Quasi-merging and Pure-arbitration on Information for the Family of Adaptive Logics $\mathbf{ADM}^*$

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Abstract

The present paper introduces two new information merging protocols for the family of adaptive logics $\mathbf{ADM}$*, for which majority merging has been defined in previous work. The new adaptive operators reflect the negotiation processes of quasi-merging and pure-arbitration known from the Integrity Constraints framework. The Adaptive Variant Counting selection provides results equivalent to the GMax family of merging operators: it selects a collective model for a multi-set of belief bases established on the number of disagreements verified by the various models. The Adaptive Minimax Counting selection is a quasi-merging operator which applies a minimax function and it obtains a larger spectrum of possibilities than the previous selection: it simulates the behaviour of the Max family of operators from the Integrity Constraints framework, but it avoids some of its counterintuitive results.*

Keywords: Information Fusion, Negotiation Protocols, Arbitration, Quasi-Merging, Adaptive Logics.

1. Introduction

The analysis of processes of intelligent interaction in multi-agent systems has grown constantly in the logical literature of the last decade, with diversificated approaches and aims. The applications vary from the formalization of interactive processes of collective deliberation, especially relevant with respect to the formulation of judgement aggregation strategies, to information fusion architectures.

*Research for this paper was supported by subventions from Ghent University and from the Research Foundation - Flanders (FWO - Vlaanderen).

The analysis of contents involved in a decision process focuses naturally on the agreements among agents, in order to perform the most satisfactory selection of common goals and judgements in the group. Obviously, such a process might not be entirely satisfactory, and the presence of disagreements expressing a certain degree of internal dissatisfaction cannot be completely ruled out by the negotiation and the consequent aggregation protocols. The formalization of selection procedures in view of such inconsistent data is the aim of the frameworks defining knowledge merging operators, also known as information fusion operators.

The merging of contents from contradictory sources, whose study goes back to [5], has applications in distributed databases and information systems. General properties for the logical approaches to merging procedures for knowledge bases containing inconsistent information have been studied in [7], [6], [4], and more recently surveyed in [11] and [10].

The first definition of an operator for merging information has been given in [21] and later considerably reworked in [14] and [15]. In the latter work, the idea of *arbitration* comes from an intuitive modification of the more standard revision operator from the AGM-paradigm in [1]: it refers to merging as the revision of an older base with the information of a newer base, without any order of priority of the latter over the former. The process requires instead preservation of information from one base in some cases and from the other in other cases. This general principle has been modified by the use of weights on the bases, to indicate the relative importance of the information rather than strict priority. Weights have been expressed as priority values (as in [9]), they have been assigned either to propositional terms (see e.g. [8]) or to the set of models of formulas (as in [21]), and finally they have been formulated as possibility values (see [23]).

A major distinction has been introduced by the definition of the *majority* protocol. In [16] an operator is defined which avoids the typical restrictions of the majority principle, formulated taking into account formulae in disjunctive form within the bases to be merged. This allows to represent bases that only partially support their contents. The ground distinction between the arbitration and majority operators - see e.g. [13] - can be reflected in the following terms: whereas majority merging operators aim at minimizing collective dissatisfaction, arbitration operators aim at maximizing individual satisfaction. This distinction is of the greatest importance with respect to the results of collective deliberation procedures and the restrictions due to results as the one of the judgement aggregation paradox.

These two sub-classes of merging operators are further defined in the more general and standard framework of information merging under intergrity constraints in [12]. This framework allows for defining families of three distinct operators:

1. the $\triangle^{\Sigma}$ operator satisfies the postulates for majority merging and it corresponds to the merging operator defined in [16];

2. the $\triangle^{GMax}$ operator satisfies a pure arbitration procedure, and it represents a new merging method;

3. finally, the $\triangle^{Max}$ operator is called a quasi-merging operator and it represents a pseudo-arbitration operator corresponding to the one defined in [15].

Moreover, in [13] it is shown that another family of operators called $\triangle^{n}$ can be defined which belongs simultaneously to the two main subclasses.

The standard approach to these protocols uses a definition of distance between the involved belief bases and the possible interpretations. The standard one is the Dalal distance from [8]: the intuitive idea behind this definition is to measure the number of atoms that have different truh values among each base and every interpretation, so to find the collective model that retains the most of each base; a variant is represented by the Satoh distance, defined in [22]. The various merging protocols apply an ordering on the values resulting from the definition of distance according to different functions, in order to obtain the desired negotiation process.

A different approach to the resolution of merging processes of conflicting belief bases has been introduced in [17] in view of the dynamic semantics of adaptive logics (see [2, 3] for a general introduction to the standard format of Adaptive Logics). The crucial change of perspective given by this new approach is represented by the focus on disagreements occurring among the agents involved in the negotiation process: the explicit derivation of conflicts in the collective decision process allows for the formulation of a consequence set that reflects the various aggregation methods in terms of unavoidable disagreements. The resulting framework is the family of logics $\mathbf{ADM}$, for *Adaptive Doxastic Merging*.

The first effective result obtained for adaptive merging is the majority protocol for bases with partial support defined in terms of the logic $\mathbf{ADM}^{c}$, for *Adaptive Doxastic Merging by Counting*, formulated in [20]. As it is shown in [16], the protocol which satisfies all the due postulates for majority has to take into account the requirements on partially supported contents, and the family $\mathbf{ADM}$ makes use of so called abnormal formulas that are designed precisely to accomplish this aim. The Counting strategy selects from the set of models of a given premise set, providing a protocol of majority merging corresponding to the generalization under Integrity Constraints represented by the $\triangle^{\Sigma}$ operator. Moreover, the use of fully versus partially supported contents allows for the mentioned notion of weights to be reformulated in a new light: weights express the support each agent gives to contents, in order for his or her beliefs to be accepted by the group in the fusion procedure. This allows for commutativity to be entirely preserved also among weighted bases.

The application of the majority protocol to the judgment aggregation paradox is considered in [18]: it provides a non-paradoxical though inefficient solution. Effectivity is obtained by modifying the agenda of interaction, which in turn amounts to give up the Universal Domain condition. To this aim the formulation of the logic $\mathbf{ADM}^{c}$ is slightly more simple in view of the fact that all bases in the case of the paradox express full support to their contents.

The next step in this resarch is represented by the formulation of an arbitration protocol for the family of logics $\mathbf{ADM}$, mimicking the results of the $\triangle^{GMax}$ operator. This result is first presented in [19], where the logic $\mathbf{ADM}^{c+}$ for *Adaptive Doxastic Merging by Variant Counting* is introduced. The semantic selection defined for this adaptive logic in standard format shows a basic correspondance between a pre-order on satisfied disagreements among the agents and a lexicographic order of Dalal's distances. The formulation of the arbitration protocol is considered in the light of the problem of fusion of information from heterogeneous databases: the fusion architecture based on this protocol shows its potential in applications where the treatment of incomplete or only partially verified data might be crucial to an effective fusion procedure.

In the present paper we shall recover the basics of the logic $\mathbf{ADM}^{c+}$ and of its selection procedure. Starting from its basis, it will be possible to define a third selection procedure on the models of a premise set, derived from the minimax rule for decision theory, and thus performing the same results as the $\triangle^{Max}$ quasi-merging operator. This selection procedure shall be introduced as the logic $\mathbf{ADM}^{c-mm}$, for *Adaptive Doxastic Merging by Minimax Counting*. With this last result the family of logics $\mathbf{ADM}$ is shown to be a general framework to define all the various negotiation processes modelled by the standard merging operators, in particular those of the general Integrity Constraints framework.

The structure of this paper is as follows. In section 2 we will consider briefly the quasi-merging and arbitration Integrity Constraints operators. In section 3 we will introduce the preliminaries needed for the adaptive logics of the family $\mathbf{ADM}$, and in section 4 and 5 we will respectively define the semantic selection procedures that give rise to the logics $\mathbf{ADM}^{c+}$ and $\mathbf{ADM}^{c-mm}$. Section 6 presents a standard example where both strategies are applied. In the final section further steps for the research on the adaptive procedures of merging are surveyed.

2. Integrity Constraints Merging

In this section we introduce the Integrity Constraints (IC) merging protocols from [12] that are going to be mimicked by different strategies in the $\mathbf{ADM}$ family of adaptive logics. In the following of this paper $\mathcal{L}$ will refer to the standard language of classical propositional logic (henceforth $\mathbf{CL}$) that is formed from a finite set of atoms $\mathcal{P}$ in the usual way. The set of literals $\mathcal{P}^{\pm}$ contains atoms and their negations. Letters from the greek alphabet $\varphi, \psi, \ldots$ are used as metavariables for sentences of $\mathcal{L}$. As is common, the abbreviation $\bigvee(\Delta)$ will stand for the disjunction of the members of Δ, where Δ is a set of formulas. A *belief base* T is a finite set of sentences of $\mathcal{L}$. Integrity constraints μ are a finite set of sentences, i.e. a belief base with respect to which the final merged state must be consistent. By Γ one refers to a multi-set consisting of n belief bases, $\Gamma = \{T_1, \ldots, T_n\}$. The formula $\bigwedge \Gamma$ denotes the conjunction of the belief bases of Γ, i.e. $\bigwedge \Gamma = \bigwedge \{T_1, \ldots, T_n\}$. A $\mathbf{CL}$-model is a function $\mathcal{P} \rightarrow \{0, 1\}$. We shall use $\mathcal{M}$ to denote the set of all $\mathbf{CL}$-models. A model M is a model of T iff all the members of T are true in it. $Mod(\Gamma)$ will be the set of models of the multi-set Γ and $Cn(\Gamma)$ will denote the consequence set of Γ. The result of a merging procedure on a multi-set Γ under constraints μ shall be denoted as $\triangle_\mu(\Gamma)$. The union of multi-sets will be denoted by $\sqcup$.

2.1 IC Pure-Arbitration

The IC framework defines selection methods of the collective models of various belief bases by operators satisfying the following postulates:

IC0 $\triangle_\mu(\Gamma) \vdash \mu$;

IC1 If μ is consistent, then $\triangle_\mu(\Gamma)$ is consistent;

IC2 If $\bigwedge \Gamma$ is consistent with μ, then $\triangle_\mu(\Gamma) = \bigwedge \Gamma \wedge \mu$;

IC3 If $\Gamma_1 \leftrightarrow \Gamma_2$ and $\mu_1 \leftrightarrow \mu_2$, then $\triangle_{\mu_1}(\Gamma_1) \leftrightarrow \triangle_{\mu_2}(\Gamma_2)$;

IC4 If $T \vdash \mu$ and $T' \vdash \mu$, then $\triangle_\mu(T \sqcup T') \wedge T \nvdash \mu$ implies that $\triangle_\mu(T \sqcup T') \wedge T' \nvdash \mu$;

IC5 $\triangle_\mu(\Gamma_1) \wedge \triangle_\mu(\Gamma_2) \vdash \triangle_\mu(\Gamma_1 \sqcup \Gamma_2)$;

IC6 $\triangle_\mu(\Gamma_1) \wedge \triangle_\mu(\Gamma_2)$ is consistent, then $\triangle_\mu(\Gamma_1 \sqcup \Gamma_2) \vdash \triangle_\mu(\Gamma_1) \wedge \triangle_\mu(\Gamma_2)$;

IC7 $\triangle_{\mu_1}(\Gamma) \wedge \mu_2 \vdash \triangle_{\mu_1 \wedge \mu_2}(\Gamma)$;

IC8 If $\triangle_{\mu_1}(\Gamma) \wedge \mu_2$ is consistent, then $\triangle_{\mu_1 \wedge \mu_2}(\Gamma) \vdash \triangle_{\mu_1}(\Gamma) \wedge \mu_2$.

On the basis of these postulates, the consistency on merging and the irrelevance of syntax are principles of the greatest importance to define quasi-merging and pure-arbitration. The former is given in the following informal definition:

Definition 2.1 [Principle of Consistency on Merging] If two subgroups agree on at least one alternative, the result of global merging will be exactly those alternatives the two groups agree on. $\triangleleft$

and it is formally obtained by the combination of postulates **IC5** and **IC6**. The principle of syntax irrelevance says informally:

Definition 2.2 [Principle of Syntax Irrelevance on Merging] If two bases are syntactically equivalent and so are their integrity constraints, then the merging of one base under one set of integrity constraints shall be equivalent to the merging of the other base under the other set of constraints. $\triangleleft$

and it is formally given by postulate **IC3**.

For the introduction and explanation of the pure-arbitration and quasi-merging protocols we will refer to a preorder on the set of models of a premise set. A preorder over the set of $\mathbf{CL}$-models is a reflexive and transitive relation on $\mathcal{M}$. Where $\leq$ is a preorder, $<$ is defined as: $M < M'$ iff $M \leq M'$ and $M' \nleq M$. Where M is a subset of $\mathcal{M}$, a model M is said *minimal* in M with respect to $\leq$ iff $M \in \mathsf{M}$ and there is no $M' \in \mathsf{M}$ such that $M' < M$. $Min(\mathsf{M}, \leq)$ shall denote the set of models that are minimal in M with respect to $\leq$.

Given two models M_1, M_2 and a belief base T, a preorder $M_1 \leq M_2$ holds if and only if $dist(M_1, T) \leq dist(M_2, T)$. The value of $dist(M_1, M_2)$ between two models M_1 and M_2 according to the Dalal distance refers to the number of atoms whose valuation differs in the two models. Given the set $Mod(T)$ of possible models of the base T, the distance between a $\mathbf{CL}$-model M and T is given as follows:

$dist(M, T) = min(dist(M, M'))$ for each $M' \in Mod(T)$. (1)

The selection of collective models $Mod(\triangle_{\mu}^{GMax}(\Gamma))$ performed according to the IC arbitration operator using this notion of distance works in the following way. Consider belief bases T_1, T_2 whose alternatives are preferred respectively under Integrity Contraints μ_1, μ_2; assume that each of the set of alternatives is equally preferred under the union of the bases $T_1 \sqcup T_2$; the subset of preferred alternatives under the disjunction of the integrity constraints coincides with the preferred alternatives of each base. Model-theoretically this means that there is a total preorder on the plausibility of the models with respect to the belief bases. Plausibility is obtained as an ordering by a notion of distance as Dalal's one and an aggregation function $\oplus$. Such an ordering says that if M_1 is more plausible than M_2 for T_1 and more plausible than M_3 for T_2, and M_2 and M_3 are equally plausible for the union of bases $T_1 \sqcup T_2$, than M_1 has to be more plausible than both M_2 and M_3 for $T_1 \sqcup T_2$. The result of the merging procedure is the belief base whose models are the most plausible ones for the given set of individual bases, according to given rationality criteria.

In [12], it is shown that the aggregation function $\oplus$ satisfying the arbitration protocol is the *leximax* function. Consider the multi-set $\Gamma = \{T_1, \dots, T_n\}$; for each model M consider the list $D = (dist_1^M, \dots, dist_n^M)$ of distances between M and the n belief bases in Γ, i.e. the list of distances $dist_i^M = dist(M, T_i)$. Let L_{Γ}^M be the list obtained from D by sorting its members in descending order. Denote now by $\leq_{lex}$ the lexicographic order among sequences of integers of the same length. For any two models M_1 and M_2, a total preorder $M_1 \leq_{\Gamma} M_2$ holds in view of Γ if and only if $L_{\Gamma}^{M_1} \leq_{lex} L_{\Gamma}^{M_2}$. Given a multi-set Γ holding under contraints μ, the $\triangle_{\mu}^{GMax}$ operator is then defined as follows:

$$Mod(\triangle_{\mu}^{GMax}(\Gamma)) = Min(Mod(\mu), \leq_{\Gamma}). \quad (2)$$

This operator satisfies the typical postulate for arbitration:

$$\left.\begin{array}{r} \triangle_{\mu_1}(T_1) \leftrightarrow \triangle_{\mu_2}(T_2) \\ \triangle_{\mu_1 \leftrightarrow \neg\mu_2}(T_1 \sqcup T_2) \leftrightarrow (\mu_1 \leftrightarrow \neg\mu_2) \\ \mu_1 \nvdash \mu_2 \\ \mu_2 \nvdash \mu_1 \end{array}\right\} \begin{array}{l} \Rightarrow \triangle_{\mu_1 \vee \mu_2} \\ (T_1 \sqcup T_2 \leftrightarrow \\ \triangle_{\mu_1}(T_1)) \end{array} \quad (3)$$

which says that if a set of alternatives preferred among one set of integrity constraints μ_1 for a belief base T_1 corresponds to the set of alternatives preferred among another set of integrity constraints μ_2 for base T_2, and if the alternatives that belong to a set of integrity constraints but not to the other are equally preferred for the whole group $(T_1 \sqcup T_2)$, then the subset of preferred alternatives among the disjunction of integrity constraints coincides with the preferred alternatives of each base among their respective integrity constraints (see [12], p.778).

2.2 IC Quasi-Merging

The second family of merging operators considered is a less fine-grained one and it is defined by the so-called quasi-merging $\triangle^{Max}$ operator in terms of the $minimax$ function. Let $\Gamma = \{T_1, \dots, T_n\}$ be the usual belief set, M a model and d the standard Dalal's distance value. The Max operator considers first the maximal distance between an interpretation and a belief base

$$d_{Max}(M, \Gamma) = Max_{T \in \Gamma} dist(M, T); \quad (4)$$

then a preorder on the set of interpretations M is defined:

$$M_1 \leq_{\Gamma}^{Max} M_2 \text{ iff } d_{Max}(M_1, \Gamma) \leq d_{Max}(M_2, \Gamma) \quad (5)$$

which says that a model M_1 comes before in the preorder than a model M_2 if and only if the maximal distance between the former and the multi-set Γ is lower than the same distance between the latter and Γ. The resulting $\triangle_{\mu}^{Max}(\Gamma)$ operator is obtained as the one with lower position (minimal value) in the obtained pre-order:

$$Mod(\triangle_{\mu}^{Max}(\Gamma)) = Min(Mod(\mu), \leq_{\Gamma}^{Max}). \quad (6)$$

In the following sections we shall introduce the Adaptive Logic **ADM** with two adaptive strategies, namely *Variant Counting* and *Minimax Counting*: their role is to formulate adaptive merging procedures whose results are comparable to those of the $\triangle^{GMax}$ and $\triangle^{Max}$ operators.

3. The Adaptive Logic for Merging

The formulation of the logics belonging to the family **ADM** is based on the langugae $\mathcal{L}^B$, which enables one to represent a *set* of belief bases by a single set of premises. It also enables one to consider (modal) models that validate all the premises, rather than having to consider models for each of the belief bases separately. Where $\mathcal{I} = \{0, 1, \dots\}$ is a set of indexes, the multi-modal language $\mathcal{L}^{\mathcal{B}}$ is $\mathcal{L}$ extended with a belief operator b_i, for any $i \in \mathcal{I}$. Each different base is given an index b_i with $i \in \mathcal{I} \setminus 0$. The operator b_0 is used exclusively for the beliefs selected for the merging state, or for the constraints holding in such state. Intuitively, $b_i\varphi$ (for $i > 0$) will express that agent i believes or supports φ; the formula $b_0\varphi$ means that all agents agree on φ or that their decision is constrained by the holding of φ. The premise set Γ refers to a multi-set of indexed belief bases $\Gamma = \{T_1, \dots, T_n\}$. When the two adaptive strategies are introduced, the operator $\triangle^{c+}$ (eventually $\triangle_{\mu}^{c+}$ when some set of constraints μ is given) is used for the result of the Variant Counting strategy and $Mod(\triangle^{c+}(\Gamma))$ to refer to the subset of $Mod(\Gamma)$ correspondingly selected; the operator $\triangle^{c-mm}$ ($\triangle^{c-mm}_{\mu}$ respectively) is used for the result obtained by the Minimax Counting Strategy, $Mod(\triangle^{c-mm}(\Gamma))$ referring to the subset of $Mod(\Gamma)$ selected by that strategy.

Let us consider as an example a set of belief bases

$$T_1 = \{p \vee q\}$$
$$T_2 = \{\neg p\}$$
$$T_3 = \{\neg q\}.$$

These belief bases are given a modal translation in any of the logics belonging to the family $\mathbf{ADM}$ as the premise set $\Gamma = \{b_1(p \vee q), b_2\neg p, b_3\neg q\}$. This means that for any $T_i \models \phi$, in a $\mathbf{DM}$ premise set there is a doxastic formula $b_i\phi$ holding in $\mathcal{L}^B$. A literal φ is *fully supported* by some belief base T if $T \vDash \varphi$. A literal φ is *partially supported* by a belief base T if there is a set of literals Δ such that $\varphi \in \Delta$, $T \vDash \bigvee(\Delta)$, $\nvDash \bigvee(\Delta)$, and there is no $\Delta' \subset \Delta$ such that $T \vDash \bigvee(\Delta')$. As usual $\bigvee(\Delta)$ stands for the disjunction of the members of the set of literals Δ. So, for the previous premise set where $T_1 = \{p \vee q\}$, T_1 partially supports p and q; whereas T_2 fully supports $\neg p$ and T_3 fully supports $\neg q$.

All the logics belonging to the family $\mathbf{ADM}$ are adaptive logics in standard format. This format is extensively discussed in [3]. They all share the same first element needed for their definition, i.e. the lower limit logic ($\mathbf{LLL}$); they all share the same second element in the definition, i.e. the set of abnormal formulas; and they all differ for the last element, i.e. the adaptive strategy which selects the abnormal models holding for a given premise set.

The basis of the adaptive logics of the $\mathbf{ADM}$ family is the so-called lower limit logic $\mathbf{DM}$: this is a multi-modal version of the modal logic $\mathbf{D}$. In addition to all $\mathbf{CL}$-axioms, the logic $\mathbf{DM}$ validates

- Necessitation Rule: if $\vdash_{\mathbf{CL}} \varphi$ then $\vdash_{\mathbf{DM}} b_i\varphi$;
- Distribution: $b_i(\varphi \supset \psi) \supset (b_i\varphi \supset b_i\psi)$;
- Consistency: $b_i\varphi \supset \neg b_i\neg\varphi$.

Semantically, the models of each logic ($\mathbf{ADM}^c$, $\mathbf{ADM}^{c+}$, $\mathbf{ADM}^{c-mm}$) of a given premise set Γ are obtained by making a selection of the $\mathbf{DM}$-models of Γ. This selection will establish the valid models, and the contents of the corresponding consequence sets are the result of the negotiation procedure.

The semantics of the lower limit logic $\mathbf{DM}$ is a standard possible world semantics, with multiple accessibility relations. A $\mathbf{DM}$-model is a quadruple $\langle \mathcal{W}, w_o, \mathcal{R}, v \rangle$ where $\mathcal{W}$ is a set of possible worlds, $w_o \in \mathcal{W}$ is the actual world, $\mathcal{R}$ is a set of serial accessibility relations R_i ($i \in \mathcal{I}$) over $\mathcal{W}$, and $v : \mathcal{P} \times \mathcal{W} \to \{0, 1\}$ is an assignment function.

The valuation function defined by a model M is characterized as follows:

C1 where $A \in \mathcal{P}$, $v_M(A, w) = v(A, w)$;

C2 $v_M(\neg A, w) = 1$ iff $v_M(A, w) = 0$;

C3 $v_M(A \vee B, w) = 1$ iff $v_M(A, w) = 1$ or $v_M(B, w) = 1$;

C4 $v_M(A \wedge B, w) = 1$ iff $v_M(A, w) = 1$ and $v_M(B, w) = 1$;

C5 $v_M(A \supset B, w) = 1$ iff $v_M(A, w) = 0$ or $v_M(B, w) = 1$;

C6 $v_M(b_i\varphi, w) = 1$ iff $v_M(\varphi, w') = 1$ for all w' such that R_iww'.

The standard semantic notions are defined as usual: a model M verifies A iff $v_M(A, w_0) = 1$, $\Gamma \models_{\mathbf{DM}} A$ iff all $\mathbf{DM}$-models of Γ verify A, and $\models_{\mathbf{DM}} A$ iff all $\mathbf{DM}$-models verify A.

In order to establish which contents of the premise set are finally merged, the adaptive machinery formulates all the disagreements that occurr in view of each agent's belief base. These are formalised in terms of a special class of formulas, called *abnormalities*, that are eventually verified by some models of the given premise set in the lower limit logic. In the case of the previously introduced premise set $\Gamma = \{b_1(p \vee q), b_2\neg p, b_3\neg q\}$, and in view of the fact that one tries to merge as much as possible of its content, some of the $\mathbf{DM}$-models of Γ verify the formula $b_3\neg q \supset b_0\neg q$, whereas others falsify it; or, what comes to the same, verify $b_3\neg q \wedge \neg b_0\neg q$. An abnormality is precisely a formula of the form

$$b_i\varphi \wedge \neg b_0\varphi \tag{7}$$

i.e. a formula expressing a (full) support by some agent i for a literal φ which is not merged in view of someone's disagreement. In all $\mathbf{DM}$-models of Γ, at least one instance of such an abnormality is verified. In a simple example, where $\Gamma = \{b_1p, b_2p, b_3\neg p\}$, there will be two types of $\mathbf{DM}$-models: those that verify b_0p and those that verify $\neg b_0p$. Models that verify b_0p, necessarily verify the abnormality $b_3\neg p \wedge \neg b_0p$; those that verify $b_0\neg p$ necessarily verify $b_1p \wedge \neg b_0p$ and $b_2p \wedge \neg b_0p$. The selection tells us which type of models should be chosen.

As one is considering arbitration on bases that express also partial support, among the $\mathbf{DM}$-models of Γ there are models verifying a different kind of abnormalities. As far as an abnormality with respect to T_1 from the previous example is concerned, one has to account for the rejection of a partially supported content. An abnormality involving a base expressing partial support might be due to conflicts arising with respect to each partially supported literal. This is formulated in the following form:

$$b_i(\varphi_1 \vee \ldots \vee \varphi_n) \wedge (\neg b_i\varphi_1 \wedge \ldots \wedge \neg b_i\varphi_n)\wedge \neg b_0(\varphi_1 \vee \ldots \vee \varphi_n) \quad (8)$$

where all φ_i are literals. The union of sets of disagreements for fully and partially supported literals will form our set of abnormalities:

Definition 3.1 [Set of Abnormalities] $\Omega = \{b_i\varphi \wedge \neg b_0\varphi \mid i \in \mathcal{I} \setminus 0, \varphi \in \mathcal{P}^{\pm}\} \cup \{b_i(\varphi_1 \vee \ldots \vee \varphi_n) \wedge (\neg b_i\varphi_1 \wedge \ldots \wedge \neg b_i\varphi_n) \wedge \neg b_0(\varphi_1 \vee \ldots \vee \varphi_n) \mid i \in \mathcal{I} \setminus 0, \varphi_1, \ldots, \varphi_n \in \mathcal{P}^{\pm}, n > 1\}$. ◁

In each adaptive logic obtained by the lower limit logic $\mathbf{DM}$, a disjunction of abnormalities may be $\mathbf{DM}$-derivable without any of its disjuncts being $\mathbf{DM}$-derivable. Consider again $\Gamma = \{b_1(p \vee q),\ b_2\neg p,\ \ b_3\neg q\}$. From this, neither $b_1p \wedge \neg b_0p$ nor $b_2\neg p \wedge \neg b_0\neg p$ is $\mathbf{DM}$-derivable, but the disjunction $(b_1p \wedge \neg b_0p) \vee (b_2\neg p \wedge \neg b_0\neg p)$ is. Disjunctions of abnormalities will be called Dab-formulas, and the abbreviation $Dab(\Delta)$ is used to refer to them:

Definition 3.2 [Disjunctions of Abnormalities] $Dab(\Delta)$ stands for $\bigvee(\Delta)$ where $\Delta \subseteq \Omega$. ◁

If Δ is a singleton, $Dab(\Delta)$ is a single abnormality; if $\Delta = \emptyset$, any disjunction $A \vee Dab(\Delta)$ corresponds to A. A Dab-formula that is $\mathbf{DM}$-derivable from Γ will be called a Dab-consequence of Γ:

Definition 3.3 [Dab-Consequence] $Dab(\Delta)$ is a Dab-consequence of a premise set Γ iff $\Gamma \models_{\mathbf{DM}} Dab(\Delta)$. ◁

If $Dab(\Delta)$ is a Dab-consequence of a set Γ, then so is any $Dab(\Delta')$ such that $\Delta' \supset \Delta$. This is why a further definition is needed:

Definition 3.4 [Minimal Dab-Consequence] A disjunction of abnormalities $Dab(\Delta)$ is a *minimal* Dab-consequence of Γ iff $\Gamma \models_{\mathbf{DM}} Dab(\Delta)$ and there is no $\Delta' \subset \Delta$ such that $\Gamma \models_{\mathbf{DM}} Dab(\Delta')$. ◁

It is in view of the derivability of Dab-formulas from a premise set that the *adaptive strategy* is needed. Intuitively, the adaptive strategy specifies what it means, in the case of disjunctions of abnormalities, that the abnormalities are false *unless and until proven otherwise*. Given the same lower limit logic and the same set of abnormalities, there are different ways to interpret a set of premises *as normally as possible*: the precise interpretation of this ambiguous phrase is determined by the adaptive strategy. In the present case, one will distinguish between the interpretation of a premise set as normally as possible in view of the Variant Counting Strategy in $\mathbf{ADM}^{c+}$, and the interpretation in view of the Minimax Counting Strategy in $\mathbf{ADM}^{c-mm}$.

4. Variant Counting for Arbitration

The selection by variant Counting is applied to the consequence set of the lower limit logic $\mathbf{DM}$ and it gives rise to the adaptive logic $\mathbf{ADM}^{c+}$. It considers the various Dab-consequences of a premise set Γ in view of the number of disagreements involving each agent. This corresponds to a selection of the formulas verified in any given model on the basis of the number of contents held true by each agent and involving a disagreement with another agent.

Consider first all the formulas $A \in Dab(\Delta)$ such that $\Gamma \models_{\mathbf{DM}} Dab(\Delta)$ and the b-operator indexed 1 occurs in A: typically, this will be the set of all the abnormalities derivable from a premise set Γ that are of the form $b_1\phi \wedge \neg b_0\phi$ or of the form $b_1(\phi_1 \vee \ldots \vee \phi_n) \wedge (\neg b_1\phi_1 \wedge \ldots \wedge \neg b_1\phi_n) \wedge \neg b_0(\phi_1 \vee \ldots \vee \phi_n)$. Call this set Ω^1. Then consider the set of all formulas of the same kind occurring with b-operator indexed 2 and call this set Ω^2, and so on up to index n. The set Ω is in turn the union of all the various Ω^i sets:

Definition 4.1 [The set of indexed abnormalities]

$$\Omega = \bigcup_{i=1}^{n} \Omega^i. \quad (9)$$

◁

It is obvious that one can consider now the set of abnormalities with a given index as a proper subset of Ω.

For each model of a given premise set, consider now the abnormal formulas of a certain Ω^i verified by that model:

Definition 4.2 [The abnormal part of a model with index i] $Ab^i(M) = \{A \mid A \in \Omega^i \text{ and } M \models A\}$. ◁

For any model M_j of a given premise set, let $\mathcal{C}^i_{M_j} = |Ab^i(M_j)|$ denote the cardinality of its abnormal part with respect to Ω^i:

Definition 4.3 [Abnormal cardinality of a model] Given a model M_j of a premise set Γ and its abnormal part $Ab^i(M_j)$, its abnormal cardinality $\mathcal{C}^i_{M_j}$ is the number of abnormal formulas $A \in \Omega^i$ verified in the model M_j. ◁

The abnormal cardinality $\mathcal{C}^i_{M_j}$ expresses the number of disagreements that agent i faces with respect to the literals verified by the model M_j. For each model M, we construct the list $(\mathcal{C}^1_M, \ldots, \mathcal{C}^n_M)$, where n is the number of elements of $\mathcal{I}$. Let L^M_Γ be the list obtained by $(\mathcal{C}^1_M, \ldots, \mathcal{C}^n_M)$ by sorting its elements in descending order. Let now $\leq_{lex}$ be the lexicographic order between sequences of integers of the same length. On the basis of the ordering $\leq_{lex}$, a total preorder $\leq^{\mathcal{C}}_{\Gamma}$ holds among the various models $M_1, \ldots, M_n$ of Γ in the following way:

Definition 4.4 [Preorder by Minimal Abnormal Cardinality] A total preorder $\leq^{\mathcal{C}}_{\Gamma}$ holds between models of a premise set Γ according to the following definition

$$M_i \leq^{\mathcal{C}}_{\Gamma} M_j \text{ iff } L^{M_i}_{\Gamma} \leq_{lex} L^{M_j}_{\Gamma}. \tag{10}$$

◁

According to this definition, the pre-order on the models of a premiset set Γ is obtained by ordering models according to their abnormal cardinalities. Where M_{Γ} stands for the set of $\mathbf{DM}$-models of Γ, the Variant Counting strategy $\triangle^{c+}(\Gamma)$ will select among those models the minimal ones with respect to the ordering obtained by $\leq^{\mathcal{C}}_{\Gamma}$:

Definition 4.5 [Selection of Models by $\mathbf{ADM}^{c+}$]

$$Mod(\triangle^{c+}(\Gamma)) = Min(\mathsf{M}_{\Gamma}, \leq^{\mathcal{C}}_{\Gamma}). \tag{11}$$

◁

The definition of the merging operator $\triangle^{c+}$ reflects a selection of abnormal models of the premise set that corresponds to the models satisfying the median possible choices that are preferred. In terms of the fair syncretic assignment presented in [13], the arbitration protocol satisfies the following conditions: the models of the premise set are the more plausible interpretations for the pre-order associated to that set; two equivalent knowledge sets have the same associated pre-orders. In the case of the adaptive selection this means that the abnormal models of a premise set selected by $\triangle^{c+}$ are those with lower position in the pre-order given by abnormal cardinalities and that two equivalent premise sets have the same pre-orders of abnormal cardinalities for their models.

The main condition of arbitration as fair syncretic assignment is satisfied as follows: if the ordering of abnormal cardinalities of M_i for base T_1 is lower than that of M_j for the same base (i.e. $M_i <^{\mathcal{C}}_{T_1} M_j$) and the same holds for M_i with respect to M_k for T_2 (i.e. $M_i <^{\mathcal{C}}_{T_2} M_k$), and if M_j and M_k are equally abnormal for $T_1 \sqcup T_2$ (i.e. $M_j \simeq^{\mathcal{C}}_{T_1 \sqcup T_2} M_k$), then M_i is less abnormal than M_j and M_k for $T_1 \sqcup T_2$ (i.e. $M_i <^{\mathcal{C}}_{T_1 \sqcup T_2} M_{j,k}$). Correspondingly, the following principle is formulated:

Definition 4.6 [Arbitration by Ordering on Abnormal Cardinalities] If for models M_i, M_j, M_k holds that $|Ab^i(M_i)| < |Ab^i(M_j)|$ and $|Ab^i(M_i)| < |Ab^i(M_k)|$; and if $|Ab^i(M_j)| = |Ab^i(M_k)|$; then $(M_i) <^{\mathcal{C}}_{\Gamma} M_{j,k}$ and $M(\triangle^{c+}(\Gamma)) = M_i$.

$$\left.\begin{array}{r} |Ab^i(M_i)| <_{T_1 \in \Gamma} |Ab^i(M_j)| \\ |Ab^i(M_i)| <_{T_2 \in \Gamma} |Ab^i(M_k)| \\ |Ab^i(M_j)| \simeq_{T_1 \sqcup T_2 \in \Gamma} |Ab^i(M_k)| \end{array}\right\} \Rightarrow M_i <^{\mathcal{C}}_{\Gamma} M_{j,k}.$$

◁

5. Minimax Counting for Quasi-Merging

The pseudo-arbitration operator from [15] has the main property of being constrained to only two bases and to require consistency to be obtained without the principle of average on bases to be preserved. This means that the negotiation procedure is performed among the belief bases rather than among the propositional letters having different truth values. If the operator is applied to two bases that support only respectively inconsistent literals, it will provide their disjunction without taking into account any combination of consistent contents. The $\triangle^{Max}$ operator from [12] is meant to model the very same procedure of arbitration, without the restriction imposed on the number of belief bases involved in the negotiation process. This operator is a less fine-grained one than the $\triangle^{GMax}$, because it provides a larger spectrum of possible results, and therefore it is called a quasi-merging operator.

In the present section a new adaptive semantic selection for $\mathbf{ADM}$ is introduced: it is called *Minimax Counting*, it gives rise to the adaptive logic $\mathbf{ADM}^{c-mm}$ and it aims at providing the same kind of negotiation process that is reflected by the result of the $\triangle^{Max}$ operator. The resulting $\triangle^{c-mm}$ operator for the Minimax Adaptive Counting applies the minimax rule to the selection of $\mathbf{DM}$-models of a premise set in view of their abnormal cardinality. The *Minimax Counting* selection presents an important difference with the standard $\triangle^{Max}$ operator: the latter, as any IC merging operator, does not satisfy the Majority Independence postulate (see [12], p.779). This postulate states that the result of merging is fully independent of the popularity of the views and it simply takes into account each different view:

$$\forall n \triangle_{\mu} (\Gamma \sqcup \Gamma'_1, \ldots, \Gamma'_n) \leftrightarrow \triangle_{\mu}(\Gamma \sqcup \Gamma'). \tag{12}$$

From this follows that the $\triangle^{Max}$ operator does not satisfy the **IC6** postulate, which togheter with its counterpart the **IC5** postulate allows the merging to satisfy always the alternatives for which there is no disagreement (consistency). The selection performed according to $\triangle^{Max}$ provides therefore a range of alternatives that contains also some of the choices for which none of the agents has expressed explicit preference. On the other hand, $\triangle^{c-mm}$ is based on the formulation of disagreements and their selection: anything which is not involved in any disagreement is obviously merged. This restricts slightly the range of results offered by the previous pseudo-arbitration operator (because it avoids some counter-intuitive results), but it still provides a larger spectrum of possibilities than the $\triangle^{c+}$ operator by using the minimax function.

The selection still makes use of the notion of abnormal cardinality $\mathcal{C}^i_{M_j}$ as given in Definition 4.3; it moreover refers for each model M to the list L^M_Γ obtained by ordering in descending order the list $(\mathcal{C}^1_M, \ldots, \mathcal{C}^n_M)$, where n is the number of elements of $\mathcal{I}$. A new maximal distance Max is defined as the first element in each list L^M_Γ for each model M:

Definition 5.1 [Maximal Abnormal Distance] $Max(M, \Gamma) = \mathcal{C}^i_M$ and there is no index k such that $|Ab^k(M)| > |Ab^i(M)|$. ◁

i.e. the highest of the values $\mathcal{C}^i$ for each model M and the first element of each L^M_Γ list. The Maximal Abnormal Distance expresses the highest number of disagreements possible in each abnormal model for any given agent. On its basis one derives a new total pre-order for the abnormal models in the following way:

Definition 5.2 [Preorder by Maximal Abnormal Distance] A total preorder $\leq^{Max}_\Gamma$ holds between models of a premise set Γ according to the following definition

$$M_i \leq^{Max}_\Gamma M_j \text{ iff } Max(M_i, \Gamma) \leq Max(M_j, \Gamma). \quad (13)$$

◁

Where M_Γ stands for the set of $\mathbf{DM}$-models of Γ, the Minimax Counting strategy of $\mathbf{ADM}^{c-mm}$ will select the minimal models with respect to the ordering obtained by $\leq^{Max}_\Gamma$:

Definition 5.3 [Selection of Models by $\mathbf{ADM}^{c-mm}$]

$$Mod(\triangle^{c-mm}(\Gamma)) = Min(\mathsf{M}_\Gamma, \leq^{Max}_\Gamma). \quad (14)$$

◁

The result of this selection is therefore obtained by restricting the $\mathbf{DM}$-models to their highest abnormal part and then selecting those that verify the minimal number of disagreements. In this way the result expresses a negotiation procedure that accounts for all the possible consistent combinations of contents, in view of full agreements and constraints.

6. An Example

The application of the various IC operators is shown in [12] in terms of an example which will now be considered for the operators $\triangle^{c+}$ and $\triangle^{c-mm}$. The formulation of the example is the following:

> At a meeting of a block of flat co-owners, the chairman proposes for the coming year the construction of a swimming pool, of a tennis court and a private car park. But if two of these three items are built, the rent will increase significantly ([12], p.787).

In the following, the letters p, q, r stand respectively for the construction of the swimming pool, the tennis court and the private car park. The rent increase will be denoted by s, which is implied by each conjunction of two out of the three items: $\mu = ((p \wedge q) \vee (p \wedge r) \vee (q \wedge r)) \rightarrow s$. The set of chioces of the co-owners is represented by $\Gamma = \{T_1 \sqcup T_2 \sqcup T_3 \sqcup T_4\}$. The first two of the co-owners want to build the three items and do not care about the rent (i.e. $(s \vee \neg s)$ holds in T_1 and T_2); the third does not want the rent increase nor anything built; the fourth wants the last two items (i.e. $(p \vee \neg p)$ holds in T_4), though he does not want the rent to increase:

$$\begin{aligned} T_1 &= \{p \wedge q \wedge r\} \\ T_2 &= \{p \wedge q \wedge r\} \\ T_3 &= \{\neg p \wedge \neg q \wedge \neg r \wedge \neg s\} \\ T_4 &= \{q \wedge r \wedge \neg s\}. \end{aligned}$$

Our premise set in $\mathbf{DM}$ is of the form $\Gamma = \{b_1(p \wedge q \wedge r), b_2(p \wedge q \wedge r), b_3(\neg p \wedge \neg q \wedge \neg r \wedge \neg s), b_4(q \wedge r \wedge \neg s)\}$. The adaptive procedure requires in the first instance the formulation of the disagreements in terms of Dab-consequences of Γ:

$$\begin{aligned} Dab(\Delta_1) &= (b_1 p \wedge \neg b_0 p) \vee (b_3 \neg p \wedge \neg b_0 \neg p) \\ Dab(\Delta_2) &= (b_1 q \wedge \neg b_0 q) \vee (b_3 \neg q \wedge \neg b_0 \neg q) \\ Dab(\Delta_3) &= (b_1 r \wedge \neg b_0 r) \vee (b_3 \neg r \wedge \neg b_0 \neg r) \\ Dab(\Delta_4) &= (b_2 p \wedge \neg b_0 p) \vee (b_3 \neg p \wedge \neg b_0 \neg p) \\ Dab(\Delta_5) &= (b_2 q \wedge \neg b_0 q) \vee (b_3 \neg q \wedge \neg b_0 \neg q) \\ Dab(\Delta_6) &= (b_2 r \wedge \neg b_0 r) \vee (b_3 \neg r \wedge \neg b_0 \neg r) \\ Dab(\Delta_7) &= (b_4 q \wedge \neg b_0 q) \vee (b_3 \neg q \wedge \neg b_0 \neg q) \\ Dab(\Delta_8) &= (b_4 r \wedge \neg b_0 r) \vee (b_3 \neg r \wedge \neg b_0 \neg r) \end{aligned}$$

These provide the following Ω^i sets of indexed abnormalities (where $!b_i\varphi$ will abbreviate $b_i\varphi \wedge \neg b_0\varphi$ provided $\varphi \in \mathcal{P}^\pm$ and $!b_i(\varphi_1 \vee \ldots \vee \varphi_n)$ will abbreviate $b_i(\varphi_1 \vee \ldots \vee \varphi_n) \wedge (\neg b_i\varphi_1 \wedge \ldots \wedge \neg b_i\varphi_n) \wedge \neg b_0(\varphi_1 \vee \ldots \vee \varphi_n)$ provided each $\varphi_i \in \mathcal{P}^\pm$ and $n > 1$):

$$\begin{aligned} \Omega^1 &= \{!b_1 p, !b_1 q, !b_1 r\} \\ \Omega^2 &= \{!b_2 p, !b_2 q, !b_2 r\} \\ \Omega^3 &= \{!b_3 \neg p, !b_3 \neg q, !b_3 \neg r\} \\ \Omega^4 &= \{!b_4 q, !b_4 r\}. \end{aligned}$$

Let us now consider our models, with respect to which abnormal cardinalities shall be calculated:

$$\begin{aligned}
M_1 &= b_0p, b_0q, b_0r, b_0s \\
M_2 &= b_0p, b_0q, b_0r, b_0\neg s \\
M_3 &= b_0p, b_0q, b_0\neg r, b_0s \\
M_4 &= b_0p, b_0q, b_0\neg r, b_0\neg s \\
M_5 &= b_0p, b_0\neg q, b_0r, b_0s \\
M_6 &= b_0p, b_0\neg q, b_0r, b_0\neg s \\
M_7 &= b_0p, b_0\neg q, b_0\neg r, b_0s \\
M_8 &= b_0p, b_0\neg q, b_0\neg r, b_0\neg s \\
M_9 &= b_0\neg p, b_0q, b_0r, b_0s \\
M_{10} &= b_0\neg p, b_0q, b_0r, b_0\neg s \\
M_{11} &= b_0\neg p, b_0q, b_0\neg r, b_0s \\
M_{12} &= b_0\neg p, b_0q, b_0\neg r, b_0\neg s \\
M_{13} &= b_0\neg p, b_0\neg q, b_0r, b_0s \\
M_{14} &= b_0\neg p, b_0\neg q, b_0r, b_0\neg s \\
M_{15} &= b_0\neg p, b_0\neg q, b_0\neg r, b_0s \\
M_{16} &= b_0\neg p, b_0\neg q, b_0\neg r, b_0\neg s
\end{aligned}$$

In view of μ the models M_2, M_4, M_6, M_{10} are rejected, i.e. any model satisfying $((p \wedge q) \vee (p \wedge r) \vee (q \wedge r)) \wedge \neg s$ is ignored. The initial assumption that $s \vee \neg s$ holds for T_1 and T_2, i.e. that though these agents express preference for the construction of all the three items, they still would approve if the three items might be built without increasing the rent ($\neg s$), means that with respect to $\neg s$ there is no disagreement, and none can be explicitely formulated within Γ. This in turn means that $\Gamma \models_{\mathbf{DM}} b_0\neg s$ holds, and the result of the selection shall be consistent with it. Hence, from the previous list all the models that still verify b_0s shall be removed as well. This leaves the following list:

$$\begin{aligned}
M_8 &= b_0p, b_0\neg q, b_0\neg r, b_0\neg s \\
M_{12} &= b_0\neg p, b_0q, b_0\neg r, b_0\neg s \\
M_{14} &= b_0\neg p, b_0\neg q, b_0r, b_0\neg s \\
M_{16} &= b_0\neg p, b_0\neg q, b_0\neg r, b_0\neg s.
\end{aligned}$$

6.1. Arbitration

For each of the remaining models one calculates the abnormal cardinality with respect to the indexed sets of abnormalities. For each model M_j and any indexed set of abnormalities Ω^i there will be a value to $\mathcal{C}^i_{M_j}$. These values are listed in the following table, where at the intersection of each M_j and Ω^i one has the value of $\mathcal{C}^i_{M_j}$, and in the last column each list $L^{M_j}_{\Gamma}$ obtained by sorting the elements of $(\mathcal{C}^1_{M_j}, \ldots, \mathcal{C}^n_{M_j})$ in descending oder:

	Ω^1	Ω^2	Ω^3	Ω^4	$L^{M_j}_{\Gamma}$
M_8	2	2	1	2	$(2,2,2,1)$
M_{12}	2	2	1	1	$(2,2,1,1)$
M_{14}	2	2	1	1	$(2,2,1,1)$
M_{16}	3	3	0	2	$(3,3,2,0)$

The lexicographic order $\leq^{\mathcal{C}}_{\Gamma}$ among the sequences of each $L^{M_j}_{\Gamma}$ gives the total preorder among the various models:

$$M_{12,14} \leq^{\mathcal{C}}_{\Gamma} M_8 \leq^{\mathcal{C}}_{\Gamma} M_{16}. \qquad (15)$$

The result of merging according to $Min(\mathsf{M}_{\Gamma}, \leq^{\mathcal{C}}_{\Gamma})$ is:

$$\triangle^{c+}_{\mu}(\Gamma) = b_0((\neg p \wedge q \wedge \neg r \wedge \neg s) \vee (\neg p \wedge \neg q \wedge r \wedge \neg s)). \qquad (16)$$

The preferred choice by the group of co-owners is therefore to build either the tennis court or the private car park without increasing the rent. This is also the result of the pure arbitration $\triangle^{GMax}$ operator from [12].

6.2. Quasi-merging

By the same example it will be shown now how the $\triangle^{c-mm}$ operator for Minimax Adaptive Counting works. From the very same premise set $\Gamma = \{b_1(p \wedge q \wedge r), b_2(p \wedge q \wedge r), b_3(\neg p \wedge \neg q \wedge \neg r \wedge \neg s), b_4(q \wedge r \wedge \neg s)\}$, the same derivable disjunctions of abnormalities and list of Ω^i sets, one derives the same list of values for abnormal cardinalities in each of the possible models, and the same lexicographic order of these values.

The models that allow the combination $b_0((p \wedge q) \vee (p \wedge r) \vee (q \wedge r) \wedge s)$ are obviously still rejected in view of the constraint μ; and it still holds in the merging state $b_0\neg s$ in view of the absence of disagreements with respect to this literal. The rejection of any other model in which s holds – which leaves only models $M_8, M_{12}, M_{14}, M_{16}$ – is of the greatest importance in order to show the result of our minimax selection.

By the original $\triangle^{Max}$ operator from [12], one cannot avoid that some of the models are selected in which at least two between p, q, r are negated (i.e. only one of the item is allowed to be built by the group of co-owners), and nonetheless s is satisfied (i.e. the rent is increased). This result is counterintuitive in view of the required constraint, but it is also undesirable in view of intelligent interaction by our agents. Our $\triangle^{c-mm}$ operator avoids this undesirable result.

According to Definition 5.1, one selects the Maximal Abnormal Distance for each of the allowed models out of the lexicographic order of abnormal cardinalities:

	Ω^1	Ω^2	Ω^3	Ω^4	$Max(M_j, \Gamma)$
M_8	2	2	1	2	2
M_{12}	2	2	1	1	2
M_{14}	2	2	1	1	2
M_{16}	3	3	0	2	3

from which the following preorder based on $\leq^{Max}_{\Gamma}$ is obtained:

$$M_{8,12,14} \leq^{Max}_{\Gamma} M_{16}. \qquad (17)$$

The selection of models M_8, M_{12}, M_{14} with the minimal values provides the following alternatives:

$$\triangle_{\mu}^{c-mm}(\Gamma) = b_0((p \wedge \neg q \wedge \neg r \wedge \neg s) \vee (\neg p \wedge q \wedge \neg r \wedge s) \vee (\neg p \wedge \neg q \wedge r \wedge \neg s)). \quad (18)$$

The preferred choice by the group of co-owners is therefore to build one of the three items without increasing the rent. This results avoids the other alternatives allowed by the $\triangle^{Max}$ operator according to which one among the tennis court or the private park is built and the rent is increased (the latter condition being not necessary in view of the constraints).

7. Conclusion

The formulation of the family of adaptive logics $\mathbf{ADM}$, started with the definition of a Majority merging selection in [20], has been in this paper further developed by the definition of selection procedures corresponding to pure-arbitration and quasi-merging protocols. A next obvious step of this research is represented by the formulation of a selection procedure for $\mathbf{ADM}$ that reflects the $\triangle^n$ operators from [13], a set of operators that belong simultaneously to the two main sub-families, majority and arbitration.

A number of application contexts, such as those presented in [19] for heterogenous databases and in [18] for judgment aggregation procedures, provide the settings for testing the computational limits and effectiveness of the procedures. With respect to these open questions, a number of positive and negative results can be formulated, in line with those valid for other general merging protocols.

References

[1] C. Alchourrón, P. Gärdernfors, and D. Makinson. On the logic of theory change: partial meet contraction and revision functions. *Journal of Symbolic Logic,*, 50:510–530, 1985.

[2] D. Batens. A general characterization of adaptive logics. *Logique & Analyse*, 173–175:45–68, 2001.

[3] D. Batens. A universal logic approach to adaptive logics. *Logica universalis*, 1:221–242, 2007.

[4] S. Benferhat, D. Dubois, and H. Prade. Some syntactic approaches to the handling of inconsistent knowledge bases: A comparative study. Part 1: The flat case. *Studia Logica*, 58:17–45, 1997.

[5] A. Borgida and T. Imielinski. Decision making in committees: a framework for dealing with inconsistency and nonmonotonicity. In *Proceedings Workshop on Nonmonotonic Reasoning*, pages 21–32, 1984.

[6] L. Cholvy. *Reasoning about merged information*, volume 3, pages 233–263. Kluwer Academic Publisher, 1998.

[7] L. Cholvy and T. Hunter. Fusion in logic: a brief overview. In *4th European Conference on Symbolic and Quantitative Approaches to reasoning with Uncertainty*, volume 1244 of *Lectures Notes in Computer Science*, pages 86–95, 1997.

[8] M. Dalal. Investigations into theory of knowledge base revision. In *Proc. AAAI-88*, pages 449–479. St. Paul, MN, 1988.

[9] R. Fagin, J. Ullman, and M. Vardi. On the semantics of updates in databases. In *Second ACM SIGACT-SIGMOD*, pages 352–365, 1983.

[10] E. Grégoire and S. Konieczny. Logic-based approaches to information fusion. *Information Fusion*, 7:4–18, 2006.

[11] S. Konieczny and R. Pino-Pérez. On the logic of merging. In *Proceedings of the Sixth International Conference on Principles of Knowledge Representation and Reasoning (KR'98)*, pages 488–498, 1998.

[12] S. Konieczny and R. Pino-Pérez. Merging information under constraints: A logical framework. *Journal of Logic and Computation*, 12(5):773–808, 2002.

[13] S. Konieczny and R. Pino-Pérez. On the frontier between arbitration and majority. In *Proceedings of the* 8^{th} *International Conference on Principles of Knowledge Representation and Reasoning*, pages 109–120, 2002.

[14] P. Liberatore and M. Schaerf. Arbitration: a commutative operator for belief revision. In *Proceedings of the Second World Conference on the Fundamentals of Articial Intelligence*, pages 217–228, 1995.

[15] P. Liberatore and M. Schaerf. Arbitration (or how to merge knowledge bases). *IEEE Transactions on Knowledge and Data Engineering*, pages 76–90, 1998.

[16] J. Lin and A. Mendelzon. Knowledge base merging by majority. In *Dynamic Worlds: from the Frame Problem to Knowledge Management*. Kluwer, 1999.

[17] G. Primiero. Belief merging based on adaptive interaction. In J. van Benthem, S. Ju, and F. Veltman, editors, *A meeting of the minds - Proceedings of the Workshop on Logic, rationality and Interaction*, volume 8 of *Texts in Computer Science*, pages 313–320. College Publications, 2007.

[18] G. Primiero. Aggregating collective judgements by selecting disagreements. Accepted paper at LOFT08, 8^{th} Conference on Logic and the Foundations of Game and Decision Theory, Amsterdam 3-5 July 2008, 2008.

[19] G. Primiero and J. Meheus. Adaptive arbitration by variant counting on commutative bases with weights. In *Proceedings of the* $11th$ *International Conference on Information Fusion*, 2008. forthcoming.

[20] G. Primiero and J. Meheus. Majority merging by adaptive counting. *Knowledge, Rationality and Action (Synthese)*, forthcoming.

[21] P. Revesz. On the semantics of arbitration. *Journal of Algebra and Computation*, 7(2):133–160, 1997.

[22] K. Satoh. Nonmonotonic reasoning by minimal belief revision. In *Proceedings of the International Conference on Fifth Generation Computer Systems*, pages 455–462, 1988.

[23] L. Zadeh. Fuzzy sets as a basis for a theory of possibility. *Fuzzy Sets and Systems*, 1(1):3–28, 1978.