$. So line i is justly A-marked.
(2.3) When phase 2 terminates, $Dab(\Theta)^{\Lambda}$ is not derived on an unmarked line for any Λ. In view of Theorem 7 and of what was said *sub* (3.1), it follows that there is no Λ for which $\Gamma \vdash_{\mathbf{CLuN}} Dab(\Theta \cup \Lambda)$ and $\Lambda \cap U(\Gamma) \neq \emptyset$. It follows that $\Theta \cap U(\Gamma) = \emptyset$. So line i is justly A-unmarked.

Phase 1. Subphase 1A. After $[G]\, G$ was introduced, one tries to derive G^{Θ} for some Θ. There were three possibilities:
(1.1) $G^{\emptyset}$ is derived. Then $\Gamma \vdash_{\mathbf{ACLuN1}} G$ in view of Theorem 5.
(1.2) G^{Θ} is derived at line i. After moving to phase 2, there are two possibilities:
(1.2.1) Line i is not A-marked. So $\Theta \cap U(\Gamma) = \emptyset$—see (2.3). Hence $\Gamma \vdash_{\mathbf{ACLuN1}} G$.
(1.2.2) Line i is A-marked. It is possible that $G^{\Theta'}$ is derivable for some $\Theta' \not\supseteq \Theta$.[28] So one goes on looking for such a $G^{\Theta'}$.
(1.3) The procedure terminates and G^{Θ} is not derived at an unmarked line for any Θ. In view of Theorem 7, G^{Θ} was derived, for all Θ, whenever $\Gamma \vdash_{\mathbf{CLuN}} G \vee Dab(Theta)$ for which there is no $\Theta' \subset \{G\} \cup \Theta$ such that $\Gamma \vdash_{\mathbf{CLuN}} \Delta$. Moreover, all lines at which a G^{Θ} was derived have been justly marked. So, in view of Theorem 1, $\Gamma \vdash_{\mathbf{ACLuN1}} G$ unless if Γ is $\neg$-inconsistent, in which case $\Gamma \vdash_{\mathbf{ACLuN1}} G$ can be demonstrated by EFQ.

Subphase 1B. One tries to derive $G^{\emptyset}$ by applications of EFQ as well as well of the other **CLuN**-rules. In view of Theorem 3, if Γ is inconsistent, then $G^{\emptyset}$ is derived iff $\Gamma \vdash_{\mathbf{ACLuN1}} G$. ■

5 In Conclusion

The 'defeasible' conditions that occur in dynamic proofs of adaptive logics suggested a kind of dynamic proofs with 'prospective' conditions. This led to a specific form of goal directed proofs. Later, these goal directed proofs turned out to provide a proof procedure that forms an algorithm for final derivability at the propositional level. As remarked in Section 1, the central interest of the procedure is that it provides a criterion at the predicative level if it stops.

The dynamic proofs explicate actual reasoning. The prospective proofs do not, but there is an algorithm for turning them into standard dynamic proofs (by reordering and replacing lines). So, after finding out that some formula

[27] If $\Lambda' \supseteq \Lambda$, then $\Lambda' \cap U(\Gamma) \neq \emptyset$ and hence a line at which $Dab(\Theta)^{\Lambda'}$ would be derived would be A-marked anyway.

[28] Compare footnote 27 for the case in which $\Theta' \supseteq \Theta$.

is derivable at a stage from the premises, one may switch to the goal directed format in order to find out whether the formula is finally derivable. If a decision is reached, one may transform the result to a regular dynamic proof, if desired. After this, the proof may proceed and, if a further interesting formula is derived at a stage, one may again switch to the goal directed format to settle its final derivability.

Given the present standard characterization (from [16]) of flat adaptive logics, some minimal changes to the aforementioned rules will result in a prospective procedure for any other adaptive logic. Basically, one replaces the rules that pertain to the abnormalities—in the case of **ACLuN1**, the rules containing the paraconsistent negation $\sim$.

While these replacements are straightforward, further research is required for the predicative level. Devising sensible rules is unproblematic—the relevant research was finished. However, more work is needed to improve the efficiency of the procedure and to avoid infinite loops whenever possible. It is easily seen that known techniques from tableau methods and resolution methods may easily be transposed to avoid infinite loops in prospective proofs.[29]

References

[1] Atocha Aliseda. *Seeking Explanations: Abduction in Logic, Philosophy of Science and Artifical Intelligence.* Phd thesis, Institute for Logic, Language and Computation (ILLC), University of Amsterdam, The Netherlands, 1997.

[2] Alan Ross Anderson and Nuel D. Belnap, Jr. *Entailment. The Logic of Relevance and Necessity*, volume 1. Princeton University Press, 1975.

[3] Grigoris Antoniou. *Nonmonotonic Reasoning.* MIT Press, Cambridge, Mass., 1996.

[4] Arnon Avron and Iddo Lev. A formula-preferential base for paraconsistent and plausible non-monotonic reasoning. In *Proceedings of the Workshop on Inconsistency in Data and Knowledge (KRR-4) Int. Joint Conf. on AI (Ijcai 2001)*, pages 60–70, 2001.

[5] Diderik Batens. Dynamic dialectical logics as a tool to deal with and partly eliminate unexpected inconsistencies. In J. Hintikka and F. Vandamme, editors, *The Logic of Discovery and the Logic of Discourse*, pages 263–271. Plenum Press, New York, 1985.

[6] Diderik Batens. Meaning, acceptance, and dialectics. In Joseph C. Pitt, editor, *Change and Progress in Modern Science*, pages 333–360. Reidel, Dordrecht, 1985.

[7] Diderik Batens. Static and dynamic paraconsistent logics and their use in expert systems. *CC-AI*, 3:33–50, 1986.

[29] Unpublished papers by members of our research group are available from the internet address `http://logica.ugent.be/centrum/writings/`.

[8] Diderik Batens. Dynamic dialectical logics. In Graham Priest, Richard Routley, and Jean Norman, editors, *Paraconsistent Logic. Essays on the Inconsistent*, pages 187–217. Philosophia Verlag, München, 1989.

[9] Diderik Batens. Inconsistency-adaptive logics and the foundation of non-monotonic logics. *Logique et Analyse*, 145:57–94, 1994. Appeared 1996.

[10] Diderik Batens. Blocks. The clue to dynamic aspects of logic. *Logique et Analyse*, 150–152:285–328, 1995. Appeared 1997.

[11] Diderik Batens. Inconsistency-adaptive logics. In Orłowska [42], pages 445–472.

[12] Diderik Batens. Zero logic adding up to classical logic. *Logical Studies*, 2:15, 1999. (Electronic Journal: `http://www.logic.ru/LogStud/02/LS2.html`).

[13] Diderik Batens. A survey of inconsistency-adaptive logics. In Diderik Batens, Chris Mortensen, Graham Priest, and Jean Paul Van Bendegem, editors, *Frontiers of Paraconsistent Logic*, pages 49–73. Research Studies Press, Baldock, UK, 2000.

[14] Diderik Batens. Towards the unification of inconsistency handling mechanisms. *Logic and Logical Philosophy*, 8:5–31, 2000. Appeared 2002.

[15] Diderik Batens. A dynamic characterization of the pure logic of relevant implication. *Journal of Philosophical Logic*, 30:267–280, 2001.

[16] Diderik Batens. A general characterization of adaptive logics. *Logique et Analyse*, 173–175:45–68, 2001. Appeared 2003.

[17] Diderik Batens. The need for adaptive logics in epistemology. To appear.

[18] Diderik Batens. A strengthening of the Rescher–Manor consequence relations. *Logique et Analyse*, To appear.

[19] Diderik Batens and Joke Meheus. A tableau method for inconsistency-adaptive logics. In Roy Dyckhoff, editor, *Automated Reasoning with Analytic Tableaux and Related Methods*, Lecture Notes in Artificial Intelligence Vol. 1847, pages 127–142. Springer, 2000.

[20] Diderik Batens and Joke Meheus. Shortcuts and dynamic marking in the tableau method for adaptive logics. *Studia Logica*, 69:221–248, 2001.

[21] Diderik Batens and Dagmar Provijn. Pushing the search paths in the proofs. A study in proof heuristics. *Logique et Analyse*, 173–175:113–134, 2001. Appeared 2003.

[22] Diderik Batens and Timothy Vermeir. Direct dynamic proofs for the Rescher–Manor consequence relations: The flat case. *Journal of Applied Non-Classical Logics*, 12:63–84, 2002.

[23] Salem Benferhat, Didier Dubois, and Henri Prade. Some syntactic approaches to the handling of inconsistent knowledge bases: A comparative study. Part 1: The flat case. *Studia Logica*, 58:17–45, 1997.

[24] Salem Benferhat, Didier Dubois, and Henri Prade. Some syntactic approaches to the handling of inconsistent knowledge bases: A comparative study. Part 2: The prioritized case. In Orłowska [42], pages 473–511.

[25] Philippe Besnard and Torsten Schaub. Signed systems for paraconsistent reasoning. *Journal of Automated Reasoning*, 20:191–213, 1998.

[26] George S. Boolos and Richard J. Jeffrey. *Computability and Logic*. Cambridge University Press, 1989. (Third edition).

[27] Gerhard Brewka. *Nonmonotonic Reasoning: Logical Foundations of Commonsense*. Cambridge University Press, Cambridge, Mass., 1991.

[28] Newton C.A. da Costa, Otávio Bueno, and Steven French. The logic of pragmatic truth. *Journal of Philosophical Logic*, 27:603–620, 1998.

[29] Peter Damerow, Gideon Freudenthal, Peter McLaughlin, and Jürgen Renn. *Exploring the Limits of Preclassical Mechanics*. Springer-Verlag, Berlin, 1992.

[30] Kristof De Clercq. Two new strategies for inconsistency-adaptive logics. *Logic and Logical Philosophy*, 8:65–80, 2000. Appeared 2002.

[31] Theo A. F. Kuipers. *From Instrumentalism to Constructive Realism*, volume 287 of *Synthese Library*. Kluwer, Dordrecht, 2000.

[32] Iddo Lev. Preferential systems for plausible non-classical reasoning. Master's thesis, Department of Computer Science, Tel-Aviv University, 2000.

[33] Witold Łukaszewicz. *Non-Monotonic Reasoning. Formalization of Commonsense Reasoning*. Ellis Horwood, New York, 1990.

[34] Joke Meheus. Adaptive logic in scientific discovery: the case of Clausius. *Logique et Analyse*, 143–144:359–389, 1993. Appeared 1996.

[35] Joke Meheus. Erotetic arguments from inconsistent premises. *Logique et Analyse*, 165–166:49–80, 1999. Appeared 2002.

[36] Joke Meheus. Adaptive logics for question evocation. *Logique et Analyse*, 173–175:135–164, 2001. Appeared 2003.

[37] Joke Meheus. An adaptive logic for pragmatic truth. In Walter A. Carnielli, Marcelo E. Coniglio, and Itala M. Loffredo D'Ottaviano, editors, *Paraconsistency. The Logical Way to the Inconsistent*, pages 167–185. Marcel Dekker, New York / Basel, 2002.

[38] Joke Meheus. Adaptive logics and the integration of induction and deduction. In F. Stadler, editor, *Induction and Deduction in the Sciences*, pages 93–120. Kluwer, Dordrecht, 2004.

[39] Joke Meheus. Empirical progress and ampliative adaptive logics. in print.

[40] Joke Meheus, Liza Verhoeven, Maarten Van Dyck, and Dagmar Provijn. Ampliative adaptive logics and the foundation of logic-based approaches to abduction. In Lorenzo Magnani, Nancy J. Nersessian, and Claudio Pizzi, editors, *Logical and Computational Aspects of Model-Based Reasoning*, pages 39–71. Kluwer, Dordrecht, 2002.

[41] Irene Mikenberg, Newton C. A. da Costa, and Rolando Chuaqui. Pragmatic truth and approximation to truth. *Journal of Symbolic Logic*, 51:201–221, 1986.

[42] Ewa Orłowska, editor. *Logic at Work. Essays Dedicated to the Memory of Helena Rasiowa.* Physica Verlag (Springer), Heidelberg, New York, 1999.

[43] Gabriele Paul. AI approaches to abduction. In Dov M. Gabbay and Philippe Smets, editors, *Handbook of Defeasible Reasoning and Uncertainty Management Systems. Volume 4: Abductive Reasoning and learning*, pages 35–98. Kluwer Academic Publishers, Dordrecht, 2000.

[44] Nicholas Rescher. *Hypothetical Reasoning.* North-Holland, Amsterdam, 1964.

[45] Nicholas Rescher and Ruth Manor. On inference from inconsistent premises. *Theory and Decision*, 1:179–217, 1970.

[46] Kurt Schütte. *Beweistheorie.* Springer, Berlin, 1960.

[47] Raymond M. Smullyan. *First Order Logic.* Dover, New York, 1995. Original edition: Springer, 1968.

[48] Guido Vanackere. Preferences as inconsistency-resolvers: the inconsistency-adaptive logic **PRL**. *Logic and Logical Philosophy*, 8:47–63, 2000. Appeared 2002.

[49] Guido Vanackere. **HL2**. An inconsistency-adaptive and inconsistency-resolving logic for general statements that might have exceptions. *Journal of Applied Non-Classical Logics*, 10:317–338, 2000.

[50] Liza Verhoeven. Proof theories for some prioritized consequence relations. To appear.

[51] Andrzej Wiśniewski. *The Posing of Questions. Logical Foundations of Erotetic Inferences.* Kluwer, Dordrecht, 1995.