Solvability of the Halting and Reachability Problem for Tag systems with $\mu = v = 2$

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Abstract

In this report we will provide a detailed proof of the solvability of the halting and reachability problem for 2-symbolic tag systems with a shiftnumber $v = 2$.

1 Introduction

> **PROBLEM**. Choose any two-symbol, two-state machine and show that it is *not* universal. Hint: Show that its halting problem is decidable by describing a procedure that decides whether or not it will stop on any given tape. D. G. Bobrow and the author did this for all (2,2) machines [1961, unpublished] by a tedious reduction to thirty-odd cases (unpublishable).
>
> Marvin Minsky, 1967.[1]

A tag system T, consists of a finite alphabet $\Sigma = \{a_0, a_1, ..., a_{\mu-1}\}$ of μ symbols, a shift number $v \in \mathbb{N}$ and a finite set of μ words, $w_0, w_1, ..., w_{\mu-1}$ defined over the alphabet, including the empty word ϵ. Each of these words corresponds with one of the letters from the alphabet as follows:

$$\begin{array}{lll} a_0 & \rightarrow & a_{0,1}a_{0,2}...a_{0,n_0} \\ a_1 & \rightarrow & a_{1,1}a_{1,2}...a_{1,n_1} \\ ... & ... & ... \\ a_{\mu-1} & \rightarrow & a_{\mu-1,1}a_{\mu-1,2}...a_{\mu-1,n_{\mu-1}} \end{array}$$

where each $a_{i,j} \in \Sigma, 0 \leq i < \mu$. Given an initial string A_0, the tag system tags the word associated with the leftmost letter of A_0 at the end of A_0,

[1] [5], p. 281

and deletes its first v symbols. This process is iterated until the tag system halts, i.e. produces the empty string ϵ. If this does not happen the tag system can become periodic or show divergent behaviour.
To give an example, consider the tag system T with $v = 3$, $0 \rightarrow 00$, $1 \rightarrow 1101$. This tag system was mentioned as an example by Post [7], [8]. Suppose $A_0 = 11100110100$. We then get:

11100110100
001101001101
10100110100
001101001101

After 4 iterations, the string A_4 is produced, which repeats itself every two iterations. Despite the formal simplicity of this example, it is still not known whether this particular tag system is recursively solvable. Some researchers have studied this it [4], [3], [5], [10], [12] but without any definite results.

2 Proof of the solvability of the halting and reachability problem for tag systems with $\mu = v = 2$.

In [8] Post remarks that he has proven the solvability of the halting and reachability problem of the class of tag systems $\mu = v = 2$, a proof which, according to Post, involved "*considerable labor*". As will become clear from the proof we will give here, this is indeed true. The proof involves a study of a rather large number of subcases.
Post differentiates between three different classes of behaviour a tag system can converge to, i.e., a tag system can halt, it can become periodic, or it can show unbounded growth. The reachability and halting problem can be proven solvable, if one can determine for any initial condition, for a given tag system, that it will lead to one of these three classes of behaviour after a finite number of steps. In case of unbounded growth, one should be able to prove that for any given number n the tag system will always produce a string A_i of length $l_{A_i} > n$ after a finite number of iterations i, such that no string $A_j, j > i$, will ever be produced again for which $l_{A_j} \leq n$.
In our proof, we have indeed been able to show that one can determine for any tag system T from the class TS(2, 2) and any initial condition over the alphabet $\Sigma = \{0, 1\}$, that T will always become periodic, halt or show unbounded growth after a finite number of steps. We have thus been able to prove the following theorem:

Theorem 1 *For any given tag system T, if $\mu = v = 2$ then the halting problem and the reachability problem for T are solvable.*

First of all, it should be noted that we only have to consider those cases with $l_{\mathbf{min}} < 2, l_{\mathbf{max}} > 2$, due to a decidability criterium proven by Wang [11] which states that any tag system with $l_{\mathbf{min}} \geq v$ or $l_{\mathbf{max}} < v$, has a solvable halting and reachability problem. In the remainder, we assume that $l_{\mathbf{max}} = l_{w_1}, l_{\mathbf{min}} = l_{w_0}$, the symmetrical case of course being equivalent to this case.[2]

There are three global cases to be taken into account, i.e., $w_0 = \epsilon, w_0 = 1, w_0 = 0$. Each of these cases is subdivided into several subcases, determined by the following parameters: the parity of w_1,[3] the length l_{w_1} of w_1 and the total number of 1's in w_0 and w_1 (indicated as #1). It should be noted that, contrary to classes of Turing machines TM(m, n), the three global cases to be taken into account contain an infinite number of tag systems. In this sense it has been basic for this proof that it is possible to determine certain threshold values for the last two of these parameters, i.e., l_{w_1} and #1. If the values of these parameters are larger than a given number the infinite class of tag systems determined by the parameters will always show unbounded growth except for a specific class of initial strings. If these values are smaller or equal to these parameters, the tag systems will always halt or become periodic, except for a determined class of initial conditions.

There is one specific method that has been basic to solve the majority of cases to be considered, which has been called the *table method* elsewhere [6]. What one basically does with this method is to look at a certain number of substrings that can be produced theoretically in a given tag system, by starting from the possible productions from the respective words $w_0, ..., w_{\mu-1}$. Given a tag system T with a shift number v, it is clear that given a word $w_i = a_{i,1}a_{i,2}...a_{i,l_{w_i}}$, some letters in w_i will be 'scanned', others not. The sequence of letters that is scanned is determined by the number n, $0 \leq n \leq v-1$, of leading letters of w_i that is erased but not scanned by the tag system and which leads to the concatenation or tagging of the words corresponding to the letters from the sequence at the tail of a given string. For example, if $v = 3$, there are three different sequences of letters in w_i that might be scanned by the tag system: $a_{i,1}a_{i,4}...a_{i,t_0}$, $a_{i,2}a_{i,5}...a_{i,t_1}$, $a_{i,3}a_{i,6}...a_{i,t_2}$, with:

$$t_j = l_{w_i} - [(l_{w_i} - j) \bmod 3]$$

[2] $l_{\mathbf{min}}$ is the length of the shortest word corresponding to a given symbol for a given tag system, $l_{\mathbf{min}}$ is the length of the longest word of a given tag system.

[3] The parity of a number x is the property of being even or odd.

Now, given a tag system T, with shift number v and μ letters. The table method is applied to the tag system by first looking at all the possible strings v that can be produced from each of the words w_i, $0 \leq i < \mu$, by concatenating the words corresponding to the letters of each of the different sequences in each of the w_i, determined as above. If one of these new strings produced is equal to one of the words w_i it is marked. If all the strings produced in this way are marked or equal to ϵ it follows that the tag system will always halt or become periodic, since the length of the strings that can be produced from the respective words is bounded. If this is not the case, the same procedure is applied to all the strings left unmarked and not equal to ϵ,... If we, for instance, apply this method to the two words 00 and 1101 of the tag system mentioned above (Sec. 1), we get the following strings: 00, 00, ϵ, 11011101, 1101, 00. As is clear only one (11011101) of the 6 possible strings produced will be left unmarked, and differs from ϵ. If we then apply the method to this one string, and then again to the 3 unmarked strings that can be produced from 11011101, it becomes clear very soon that the method will never come to a halt, i.e., there will always remain strings left unmarked.

The method is called the table method, because the results from the method can best be represented through tables. We will explain how such a table should be read, in the first application of the method in the proof.

As will become clear in the proof, the table method is not only useful if, for a given tag system, all the strings become marked or are equal to ϵ at a given time, but can also be used to e.g. prove that a tag system will either halt or show unbounded growth. In general, it should be noted here that, although this method is very simple, it is an important instrument to study tag systems.

In should be noted that from now on, $\dot{x}$ denotes that x is odd, similarly, a non-dotted number x denotes an even number. Furthermore l_{w_0} and l_{w_1} are abbreviated as l_0 rsp. l_1. In our proof, we will separate the three global cases, which are in their turn to be subdivided into the respective subcases. For each of the cases we will only provide detailed proofs when the method used is considerably different from the ones already used.

2.0.1 Case 1. $w_0 = \epsilon$

Case 1.1. $\#1 = 0$. Irrespective of the length of w_1 it is trivial to prove that tag systems from this class will always halt, since only 0's can be scanned.

Case 1.2. $\#1 = 1$, $l_1 \equiv 0 \bmod 2$. Let $w_1 = 0^{\dot{x}_1}10^{y_1}$. The following table proves the case:

	w_1
S_0	HALT
S_1	w_1✓

The row headed with S_0 (shift 0) gives the string produced from a given string S (in this case w_0 or w_1) when the first letter of the string S is scanned by the tag system. Similarly, the row headed S_1 (shift 1) gives the possible productions from a given string S when its first letter is erased without being scanned.
As is clear from the table, a tag system from this class will either halt or become periodic. It will always become periodic when at least one 1 is scanned in the initial condition, such that the first letter in w_1 resulting from this 1 will not be scanned. This can be determined through the parity of the length of the initial condition. In all other cases, tag systems from this class always halt. A similar proof can be given for the case $w_1 = 0^{x_1}10^{\dot{y}_1}$.

Case 1.3. $\#1 = 1$, $l_1 \equiv 1 \bmod 2$ The table that can be constructed for this class of tag systems, is identical to the previous table, with $w_1 = 0^{\dot{x}_1}10^{\dot{y}_1}$. Despite the table being identical, tag systems from this class can be proven to always halt. Given an arbitrary number n of 1's scanned in the initial condition, which are separated by a certain number of 0's, such that all w_1's produced from these 1's will be entered with a shift 1. After all the letters of the initial condition have been scanned, the following string is produced:

$$\underbrace{0^{x_2}10^{\dot{y}_1}0^{\dot{x}_1}10^{\dot{y}_1}....0^{\dot{x}_1}10^{\dot{y}_1}}_{n} \tag{1}$$

where $x_2 = x_1$ or $x_2 = x_1 - 1$. From (1) the tag system will then produce the following string:

$$\underbrace{0^{x'_2}10^{\dot{y}_1}0^{\dot{x}_1}10^{\dot{y}_1}....0^{\dot{x}_1}10^{\dot{y}_1}}_{\lfloor \frac{n}{2} \rfloor} \tag{2}$$

Clearly, whatever the number of 1's in the initial condition might be, they will more or less be reduced by a factor 1/2, for each application of $\stackrel{\circ}{\rightarrow}$, thus ultimately leading to the production of ϵ. This is the case, because for every pair of w_1's one of them will be erased due to the fact that $\dot{y}_1 + \dot{x}_1$

is always even. It thus follows that tag systems from this class will always halt, whatever the initial condition might be.

Case 1.4. **$\#1 = 2$, $l_1 \equiv 0 \bmod 2$.** To prove the case we have to differentiate between two subcases, i.e. the case with $w_1 = 0^{x_1}10^{y_1}10^{z_1}$ and $w_1 = 0^{\dot{x}_1}10^{\dot{y}_1}10^{z_1}$ (the proof for the case with $w_1 = 0^{\dot{x}_1}10^{y_1}10^{\dot{z}_1}$ is similar to the first case, the proof with $w_1 = 0^{x_1}10^{\dot{y}_1}10^{\dot{z}_1}$ is similar to the second case).

Subcase 1.4.1. **$w_1 = 0^{x_1}10^{y_1}10^{z_1}$.** The first case is proven through the following table:

Table 2: $w_1 = 0^{x_1}10^{y_1}10^{z_1}$

	w_1
S_0	w_1✓
S_1	w_1✓

From this proof it follows that any tag system from this class of cases will always become periodic, except when no 1 is scanned in the initial condition, then it always halts.

Subcase 1.4.2. **$w_1 = 0^{\dot{x}_1}10^{\dot{y}_1}10^{z_1}$.** The proof of the solvability of the second case follows from the following table:

Table 3: $w_1 = 0^{\dot{x}_1}10^{\dot{y}_1}10^{z_1}$

	w_1	w_1w_1	...	$(w_1w_1)^{n-1}$
S_0	HALT	HALT	...	HALT
S_1	w_1w_1	$w_1w_1w_1w_1$	....	$(w_1w_1)^n$

Tag systems from this class will either halt or show unbounded growth depending on the parity of the length of the initial condition.

Case 1.5. **$\#1 = 2$, $l_1 \equiv 1 \bmod 2$.** The tables for the proof are almost identical to those for case 1.4., except that now we have to consider the cases

$w_1 = 0^{x_1}10^{\dot{y}_1}10^{z_1}$ (or similarly $w_1 = 0^{\dot{x}_1}10^{\dot{y}_1}10^{\dot{z}_1}$) and $w_1 = 0^{\dot{x}_1}10^{y_1}10^{z_1}$ (or similarly $w_1 = 0^{x_1}10^{y_1}10^{\dot{z}_1}$). Contrary to case 1.4.2., the tag system will always become periodic when $w_1 = 0^{x_1}10^{\dot{y}_1}10^{z_1}$, if at least two 1's are scanned in the initial condition. This is the case, because, for each two consecutive w_1's, the tag system produces two consecutive w_1's, since $z_1 + x_1$ is even. If only one 1 is scanned in the initial condition, the system will either halt or become periodic depending on the parity of the length of the initial condition. In case $w_1 = 0^{\dot{x}_1}10^{y_1}10^{z_1}$ tag systems from this class will always become periodic when at least one 1 is scanned in the initial condition since for every w_1 produced, one and only one w_1 will be produced. In all other cases, tag systems from this class halt.

Case 1.6. $\#1 = 3$, $l_1 \equiv 0 \bmod 2$. Again we have to consider several cases, depending on the spacings between the 1's, i.e. the number of 0's between the consecutive 1's. If all spacings are odd, i.e. if $w_1 = 0^{\dot{x}_1}10^{\dot{y}_1}10^{\dot{z}_1}10^{t_1}$ (or, $w_1 = 0^{x_1}10^{\dot{y}_1}10^{\dot{z}_1}10^{\dot{t}_1}$) the proof is similar to the one for case 1.4.2., the tag system leading to unbounded growth or a halt, depending on the parity of the initial condition and the position of the 1's in w_1. The proof of the second case follows from cases 1.4.1. and 1.4.2., since either two 1's are scanned or one 1. An example of such case is $w_1 = 0^{x_1}10^{\dot{y}_1}10^{z_1}10^{t_1}$. Based on the proofs from case 1.4., we conclude that tag systems from this second subclass will either become periodic or lead to unbounded growth, depending on the parity of the initial condition and the position of the 1's. Of course, all tag systems from this class will always halt, when no 1 is scanned in the initial condition.

Case 1.7. $\#1 = 3$, $l_1 \equiv 1 \bmod 2$. Again we have to differentiate between two cases: $w_1 = 0^{s_1}10^{x_1}10^{y_1}10^{t_1}$ (and all variants) or $w_1 = 0^{s_1}10^{\dot{x}_1}10^{\dot{y}_1}10^{t_1}$ (plus all variants).

Case 1.7.1. $w_1 = 0^{s_1}10^{\dot{x}_1}10^{\dot{y}_1}10^{t_1}$. If all 1's are oddly spaced, i.e. if one 1 is scanned in w_1 all others will also be scanned, it can be proven that the tag system will always grow, when at least one w_1 is produced from the initial condition such that all its 1's are scanned, thus producing w_1^3. The following table shows that in case w_1^3 is produced, a tag system from this class will always lead to unbounded growth.

Table 4: Case $w_1 = 0^{s_1}10^{\dot{x}_1}10^{\dot{y}_1}10^{t_1}$

	w_1	w_1^3	w_1^6	...	w_1^{2m}	w_1^{2m+1}
S_0	w_1^3	w_1^6	w_1^9	...	w_1^{3m}	w_1^{3m+3}
S_1	w_1✓	w_1^3✓	w_1^9	...	w_1^{3m}	w_1^{3m}

Note that although in shift 1, w_1^3 produces w_1^3, the tag system will not become periodic. Indeed, if the tag system produces the string w_1^3 and it is entered with a shift 1, the next time w_1^3 is produced, it will be entered with a shift 0, given the fact that its length is odd.

Case 1.7.2. $w_1 = 0^{s_1}10^{x_1}10^{y_1}10^{t_1}$**.** Tag systems from this class, i.e., those for which only two 1's will be scanned in the same shift, will always lead to unbounded growth if at least one 1 is scanned in the initial condition. The table proving the result, will not be given here, since it can be easily replaced by the following reasoning. First of all, note that once two consecutive $w_1's$ have been produced, the tag system will always lead to unbounded growth grow, since two consecutive w_1's always lead to the production of at least 3 consecutive w_1's. If only one 1 has been scanned in the initial condition, leading to the production of one time w_1, the tag system will always lead to the production of two times w_1 because of the fact that l_1 is odd. Indeed, either w_1 will immediately lead to the production of two consecutive w_1's (depending on the parity of the length of the initial condition), or two consecutive w_1's will be produced the next time w_1 is produced. If no 1 is scanned in the initial condition, tag systems from this class always lead to a halt.

Case 1.8. $\#1 > 3$**,** $l_1 \equiv 0 \bmod 2$**.** In general, any tag system with $\#1 > 3$ from this class will either halt become periodic or lead to unbounded growth. For each #1, there are always three different cases to be taken into consideration. In the first case, all 1's are separated by an odd number of 0's. In generalizing Table 3 it follows that tag systems from this class will always halt or lead to unbounded growth, depending on the length of the initial condition and the position of the 1's in w_1.
The second case applies when $w_1 = 0^{s_1}10^{x_1}10^{\dot{x}_2}10^{\dot{x}_3}1...0^{\dot{x}_n}10^{t_1}$, when either one 1 is scanned or $\#1 - 1$ 1's are scanned, depending on the parity of the length of the initial condition. It easily follows from case 1.5. that a tag system from this class will either become periodic or show unbounded growth depending on the parity of the length of the initial condition and

the position of the 1's in w_1.
The last case concerns those cases where, whatever shift w_1 is entered with, at least two 1's will be scanned. Clearly, tag systems from this class will always lead to unbounded growth, except when no 1 is scanned in the initial condition, since whatever shift w_1 is entered with, it will always lead to the production of at least two w_1's.
Of course for all cases, a halt will result when no 1 is scanned in the initial condition.

Case 1.9. $\#1 > 3$, $l_1 \equiv 1 \bmod 2$. In case all 1's are oddly spaced, all 1's being scanned if one 1 is scanned in w_1, the tag system will always show unbounded growth if at least one 1 is scanned in the initial condition. This follows from generalizing the proof from case 1.7.1. In case all 1's are oddly spaced except for 1, tag systems from this class can also be proven to always lead to unbounded growth, if at least one 1 is scanned in the initial condition. The result follows from the proof of case 1.7.2. If at least two 1's are scanned, whatever shift w_1 is entered with, it is clear that also in this case the tag systems will always lead to unbounded growth, if at least one 1 is scanned in the initial condition.
To summarize, all tag systems from this class will always lead to unbounded growth, except when no 1 is scanned in the initial condition, leading to a halt.

2.0.2 Case 2. $w_0 = 1$.

Case 2.1. $\#1 = 1$. In this case the length of w_1 is a determining factor to predict the behaviour of a tag system from this class. We have to differentiate between the following two cases: $2 < l_1 < 5$ or $5 \leq l_1$.

Subcase 2.1.1. $2 < w_1 < 5$. If $2 < l_1 < 5$ the tag system will always become periodic, except when the initial condition is equal to 0, then it will halt. Note that since $w_0 = 1, \#1 = 1, w_1$ does not contain 1, and consists merely of 0's. The result follows from the following tables. It is important to note that in the case $l_1 = 3$, although w_1 can lead to the production of w_0 and thus to a halt, this will never occur given the parity of w_1:

Table 5: Case $l_1 = 3$

	w_0	w_1	w_0w_0

S_0	w_1	w_0w_0	w_1✓
S_1	HALT	w_0✓	w_1✓

Table 6: Case $l_1 = 4$

	w_0	w_1	w_0w_0
S_0	w_1	w_0w_0	w_1✓
S_1	HALT	w_0w_0	w_1✓

Subcase 2.1.2. $5 \leq w_1$**.** If $l_1 = 5$ tag systems from this class always become periodic if the initial condition is equal to: 1, 00, 10, 01, 11, 000, 001, 110, 100, 011, 010, 0000 and 0101. If it consists of only one 0, it will halt. This can easily be checked by hand. In all other cases it leads to unbounded growth. This follows from the following table:

Table 7: Case $l_1 = 5$

	w_0	w_1	w_0^2	w_0^3	w_1^2	w_0^5	...	w_0^n
S_0	w_1	w_0^3	w_1✓	w_1^2	w_0^5	w_1^3	...	$w_1^{\lceil n/2 \rceil}$
S_1	HALT	w_0^2	w_1✓	w_1✓	w_0^5	w_1^2✓	...	$w_1^{\lceil n/2 \rceil} - 1$✓

Although the table seems to allow for periodicity, the fact that w_1 is odd guarantees that once w_1^2 is produced, the tag system will always lead to unbounded growth. This can be easily checked by hand.

Clearly, if $l_1 > 5$, the tag systems will always show unbounded growth (if the length of the initial condition is longer than 1). The proof can be found by constructing a table similar to Table 7 and is left to the reader. Note that once $l_1 > 7$, the proof becomes very simple, since whatever shift w_1 is entered with, it will always lead to the production of at least 4 1's, and thus to the production of w_1w_1.

Case 2.2. $\#1 = 2, l_1 = 3$. It can be determined for any tag system from this class that it will either halt or become periodic. There are three different tag systems to be taken into account here:

$$\begin{array}{ll} 0 \rightarrow 1 & 1 \rightarrow 100 \\ 0 \rightarrow 1 & 1 \rightarrow 010 \\ 0 \rightarrow 1 & 1 \rightarrow 001 \end{array}$$

In the following tables it is shown that all three tag systems will always become periodic, except when the initial condition is equal to 0. It should again be noted that although w_1 can lead to the production of w_0 and thus to a halt, this will never occur given the parity of w_1

Table 8: Case $0 \rightarrow 1, 1 \rightarrow 100$

	w_0	w_1	$w_1 w_0$	$w_0 w_1$
S_0	w_1	$w_1 w_0$	$w_1 w_0$✓	$w_1 w_0$✓
S_1	HALT	w_0✓	$w_0 w_1$	$w_1 w_0$✓

Table 9: Case $w_0 = 1, w_1 = 010$

	w_0	w_1	$w_0 w_0$
S_0	w_1	$w_0 w_0$	w_1✓
S_1	w_1	w_1✓	w_1✓

Table 10: Case $w_0 = 1, w_1 = 001$

	w_0	w_1	$w_0 w_1$	$w_1 w_0$
S_0	w_1	$w_0 w_1$	$w_1 w_0$	$w_0 w_1$✓
S_1	w_1	w_0✓	$w_0 w_1$✓	$w_0 w_1$✓

Case 2.3. $\#1 = 2, l_1 > 3$. For any tag system from this class it can be determined that it will either halt, become periodic or lead to unbounded

growth. To prove this, we will only consider the case $l_1 = 4$ in more detail. We will first prove that once the tag system produces $w_1 w_0$, scanning the first letter of w_1 the system will always grow. There are four different tag systems in this class, i.e. $w_1 = 1000, w_1 = 0100, w_1 = 0010, w_1 = 0001$. We will only prove the first case, the proofs of the other cases being similar to the first case. The following table illustrates why the tag system will always lead to unbounded growth, once it has produced $w_1 1$, scanning the leftmost letter in $w_1 1$.

Table 11: Case $0 \to 1, 1 \to 1000$

	$w_1 1$	11	w_1	$w_1 1 w_1$
S_0	$w_1 1 w_1$	w_1	$w_1 1 \checkmark$	$w_1 1 w_1 11$
S_1	11	w_1	$11 \checkmark$	$11 w_1 1$
	$w_1 1 w_1 11$	$w_1 1 w_1 11 w_1$	$w_1 1 w_1 11 w_1 11$	....
S_0	$w_1 1 w_1 11 w_1$	$w_1 1 w_1 11 w_1 11$	$w_1 1 w_1 11 w_1 11 w_1$	....
S_1	$11 w_1 1 w_1$	$11 w_1 1 w_1 w_1 1$	$11 w_1 1 w_1 w_1 1 w_1$	....
	$w_1 1 (w_1 11)^{n-1}$	$w_1 1 (w_1 11)^{n-1} w_1$	$11 w_1 1$	$w_1 11$
S_0	$w_1 1 (w_1 11)^{n-1} w_1$	$w_1 1 (w_1 11)^{n}$	$w_1 \underbrace{w_1 1 w_1}$	$w_1 1 w_1$
S_1	$11 w_1 (1 w_1 w_1)^{n-2} 1 w_1$	$11 w_1 (1 w_1 w_1)^{n-1} 1$	$w_1 11$	$11 w_1$
	$11 w_1$	$w_1 w_1 1$	1111	$w_1 w_1$
S_0	$w_1 w_1 1$	$\underbrace{w_1 1 w_1} 1 w_1$	$w_1 w_1$	$\underbrace{w_1 1 w_1} 1$
S_1	$w_1 11 \checkmark$	1111	$w_1 w_1$	$1111 \checkmark$
	$11 w_1 1 w_1$	$w_1 11 w_1 1$	$11 w_1 11$	$w_1 11 w_1$
S_0	$w_1 \underbrace{w_1 1 w_1} 11$	$\underbrace{w_1 1 w_1} 1 w_1 1$	$w_1 \underbrace{w_1 1 w_1}$	$\underbrace{w_1 1 w_1} w_1 1$
S_1	$w_1 11 w_1 1$	$11 w_1 11$	$w_1 11 w_1$	$11 w_1 11 \checkmark$

Although the table seems to allow for some periodicity, it should be noted that this can never occur once $w_1 1$ is produced and entered with a shift 0. Suppose our initial condition is $w_1 1$, then:

$$w_1 1 \xrightarrow{\text{S1}} w_1 1 w_1 \xrightarrow{\text{S0}} 11 w_1 1 \xrightarrow{\text{S1}} w_1 \underbrace{w_1 1 w_1}$$
$$\xrightarrow{\text{S0}} \underbrace{w_1 1}\, \underbrace{w_1 1 w_1 11} \xrightarrow{\text{S0}} \underbrace{w_1 1 w_1}\, \underbrace{11 w_1 1 w_1} \xrightarrow{\text{S0}} \underbrace{w_1 1 w_1 11}\, \underbrace{w_1 w_1 1 w_1 11}$$
$$\xrightarrow{\text{S0}} \underbrace{w_1 1 w_1 11 w_1}\, 1111\, \underbrace{w_1 1 w_1} \xrightarrow{\text{S0}} \underbrace{w_1 1 w_1 11 w_1 11}\, w_1 w_1\, \underbrace{11 w_1 1}$$
$$\xrightarrow{\text{S0}} \underbrace{w_1 1 w_1 11 w_1 11}\, 1111\, \underbrace{w_1 11}.$$

where $\xrightarrow{\text{Sx}} X \xrightarrow{\text{Sy}} Y$ means that Y results from X, if X is entered with shift Sx, i.e., its first x letters are erased without being scanned. Whatever shift the last string produced in this sequence is ever entered with, the tag system will always lead to unbounded growth. If the shift 0 remains a constant, the table shows that this must indeed happen. If the shift changes at one time to 1, it is guaranteed that the string produced will contain $w_1 1 w_1 w_1 1 w_1$ as a substring (See table). Since the length of $w_1 1 w_1$ is odd, whenever the first sequence of $w_1 1 w_1$ is entered with a shift 0, $w_1 1 w_1$ will be entered with a shift 1, and vice versa and it is thus guaranteed that the system must lead to unbounded growth (See Table).
This reasoning still does not result in a proof of the fact that this tag system will always lead to unbounded growth once $w_1 1$ is produced, and entered with a shift 0, since we have merely shown it for the case where the initial condition is equal to $w_1 1$. If we would e.g. add only one 0 to the condition, the shifts completely change. Still, our reasoning remains valid. Indeed, based on the table, one can deduce that there are only three possibilities for periodicity:

$$1111 \xrightarrow{S1} w_1 w_1 \xrightarrow{Sx} 1111$$
$$11 w_1 11 \xrightarrow{S1} w_1 11 w_1 \xrightarrow{S1} 11 w_1 11$$
$$w_1 11 \xrightarrow{S1} 11 w_1 \xrightarrow{S1} 11 w_1$$

Any other path through the table that leads to neither of these periodic strings, will lead to unbounded growth, producing a string containing $w_1 1 w_1 w_1 1 w_1$ as a substring. However, these periods can only be produced if the tag system is started with initial conditions of one of the following forms:

$$111(1111)^n$$
$$0001000(w_1 w_1)^n$$
$$1100011(11 w_1 11)^n$$
$$000111000(w_1 11 w_1)^n$$
$$00011(w_1 1)^n$$
$$11000(11 w_1^n)$$

or any combination of these strings, with $n = \{0, 1, ...\}$. As is clear, for neither of these conditions can $w_1 1$ be entered with a shift 0. Any other initial condition, containing $w_1 1$ entered with a shift 0, will lead to unbounded growth. Indeed, although any such condition might still allow for the production of a string containing one of these periodic strings as a substring, they will always be combined with strings that grow or strings for which the shift does not remain constant. This follows from the productions as given in the table. Indeed, if we look at the different paths in the table leading to periodicity, it is clear that the lengths of the strings leading to the periodic strings change from odd to even (and vice versa). It is this fact that makes it impossible for strings to become periodic, except when the initial condition is in one of the forms as described above. If the initial condition is not a combination of these periodic strings, the fluctuation of the parity of the lengths of the several substrings, makes it impossible for the whole string to become periodic.

To summarize, the tag system analyzed through the table will always show unbounded growth, once $w_1 1$ is produced by the system, scanning its left-most letter. The system only halts when the initial condition is equal to 0. The system becomes periodic for the following initial conditions: 000, 1, 00, 01 and the set of periodic strings described above. All other conditions will lead to the production of $w_1 1$, entered with a shift 0 and thus to growth. A similar proof can be found for the remaining cases for which $l_1 = 4$.

The general solvability of the class of tag systems with $\#1 = 2, w_0 = 1, l_1 > 4$ follows from the following considerations. First, suppose the 1 in w_1 is at an odd position. Then, if w_1 is entered with a shift 0, it will always lead to a string longer than w_1, consisting of w_1 and $\lfloor \frac{l_1 - 1}{2} \rfloor$ times 1. If w_1 is entered with a shift 1, it will result in a string consisting of $\lfloor \frac{l_1}{2} \rfloor$ 1's.

Then, if $l_1 = 5$, the tag system will always show unbounded growth if $w_1 w_1$ is produced at least once from the initial condition. Indeed, if $w_1 w_1$ is produced, it will always lead to the production of a string consisting of one time w_1 and four times 1 (whatever shift $w_1 w_1$ is entered with) a string which will lead to the production of at least two times w_1 and two times 11, a string which clearly leads to unbounded growth.

If $l_1 = 6$ then the tag system will also always lead to unbounded growth, if $w_1 w_1$ is produced as a substring at least once. The proof is similar to the case $l_1 = 5$ and is left to the reader.

If $l_1 > 6$, once w_1 is produced from the initial condition, tag systems from this class will always lead to unbounded growth. This is the case because, whatever shift w_1 is entered with, it always leads to the production of at least 4 1's, and thus to the production of at least two w_1's or the production of one w_1 plus at least three 1's, in its turn leading to the production of at least two w_1's. To summarize, for all tag systems from this class, with $l_1 > 4$, it is clear that all but a finite number of initial conditions will always lead to unbounded growth. The remaining (finite number of) initial conditions can be easily calculated for each of the subcases, and lead to either a halt or periodicity in a finite number of steps.

Case 2.4. **#1 > 2.** We have to separate between two cases, $l_1 = 3$ and $l_1 > 3$

Subcase 2.4.1. $l_1 > 3$ For $l_1 > 3$, if all 1's are oddly spaced – implying that if one 1 is scanned, the other will also be scanned – each of the tag systems from this class will either halt, become periodic or show unbounded growth after a finite number of steps.

In case $l_1 = 4$, there are two possible tag systems to be taken into consideration: either $w_1 = 1010$, or $w_1 = 0101$. Both tag systems will halt if the initial condition is equal to 0. For all other conditions, the tag systems will always lead to unbounded growth or become periodic. Indeed, once w_1 is produced, and this will always happen for initial conditions different from 0, the tag system cannot halt. This is the case because w_1 always leads to the production of either w_0w_0 or w_1w_1. If w_0w_0 is produced, it will always lead to the production of w_1, while w_1w_1 will lead to the production of w_0^4 or w_1^4. Whether any of these two tag systems will become periodic or show unbounded growth then depends on the length of the string S_1 produced from the initial condition after all the relevant letter of the initial condition have been processed. Clearly, in case $w_1 = 0101$, if l_{S_1} odd, the system will show unbounded growth, if l_{S_1} even, it will become periodic. For the case $w_1 = 1010$, if l_{S_1} even, the system will show unbounded growth, if l_{S_1} odd, it will become periodic. This can easily be checked through the table method.

If $l_1 > 4$, tag systems in this class, with the 1's in w_1 oddly spaced, will always lead to unbounded growth, except when the initial condition is equal to 0. This result follows from the proof of case 2.3. for those cases with $l_1 > 4$. The details of the proof are left to the reader.

If all 1's are oddly spaced except for one, it is trivial to prove that tag systems from this class will always lead to unbounded growth once w_1 is produced from the initial condition, i.e., for all initial conditions except for 0. Indeed, whatever shift w_1 is entered with, it will always lead to the production of at least one w_1 plus one 1.

If w_1 is such that whatever shift it is entered with, at least two 1's are scanned, it trivially follows that any tag system from this class will always lead to unbounded growth once w_1 is produced from the initial condition, i.e., it will lead to unbounded growth with any initial conditions except for 0. Indeed, whatever shift w_1 is entered with, it will always lead to the production of at least two w_1's.

Subcase 2.4.2. Case $l_1 = 3$ We still have to show that in case $l_1 = 3$, one can determine that a tag systems will either halt, become periodic or lead to unbounded growth. There are four different tag systems in this class:

$$\begin{array}{ll} 0 \to 1 & 1 \to 110 \\ 0 \to 1 & 1 \to 101 \\ 0 \to 1 & 1 \to 011 \\ 0 \to 1 & 1 \to 111 \end{array}$$

As for the last tag system of this list, it trivially follows that it will always lead to unbounded growth, except when the initial condition is equal to 0. To prove the remaining cases, let us look at some of the possible productions from w_1, for the first tag system from the list with $w_1 = 110$ ($\dot{n}$ denotes that n is odd):

Table 12: Case $0 \to 1, 1 \to 110$

	w_1	$w_1 1$	w_1^2	$w_1 1 w_1$
S_0	$w_1 1$	$w_1 1 \checkmark$	$w_1 1 w_1$	$(w_1 1)^2$
S_1	$w_1 \checkmark$	w_1^2	$w_1^2 1$	w_1^3
	$(w_1 1)^2$	$w_1^2 1$	$w_1 1 w_1^2$	$(w_1 1)^2 w_1$
S_0	$(w_1 1)^2 \checkmark$	$w_1 1 w_1^2$	$(w_1 1)^2 w_1$	$(w_1 1)^3$
S_1	w_1^4	$w_1^2 1 \checkmark$	$w_1^4 1$	w_1^5
	$(w_1 1)^3$	w_1^3	$w_1 1 w_1^2 1$	$(w_1 1)^2 w_1^2$
S_0	$(w_1 1)^3 \checkmark$	$w_1 1 w_1^2 1$	$(w_1 1)^2 w_1^2$	$(w_1 1)^3 w_1$
S_1	w_1^6	$w_1^2 1 w_1$	$w_1^4 1 \checkmark$	$w_1^6 1$
	$(w_1 1)^n$	$(w_1)^{2n}$	$(w_1 1 w_1)^n, \dot{n}$	$(w_1 1 w_1)^n, n$
S_0	$(w_1 1)^n \checkmark$	$(w_1 1 w_1)^n$	$((w_1 1)^2 w_1^3)^{n-1} (w_1 1)^2$	$((w_1 1)^2 w_1^3)^n$
S_1	w_1^{2n}	$(w_1^2 1)^n$	$(w_1^3 (w_1 1)^2)^{n-1} w_1^3$	$(w_1^3 (w_1 1)^2)^n$
	$(w_1^2 1)^n, n$	$(w_1^2 1)^n, \dot{n}$	$w_1^{3n}, \dot{n}$	$(w_1 1 w_1^2)^n$

S_0	$(w_1 1(w_1^2 1)^2)^{\frac{n}{2}}$	$(w_1 1(w_1^2 1)^2)^{\frac{n-1}{2}} w_1 1 w_1^2$	$(w_1 1 w_1)^{\frac{3n-1}{2}} w_1 1$	$((w_1 1)^2 w_1)^n$
S_1	$(w_1^2 1 w_1 1 w_1^2 1)^{\frac{n}{2}}$	$(w_1^2 1 w_1 1 w_1^2 1)^{\frac{n-1}{2}} w_1^2 1$	$(w_1^2 1)^{\frac{3n-1}{2}} w_1$	$(w_1^4 1)^n$
	$((w_1 1)^2 w_1)^n, n$	$((w_1 1)^2 w_1)^n, \dot{n}$		
S_0	$((w_1 1)^3 w_1^5)^{\frac{n}{2}}$	$((w_1 1)^3 w_1^5)^{\frac{n-1}{2}} (w_1 1)^3$		
S_1	$(w_1^5 (w_1 1)^3)^{\frac{n}{2}}$	$(w_1^5 (w_1 1)^3)^{\frac{n-1}{2}} w_1^5$		

As is clear from the table, the length of a string produced in this tag system, can never shrink once w_1 is produced. Although the table allows for some periodicity, the tag system will always lead to unbounded growth, except when the initial condition is equal to 0, leading to a halt, or the initial condition is of the form $(w_1 1)^n$ always leading to periodicity. Although $w_1^2 1$ is self-reproducing when entered with a shift 1, it is not a periodic string, because its length is odd, i.e. a string that is a concatenation of $w_1^2 1$ is not periodic. We thus conclude that this tag system will always grow unboundedly for all but a finite class of initial conditions. Similar proofs can be given for the other two cases of tag systems, with $l_1 = 3$. The details of the proofs are left to the reader.

2.0.3 Case 3. $w_0 = 0$.

Case 3.1. $\#1 = 0$, $l_1 > 2$. It is trivial to prove that any tag system from this class will halt, since any sequence of 0's always leads to ϵ.
It should be noted that from now on, since $w_0 = 0$, any substring of 0's part of a given string produced by a tag system from the classes considered, will always ultimately lead to ϵ. In this respect, the size of any number of consecutive 0's is in a way irrelevant. Of more significance is the parity of such sequences of 0's. In the remaining sections, the sequence of 0's preceding the first 1 in w_1 and the sequence of 0's following the last 1 in w_1 will, respectively, be denoted through the indexed variables s_n and t_n (we will not e.g. use 0^{t_n} to avoid confusing notations). The intermediate sequences of 0's, separating two 1's will be denoted through indexed variables x_n, y_n and z_n. Note that for any s_n, t_n, x_n, y_n, z_n: $s_{n+1} < s_n, t_{n+1} < t_n$, $x_{n+1} < x_n, y_{n+1} < y_n$.

Case 3.2. $\#1 = 1$, $l_1 > 2$. For all tag systems from this class it can be determined that they will always halt or become periodic. The following table proves the case.

	w_0	w_1	$s_2w_1t_2$	...	$s_nw_1t_n$
S_0	HALT	$s_3t_3 \rightarrow$ HALT	$s_5t_5 \rightarrow$ HALT	...	$\epsilon \rightarrow$ HALT
S_1	0	$s_2w_1t_2$	$s_4w_1t_4$	...	w_1

From this table, it follows that the lengths of the strings produced in these tag systems, are always bounded and we have thus solved the case.

Case 3.3. $\#1 = 2$**,** $l_1 > 2$**,** $l_1 \equiv 0 \bmod 2$**.** It can be determined for any tag system from this class that it will either halt, become periodic or lead to unbounded growth. We have to take into account two cases. The 1's can be oddly or evenly spaced i.e. $w_1 = t_11x_11s_1$ or $w_1 = \dot{t_1}1\dot{x_1}1s_1$.[4]

Subcase 3.3.1. $w_1 = t_11x_11s_1$**.** The following table proves that any tag system from this class either halts or becomes periodic after a finite number of steps.

Table 14: Case $w_1 = t_11x_11s_1$

	w_0	w_1	A_1	A_2	...	A_n
S_0	w_0	$t_2w_1x_2s_2 = A_1$	$t_4A_1x_4s_4 = A_3$	$t_6A_1x_6s_6$	...	$t_kx_kA_ns_k = t_px_pA_1s_p$✓
S_1	HALT	$t_3x_3w_1s_3 = A_2$	$t_5x_5A_2s_5 = A_4$	$t_7x_7A_2s_6$	...	$t_lA_{n+1}x_ls_l = t_qA_2x_qs_q$✓

As is clear from the table, a tag system from this class will always become periodic – except for those initial conditions in which no 1 is scanned – since the number of 0's surrounding A_1 becomes bounded, while, whatever shift w_1 is entered with, it will lead to the production of w_1.

Subcase 3.3.2. $w_1 = t_11\dot{x_1}1\dot{s_1}$ For any tag system from this class it can be determined that it will either halt, become periodic or grow. The proof of this case is more complicated, and we have to subdivide the case in two cases: t_1, $\dot{x_1}$ or $\dot{s_1}$ ¿ 1; $t_1 = 0$, $\dot{x_1} = 1$, $\dot{s_1} = 1$.

[4] The proofs for the other possible combinations, $w_1 = \dot{t_1}1x_11\dot{s_1}$ or $w_1 = t_11\dot{x_1}1\dot{s_1}$ are identical to the proofs for these two forms.

SubSubcase 3.3.2.1. t_1, $\dot{x_1}$ or $\dot{s_1} > 1$. For any tag system from this class it can be determined that it will either halt, become periodic or show unbounded growth. Set $w_1 = t_1 1 \dot{x_1} 1 \dot{s_1}$. In shift 1, the tag system will produce a sequence of 0's from w_1, ultimately leading to a halt. In shift 0, we get:

$$A_1 = t_2 w_1 \lfloor \dot{x_1}/2 \rfloor w_1 s_2 \tag{3}$$

Depending on the shift, if $\dot{s_1} + \lfloor \dot{x_1}/2 \rfloor + t_1$ is even, we get:

$$t_3 A_1 0^{n_1} \tag{4}$$

or:

$$t_3 0^{n_1} A_1 \tag{5}$$

It thus follows that if $\dot{s_1} + \lfloor \dot{x_1}/2 \rfloor + t_1$ is even, and at least one w_1 is produced such that its first 1 will be scanned, the tag system will ultimately become periodic, since the lengths of the possible strings produced from w_1 in this case are bounded, but never produce the empty string.
If $\dot{x_1} + \lfloor \dot{x_1}/2 \rfloor + t_1$ is odd, the tag system produces:

$$A_2 = t_4 A_1 \lfloor \dot{x_1}/4 \rfloor A_1 s_3 \tag{6}$$

from (3), or a string merely consisting of a certain number of 0's (ultimately converging to ϵ), depending on the shift. If $\dot{x_1} + s_2 + \lfloor \dot{x_1}/2 \rfloor + t_2 + t_1$ is even, we get:

$$t_5 A_2 0^{n_2} \tag{7}$$

or:

$$t_5 0^{n_2} A_2 \tag{8}$$

again depending on the shift. Thus if $\dot{x_1} + s_2 + \lfloor \dot{x_1}/2 \rfloor + t_2 + t_1$ is even, the tag system will always halt or become periodic. A halt occurs, if no A_2 is produced. If this is not the case, but $\dot{s_1} + s_2 + (x_1 - 1)/4$ is odd, the tag system produces:

$$A_3 = t_6 A_2 \lfloor (x_1 - 1)/8 \rfloor A_2 s_4 \tag{9}$$

from (6), or a sequence of 0's depending on the shift.
Generally, tag systems from this class will always become periodic or halt once a sequence $\dot{s_1} + s_2 + s_3 + ... + s_n + \lfloor (x_1 - 1)/2^n \rfloor + t_n + ... + t_2 + t_1$, separating two consecutive A_{n-1} in A_n ($n \in \mathbb{N}, A_0 = w_1$) becomes even. Indeed, given a string $A_n = t_i A_{n-1} \lfloor \dot{x_1}/2^n \rfloor A_{n-1} \dot{s_i}$, with $\dot{s_1} + s_2 + s_3 + ... + s_n + \lfloor \dot{x_1}/2^n \rfloor + t_n + ... + t_2 + t_1$ even, the tag system will produce either $t_i A_n 0^{n_j}$ or $t_i 0^{n_j} A_n$, with the number of 0's surrounding each A_n being

bounded. If for a given tag system, there is no n such that the sequence $\dot{s_1} + s_2 + s_3 + ... + s_n + \lfloor (x_1 - 1)/2^n \rfloor + t_n + ... + t_2 + t_1$ between a pair of A_{n-1} in A_n is even, the tag system will either halt or show unbounded growth. Now, it can be easily determined (in a finite number of steps) for any tag system from this class whether there exists an n such that $\dot{s_1} + s_2 + s_3 + ... + s_n + \lfloor (x_1 - 1)/2^n \rfloor + t_n + ... + t_2 + t_1$ between a pair of A_{n-1} in A_n is even. This follows from the following lemma:[5]

Lemma 1 *For any tag system from the class 3.3.2.1. it can be proven that there is always an n, $n \in \mathbb{N}$ such that for any $i \geq n$ the sequence of 0's $\dot{s_1} + s_2 + s_3 + ... + s_i + \lfloor (x_1 - 1)/2^i \rfloor + t_i + ... + t_2 + t_1$ between a pair of A_{i-1} in A_i is of the same length as $\dot{s_1} + s_2 + s_3 + ... + s_n + \lfloor (x_1 - 1)/2^n \rfloor + t_n + ... + t_2 + t_1$.*

Proof To prove the lemma, consider again the sequence:

$$A_2 = t_4 1 s_1 + s_2 + \lfloor \frac{x}{4} \rfloor + t_2 + t_1 1 s_3 \quad (10)$$

Since for any tag system from this class, any sequence of 0's ultimately converges to ϵ, while for every iteration, each s_i rsp. t_i is converted to s_{i+1} rsp. t_{i+1}, the tag system will ultimately produce a sequence:

$$A_n = X_{n-1} s_1 + s_2 + s_3 + \ldots + s_n + \lfloor \frac{x}{2^n} \rfloor + t_n + \ldots t_3 + t_2 + t_1 Y_{n-1} \quad (11)$$

from (10) such that $s_n = \lfloor \frac{x}{2^n} \rfloor = t_n = \epsilon$, with X_{n-1} rsp. Y_{n-1} equal to A_{n-1} minus its rightmost rsp. leftmost sequence of 0's. This string can be rewritten as:

$$A_n = X_{n-1} s_1 + s_2 + s_3 + \ldots + s_{n-1} + t_{n-1} \ldots t_3 + t_2 + t_1 Y_{n-1} \quad (12)$$

If the tag system now scans A_n it produces:

$$A_{n+1} = X_n s_1 + s_2 + s_3 + \ldots + s_{n-1} + s_n + t_n + t_{n-1} \ldots t_3 + t_2 + t_1 Y_n \quad (13)$$

However, since $t_n = s_n = \epsilon$, (12) = (13) and we have thus proven the lemma. □

It follows from this lemma that one can determine for any tag system from this class whether a sequence of 0's separating two consecutive A_{i-1} in A_i will ever become even or not, since this can only take a finite number of steps. We have thus proven the case: tag systems from this class will either halt, become periodic or show unbounded growth.

[5] We are indebted to an anonymous referee for pointing out a serious error in a previous proof of this case concerning the number of 0's separating a pair of A_{n-1} and having provided us with the necessary lemma and its proof to solve the case.

SubSubcase 3.3.2.2. $t_1 = 0$, $x_1 = 1$, $s_1 = 1$ It can be proven that the only tag system in this class, with $w_1 = 1010$, will either halt or lead to unbounded growth. Clearly, if w_1 is entered with shift 0 we get $w_1 w_1$, if entered with shift 1, it will lead to a string of two 0's, and thus ultimately to ϵ. Now given A_n. Since there is always only one 0 between two consecutive A_{n-1} in A_n, the number of 0's separating such A_{n-1} is always odd, and it is thus always the case that any A_n will either lead to the production of a string consisting of 0's, or A_{n+1} with $l_{A_{n+1}} > l_{A_n}$. In other words, this tag system will always show unbounded growth, once all 0's separating consecutive 1's from the initial condition have been erased, and there is at least one w_1 remaining in this string produced. Otherwise it will halt.

Case 3.4. $\#1 = 2$, $l_1 > 2$, $l_1 \equiv 1 \mod 2$. It can be determined for any tag system from this class that it will always halt or become periodic. Again we have to consider two cases, depending on the parity of the spacing between the two 1's, i.e. $w_1 = \dot{t}_1 1 x_1 1 s_1 s$ and $w_1 = t_1 1 \dot{x}_1 1 s_1$.[6]

Subcase 3.4.1. $w_1 = \dot{t}_1 1 x_1 1 s_1$ The table that proves the result is identical to Table 14, and it thus follows that any tag system from this class will either halt or become periodic. It will always become periodic once w_1 is produced and entered with a shift 1, in all other cases it halts. A similar proof can be given for the case $w_1 = t_1 1 x_1 1 \dot{s}_1$.

Subcase 3.4.2. $w_1 = t_1 1 \dot{x}_1 1 s_1$. For any tag system from this class, it can be determined it will either halt or become periodic. We have to differentiate between two cases: t_1, x_1 or $s_1 > 1$ and $t_1 = 0$, $x_1 = 1$, $s_1 = 0$. We will not give the proof for the first case, since it is almost identical to the (rather complicated) proof of case 3.3.2.1.
In case $t_1 = 0$, $x_1 = 1$, $s_1 = 0$ we only have to consider one tag system, with $w_1 = 101$. For this tag system it can be determined that it will either halt or become periodic. This is shown through the following table:

Table 15: Case $w_0 = 0, w_1 = 101$

	w_0	w_1	$w_1 w_1$	$w_1 w_1 w_0$

[6]The proofs for the two other possible w_1 are almost identical to the proofs of these two forms.

S_0	w_0✓	w_1w_1	$w_1w_1w_0$	$w_1w_1w_0w_0$
S_1	w_0‘✓	w_0✓	$w_0w_1w_1$	$w_0w_1w_1$✓
	$w_1w_1w_0w_0$	$w_0w_1w_1w_0$	$w_0w_0w_1w_1$	$w_0w_1w_1$
S_0	$w_1w_1w_0w_0$✓	$w_0w_0w_1w_1$	$w_0w_1w_1w_0$✓	$w_0w_0w_1w_1$✓
S_1	$w_0w_1w_1w_0$	$w_1w_1w_0w_0$✓	$w_0w_0w_1w_1$✓	$w_1w_1w_0$✓

This tag system will always become periodic, once it has produced w_1, entered with a shift 0. In all other cases it will halt.

Case 3.5. **$\#1 > 2$, $l_1 > 2$, $l_1 \equiv 0 \bmod 2$.** It can be determined for each tag system from this class that it will lead to unbounded growth, become periodic or halt. To prove this, we merely have to show this in detail for the case $\#1 = 3$. There are two possible cases here: all 1's are oddly spaced, i.e.,$w_1 = t_1 1 \dot{x}_1 1 \dot{y}_1 1 \dot{s}_1$, or, only two of them are oddly spaced, i.e., $w_1 = t_1 1 \dot{x}_1 1 y_1 1 s_1$.[7] The third case we will consider here, is the generalization of the results for $\#1 = 3$ to $\#1 > 3$.

Subcase 3.5.1. **$w_1 = t_1 1 \dot{x}_1 1 y_1 1 s_1$, $\#1 = 3$.** Depending on the shift w_1 is entered with, the tag system will produce one of the following two strings:

$$A_1 = t_2 w_1 \lfloor \dot{x}_1/2 \rfloor w_1 \lfloor y_1/2 \rfloor s_2 \tag{14}$$

or:

$$B_1 = 0^{n_1} w_1 s_2 \tag{15}$$

this second string, being again a case of w_1 (surrounded by 0's) thus allowing for the possibility of periodicity. If A_1 is produced and $s_1 + \lfloor \dot{x}_1/2 \rfloor + t_1$ is odd one of the following strings is produced from A_1:

$$A_2 = t_3 A_1 \lfloor \dot{x}_1/4 \rfloor A_1 \lfloor y_1/4 \rfloor s_4 \tag{16}$$

or:

$$B_2 = t_4 B_1 \lfloor \dot{x}_1/4 \rfloor B_1 \lfloor y_1/4 \rfloor s_5 \tag{17}$$

else, if $s_1 + \lfloor \dot{x}_1/2 \rfloor + t_1$ is even, we get:

$$C_1 = t_3 A_1 \lfloor \dot{x}_1/4 \rfloor B_1 \lfloor y_1/4 \rfloor s_5 \tag{18}$$

[7] The proofs for all other w_1's, with all or not all 1's evenly spaced, are of course almost identical to the proofs to follow.

or:

$$D_1 = t_4 B_1 \lfloor \dot{x}_1/4 \rfloor A_1 \lfloor y_1/4 \rfloor s_4 \tag{19}$$

To prove the case, let us first note that once w_1 is produced by a tag system from this class it can never halt. We now have to look at what happens to each of the possible strings A_2, B_2, C_1 and D_1.[8]
We will not discuss each of these possibilities individually. Instead we will look only at A_2 since the solutions of the remaining three are quite similar. If $s_1 + \lfloor y_1/2 \rfloor + s_2 + \lfloor \dot{x}_1/2 \rfloor + t_2 + t_1$ is odd then we get either of the two following strings from A_2

$$C_2 = t_{1,4} A_2 \lfloor \dot{x}_1/8 \rfloor B_1 \lfloor y_1/8 \rfloor s_{1,5} \tag{20}$$

or

$$D_2 = t_{2,4} B_1 \lfloor \dot{x}_1/8 \rfloor A_2 \lfloor y_1/8 \rfloor s_{2,5} \tag{21}$$

If $s_1 + \lfloor y_1/2 \rfloor + s_2 + \lfloor \dot{x}_1/2 \rfloor + t_2 + t_1$ is even, we get one of the two following strings depending on the shift:

$$A_3 = t_{3,4} A_2 \lfloor \dot{x}_1/8 \rfloor A_2 \lfloor y_1/8 \rfloor s_{3,5} \tag{22}$$

or

$$B_3 = t_{3,4} B_2 \lfloor \dot{x}_1/8 \rfloor B_2 \lfloor y_1/8 \rfloor s_{3,5} \tag{23}$$

Note that cases C_2 and D_2 are similar in form to C_1 and D_1. To understand these cases, we only have to look at what happens to C_2. Since C_2 contains A_2, and A_2 here always leads to C_2 or D_2 whatever shift it is entered with, strings of the form C_2 will always lead to unbounded growth. Indeed, C_2 always leads to the production of a string containing either C_2 (or D_2) and B_1 or C_2 (or D_2) and A_1, depending on the parity of the number of 0's between A_2 and B_1 in C_2.
If A_3 is produced, i.e., if $s_1 + \lfloor y_1/2 \rfloor + s_2 + \lfloor \dot{x}_1/2 \rfloor + t_2 + t_1$ is even, everything again depends on the parity of the number of 0's separating the two substrings A_2 in A_3, the possible strings produced from A_3 being similar to those produced from A_2. Indeed, from A_3 strings C_3, D_3, A_4 or B_4 can be produced depending on the shift A_3 is entered with and the number of 0's separating the pair of A_2's in A_3.
A similar reasoning can be applied to B_3. If the number of 0's separating the two B_2 is even we get strings similar to either A_3 or B_3. However, if this is not the case B_3 will lead to the production of a string that always consists of two times A_1 and two times B_1, irrespective of the parity of the distance

[8]The string B_1 is not taking into account since it reduces to w_1.

between a pair of B_1 in B_2. This can easily be checked by hand.[9] Thus, once a string of the form B_3 is produced and the number of 0's separating the two B_2 is odd, the tag system will lead to unbounded growth. In general, once a string of the form B_n is produced, and the number of 0's separating two consecutive B_{n-1} is odd, the tag system will show unbounded growth if w_1 is produced at least once.
In general, once a tag system from this class produces a string of the form C_n or D_n, or a string of a form B_n, with the pair of B_{n-1} separated by an odd number of 0's, the tag system will always lead to unbounded growth. Now, as should be clear, lemma 1 only requires some minor changes to be valid for this case, so it follows that we can determine in a finite number of steps whether these possibilities can occur. If this is the case, the tag system will lead to unbounded growth if w_1 is produced at least once (else it will halt). If this is not the case, the tag system will either halt (if no w_1 is produced), become periodic or show unbounded growth (depending on whether all strings produced are, after a finite of number of iterations of the form B_n or A_n once the number of 0's separating a pair of B_{n-1} rsp. A_{n-1} has become constant).

Subcase 3.5.2. $w_1 = t_1 1 \dot{x_1} 1 \dot{y_1} 1 \dot{s_1}$**, #1 = 3** We have to consider two cases.

SubSubcase 3.5.2.1. $t_1 = 0, x_1 = 1, y_1 = 1, s_1 = 1$ The first case concerns only one tag system. It can be proven that this tag system, with $w_1 = 101010$ will either halt or lead to unbounded growth. The proof is identical to Case 3.3.2.1. with $w_1 = 1010$ and we will thus not repeat this here.

SubSubcase 3.5.2.2. $t_1 > 0$ Tag systems from this class will produce the following string from w_1:

$$A_1 = t_2 w_1 \lfloor x_1/2 \rfloor w_1 \lfloor y_1/2 \rfloor w_1 s_2 \tag{24}$$

[9]Indeed if the number of 0's separating two B_1 in B_2 is odd this leads to the production of a string containing B_1 and A_1. If then the distance between the two B_2 in B_3 is odd this will indeed lead to a string containing two times A_1 and two times B_1. If the distance between two B_1 in B_2 is even this leads either to the production of a string containing two times A_1 or a string containing two times B_1. But then, given the oddness of the number of 0's separating B_2 from B_2 in B_3, it again follows that a string will be produced from B_3 containing two times A_1 and two times B_1.

or a string consisting of 0's, depending on the shift. If $\dot{s}_1 + \lfloor x_1/2 \rfloor + t_1$ is even and $\dot{s}_1 + \lfloor y_1/2 \rfloor + t_1$ is odd from A_1 we get:

$$B_{1,1} = t_3 A_1 0^{n_1} \tag{25}$$

or:

$$A_{2,1} = t_3 0^{n_2} A_1 \lfloor (y_1 - 1)/4 \rfloor A_1 s_3 \tag{26}$$

depending on the shift A_1 is entered with. Similarly, if $\dot{s}_1 + \lfloor x_1/2 \rfloor + t_1$ is odd and $\dot{s}_1 + \lfloor y_1/2 \rfloor + t_1$ is even we get:

$$B_{1,2} = t_3 0^{n_3} A_1 s_3 \tag{27}$$

or:

$$A_{2,2} = t_3 A_1 \lfloor x_1/4 \rfloor A_1 0^{n_4} \tag{28}$$

depending on the shift. If both $\dot{s}_1 + \lfloor x_1/2 \rfloor + t_1$ and $\dot{s}_1 + \lfloor y_1/2 \rfloor + t_1$ are even, the tag system produces either:

$$A_{2,3} = t_3 A_1 0^{n_5} A_1 s_3 \tag{29}$$

or:

$$B_{1,3} = t_3 0^{n_6} A_1 0^{n_7} s_3 \tag{30}$$

depending on the shift A_1 is entered with. If both $\dot{s}_1 + \lfloor x_1/2 \rfloor + t_1$ and $\dot{s}_1 + \lfloor y_1/2 \rfloor + t_1$ are odd, the tag system either produces a string solely consisting of 0's, or:

$$A_{2,4} = t_3 A_1 \lfloor x_1/4 \rfloor A_1 \lfloor y_1/4 \rfloor A_1 s_3 \tag{31}$$

As was the case for cases 3.3.2.1. and 3.5.1., the behaviour of tag systems from this class heavily depends on the parity of the number of 0's separating consecutive w_1 in A_1. There are four different combinations of pairs of w_1 being separated by an odd or even number of 0's and we have to look at each of these possibilities individually. The cases for which $\dot{s}_1 + \lfloor x_1/2 \rfloor + t_1$ is even and $\dot{s}_1 + \lfloor y_1/2 \rfloor + t_1$ is odd or $\dot{s}_1 + \lfloor x_1/2 \rfloor + t_1$ is odd and $\dot{s}_1 + \lfloor y_1/2 \rfloor + t_1$ is even, are reducible to each other so we only have to look at the first. $B_{1,1}$ reduces to A_1 so we merely have to consider $A_{2,1}$.
If $s_1 + t_2 + \lfloor (y-1)/4 \rfloor + t_2 + t_1$ is odd we get either of the two following strings, depending on the shift:

$$A_{3,1} = t_4 A_{2,1} \lfloor (y-1)/8 \rfloor A_{2,1} s_4 \tag{32}$$

or:

$$B_{2,1} = t_4 B_{1,1} \lfloor (y-1)/8 \rfloor B_{1,1} s_4 \tag{33}$$

If $s_1 + t_2 + \lfloor (y-1)/4 \rfloor + t_2 + t_1$ is even we get:

$$C_{1,1} = t_4 A_{2,1} \lfloor (y-1)/8 \rfloor B_{2,1} s_4 \tag{34}$$

or:

$$D_{1,1} = t_4 B_{2,1} \lfloor (y-1)/8 \rfloor A_{2,1} s_4 \tag{35}$$

depending on the shift. Clearly if strings of the form $C_{1,1}$ or $D_{1,1}$ are produced by a tag system, it will always show unbounded growth (See case 3.5.1.). If strings $B_{2,1}$ or $A_{3,1}$ are produced, tag systems from this class might still become periodic, but this depends on the parity of the number of 0's separating the pair of $A_{2,1}$ rsp. $B_{1,1}$. They will lead to the production of strings of a form similar to either $A_{3,1}, B_{2,1}, C_{1,1}$ or $D_{1,1}$ (See case 3.5.1.). The two remaining possibilities, i.e. $\dot{s_1} + \lfloor x_1/2 \rfloor + t_1$ and $\dot{s_1} + \lfloor y_1/2 \rfloor + t_1$ are both even or odd, can be solved in a similar way. Indeed, $B_{1,3}$ reduces to A_1, $A_{2,2}$ is of the same form as $A_{2,1}$ and $A_{2,4}$ is of the same form as w_1. In general, for each of the possibilities, it can be computed in a finite number of steps whether it gives rise to unbounded growth or not, since lemma 1 only requires some minor changes to be applicable for this case. It follows that the number of 0's separating pairs of $A_{i,j}$ or $B_{i,j}$ will become constant after a finite number of steps, so it is possible to compute whether this number will ever become odd or not. To conclude, tag systems from this class will either halt (if no w_1 is produced), become periodic or show unbounded growth.

Subcase 3.5.3. $\#1 > 3$. On the basis of the proofs of the solvability of the cases $w_1 = t_1 1 x_1 1 y_1 1 \dot{s_1}$, $\#1 = 3$ and $w_1 = t_1 1 \dot{x_1} 1 y_1 1 s_1$, $\#1 = 3$ we can now prove that we can determine for any tag system with $\#1 > 3$, $w_0 = 0$, $l_1 > 3$, $l_1 \equiv 0 \bmod 2$, that it will either halt, become periodic or lead to unbounded growth. Clearly if whatever shift w_1 is entered with, at least two 1's will be scanned, it can be determined that the tag system will either halt or show unbounded growth, a halt only occurring when no 1 is scanned in the initial condition.
The two other cases are that either all 1's are oddly spaced, or all but one are oddly spaced. The proofs of these two cases immediately follow from the two cases $w_1 = t_1 1 \dot{x_1} 1 \dot{y_1} 1 \dot{s_1}$, $\#1 = 3$ and $w_1 = t_1 1 \dot{x_1} 1 y_1 1 s_1$, $\#1 = 3$.

Case 3.6. $\#1 > 2$, $l_1 > 2$, $l_1 \equiv 1 \bmod 2$. The proof of this case is very similar to the proof of case 3.5., and is thus left to the reader.
Given Cases 1–3 we have thus proven theorem 2

□

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